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Generalized Irreducible α-Matrices and Its Applications

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Abstract

The class of generalized α-matrices is presented by Cvetković, L. (2006), and proved to be a subclass of H-matrices. In this paper, we present a new class of matrices-generalized irreducible α-matrices, and prove that a generalized irreducible α-matrix is an H-matrix. Furthermore, using the generalized arithmetic-geometric mean inequality, we obtain two new classes of H-matrices. As applications of the obtained results, three regions including all the eigenvalues of a matrix are given.

Keywords

Generalized Irreducible α-Matrices, H-Matrices, Irreducible, Nonsingular, Eigenvalues

1. Introduction

H-matrices play a very important role in Numerical Analysis, in Optimization theory and in other Applied Sciences [1]-[7]. Here we call a matrix $A=(a_{ij})\in C^{n\times n}$ an H-matrix if its comparison matrix $com(A)=(m_{ij})$ defineded by

$$m_{ii}=|a_{ii}|, m_{ij}=-|a_{ij}|, i,j\in N=\{1,2,\cdots,n\}, j\neq i$$

is an M-matrix, *i.e.*, $(com(A))^{-1}\geq 0$ [4].

One interesting problem involving on H-matrices is to identify whether or not a matrix is an H-matrix [2] [8]. But it is not easy to do this by its definition. So researchers turned to study some subclasses of H-matrices, which are easy to identify [1] [2] [3] [4] [5] [8] [9] [10]. One of the classical subclasses is strictly diagonally dominant matrices (see Definition 1) which was first presented by Lévy only for real matrices [11]. And Minkowski [12] and Desplanques [13] obtained the general complex result.

Definition 1. A matrix $A=(a_{ij})\in C^{n\times n}$ is called a strictly diagonally dominant matrix if for any $i\in N$,

$$|a_{ii}|>r_i(A)=\sum_{i\neq j}|a_{ij}|$$

As is well known, a strictly diagonally dominant matrix is nonsingular.

This can lead to the following famous Geršgorin's Theorem.

Theorem 1. [12] Let $A=(a_{ij})\in C^{n\times n}$ and $\sigma(A)$ be the spectrum of *A*. Then

$$\sigma(A)\subseteq\Gamma(A)=\bigcup_{i\in N}\Gamma_i(A)$$

where $\Gamma_i(A)=\{z\in C:|z-a_{ii}|\leq r_i(A)\}$.

By considering the irreducibility of a matrix, Taussky [14] [15] extended the notion of a strictly diagonally dominant matrix, and given the following subclass of *H*-matrices (see Definition 2). A matrix *A* is irreducible if and only if its directed graph $G(A)$ is strongly connected (for details, see [16] [17]).

Definition 2. A matrix $A=(a_{ij})\in C^{n\times n}$ is called an irreducibly diagonally dominant matrix if *A* is irreducible, if for any $i\in N$,

$$|a_{ii}|\geq r_i(A) \tag{1}$$

and if strict inequality holds in (1) for at least one *i*.

Theorem 2. ([17], Theorem 1.11) For an irreducibly diagonally dominant matrix *A*, then *A* is nonsingular.

Another one subclass of *H*-matrices is provided by Ostrowski (see [14] or Theorem 1.16 of [17]).

Theorem 3. [18] For any $A=(a_{ij})\in C^{n\times n}$, and any $\alpha\in[0,1]$, assume that

$$|a_{ii}|>(r_i(A))^{\alpha}(c_i(A))^{1-\alpha}\quad\text{for each}\quad i\in N \tag{2}$$

where $c_i(A)=r_i(A^{\mathrm{T}})$. Then *A*, which is called α_2-matrices, is nonsingular and is an *H*-matrix.

By the nonsingularity of α_2-matrices, one can easily obtain the corresponding eigenvalue localization theorem as below.

Theorem 4. [17] For any $A=(a_{ij})\in C^{n\times n}$, and any $\alpha\in[0,1]$, then

$$\sigma(A)\subseteq\{z\in C:|z-a_{ii}|\leq r_i(A)^{\alpha}c_i(A)^{1-\alpha}\}$$

For irreducible matrices, Hadjidimos in [19] gave extensions of Theorem 4 by the nonsingularity of the so-called irreducible α_2-matrices (see Theorems 5 and 6).

Definition 3. A matrix $A=(a_{ij})\in C^{n\times n}$ is called an irreducible α_2-matrix if *A* is irreducible, if for any $i\in N$,

$$|a_{ii}|\geq r_i(A)^{\alpha}c_i(A)^{1-\alpha} \tag{3}$$

hold for some $\alpha\in[0,1]$, with at least one inequality being strict.

Theorem 5. ([19], Theorem 2.1) For an irreducible α_2-matrix *A*, then *A* is nonsingular.

Theorem 6. [19] For any $A=(a_{ij})\in C^{n\times n}$, and any $\alpha\in[0,1]$, for which (3) holds, then

$$\sigma(A)\subseteq\Gamma^{\alpha 1}(A)\bigcup\Gamma^{\alpha 2}(A)$$

where

$$\Gamma^{\alpha 1}(A)=\bigcup_{i\in N}\left\{z\in C:\left|z-a_{ii}\le r_i(A)^{\alpha}c_i(A)^{1-\alpha}\right|\right\}$$

$$\Gamma^{\alpha 2}(A)=\bigcup_{i\in N\setminus N_1}\left\{z\in C:\left|z-a_{ii}<r_i(A)^{\alpha}c_i(A)^{1-\alpha}\right|\right\}$$

and N_1 is the set of indices for which strict inequality holds in (3).

We remark here that although Hadjidimos in [19] pointed out that irreducible α_2-matrices is nonsingular, he didn't give the relationship between α_2-matrices and H-matrices. In fact, the class of α_2-matrices is a subclass of H-matrices, which is showed by the following theorem.

Theorem 7. For an irreducible α_2-matrix A, then A is an H-matrix.

Proof. We let $com(A)=D-B$, where $D=diag(|a_{11}|,|a_{22}|,\cdots,|a_{nn}|)$, and prove that the spectral radius $\rho(D^{-1}B)$ of $D^{-1}B$ is less than 1. In fact, if there exists an eigenvalue λ of $D^{-1}B$ such that $|\lambda|\ge 1$, then $D(\lambda I-D^{-1}B)=\lambda D-B$, is an irreducible α_2-matrix, and hence it is nonsingular. But this contradicts the fact that λ is an eigenvalue of the matrix $D^{-1}B$. Therefore, $\rho(D^{-1}B)<1$.

According to $(com(A))^{-1}=\sum_{j=0}^{\infty}(D^{-1}B)^{j}D^{-1}\ge 0$, the conclusion follows.

Recently, Cvetković in [4] presented a new subclass of H-matrices, which is called generalized α-matrices defined as below, and given a new eigenvalue localization set by using the nonsingularity of generalized α-matrices (see Theorem 9).

Theorem 8. ([4], Theorem 16) If for a matrix $A=(a_{ij})\in C^{n\times n}$, there exists $\alpha\in[0,1]$ and $k\in N$ such that for each subset $S\subseteq N$ of cardinality k

$$|a_{ii}|>\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}+r_i^{\bar{S}}(A),\bar{S}=N\setminus S \tag{4}$$

holds, where $r_i^S(A)=\sum_{j\in S,j\ne i}|a_{ij}|$ and $c_i^S(A)=r_i^S(A^{\mathrm{T}})$, then the matrix A, which is called a generalizaed α-matrices, is nonsingular, moreover it is an H-matrix.

Theorem 9. ([5], Theorem 17) For any $A=(a_{ij})\in C^{n\times n}$, and any $\alpha\in[0,1]$, then

$$\sigma(A)\subseteq\bigcap_{k\in N/S}\bigcup_{|S|=k}\bigcup_{i\in N}\Gamma_i^{\alpha,k,S}$$

where

$$\Gamma_i^{\alpha,k,S}=\left\{z\in C:|z-a_{ii}|\le\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}+r_i^{\bar{S}}(A)\right\}$$

We now present a new class of matrices–generalized irreducible α-matrix, which is different from the class of generalized α-matrices and will be proved to be an H-matrix in Section 2.

Definition 4. A matrix $A=(a_{ij})\in C^{n\times n}$ is called a generalized irreducible α-matrix if A is irreducible and if there exists $\alpha\in[0,1]$ and $k\in N$ such that for each subset $S\subseteq N$ of cardinality k

$$|a_{ii}|\ge\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}+r_i^{\bar{S}}(A) \tag{5}$$

holds, with at least one inequality in (5) being strict.

The outline of this paper is given as follows. In Section 2, we prove that a generalized irreducible α-matrix is nonsingular, and is an H-matrix. By using its nonsingularity, we also obtain a new eigenvalue localization set. Combining with the generalized arithmetic-geometric mean inequality, we in Section 3 obtain two other subclasses of H-matrices, consequently, two corresponding eigenvalue localization set. And then the simplifications of the obtained eigenvalue localization set are given in Section 4.

2. Nonsingularity of Generalized Irreducible α-Matrices

In this section, we prove that a generalized irreducible α-matrix is nonsingular, and obtain a new eigenvalue localization set by using its nonsingularity.

Theorem 10. If a matrix $A=(a_{ij})\in C^{n\times n}$ is a generalized irreducible α-matrix, then it is nonsingular, moreover it is an H-matrix.

Proof. First, Apparent we remark that the case $k=1$ represents the class of irreducibly diagonally dominant matrices, while $k=n$ represents irreducible α_2-matrices, so in both cases the nonsingularity has already been shown in Theorem 2 and Theorem 5, respectively. So, from now on, we suppose that $1<k<n$.

Suppose on the contrary that A is singular. Then there exists a nonzero vector $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$ such that $Ax=0$, that is,

$$-a_{ii}x_i=\sum_{i\ne j,j=1}^{n}a_{ij}x_j\text{, for each } i\in N$$

Taking absolute values in the above equation and using the triangle inequality gives

$$|a_{ii}||x_i|\le\sum_{i\ne j,j=1}^{n}|a_{ij}||x_j|=\sum_{i\ne j,j\in S}|a_{ij}||x_j|+\sum_{i\ne j,j\in\bar{S}}|a_{ij}||x_j|\quad\text{for each } i\in N$$

Note that for the nonzero vector $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$ there always exists a subset $S\subset N$ of cardinality k such that $|x_i|\ge|x_j|$ and $|x_i|>0$ for each $i\in S$ and each $j\in\bar{S}$. Hence, for each $i\in S$.

$$|a_{ii}||x_i|\le\sum_{i\ne j,j=1}^{n}|a_{ij}||x_j|\le\sum_{i\ne j,j\in S}|a_{ij}||x_j|+r_i^{\bar{S}}(A)|x_i| \tag{6}$$

equivalently,

$$\left(|a_{ii}|-r_i^{\bar{S}}(A)\right)|x_i|\le\sum_{i\ne j,j=1}^{n}|a_{ij}||x_j|$$

Furthermore, by (5) in Definition 4, we have

$$\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}|x_i| \le \left(|a_{ii}| - r_i^{\bar{S}}(A)\right)|x_i| \le \sum_{j\in S, j\ne i}|a_{ij}||x_j|, i \in S \tag{7}$$

with at least one strict inequality holds above. Using Höder's inequality (see Lemma 2.1 in [19]) we get

$$\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}|x_i| \le \left(\sum_{j\in S, j\ne i}|a_{ij}|\right)^{\alpha}\left(\sum_{j\in S, j\ne i}|a_{ij}||x_j|^{\frac{1}{1-\alpha}}\right)^{1-\alpha}, \quad i \in S$$

that is

$$\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}|x_i| \le \left(r_i^S(A)\right)^{\alpha}\left(\sum_{j\in S, j\ne i}|a_{ij}||x_j|^{\frac{1}{1-\alpha}}\right)^{1-\alpha}, \quad i \in S \tag{8}$$

without loss of generality, suppose that for any $i \in S$, $r_i^S(A) \ne 0$. In fact, if there exists $i_0 \in S$ such that $r_{i_0}^S(A) = 0$, *i.e.*, $a_{i_0 k} = 0$ for each $k \in S$, $k \ne i_0$, then from (7),we have

$$\left(|a_{i_0 i_0}| - r_{i_0}^{\bar{S}}(A)\right)|x_{i_0}| \le 0.$$

Note that $|x_i| \ne 0$ for each $i \in S$. then

$$|a_{i_0 i_0}| \le r_{i_0}^{\bar{S}}(A) = r_{i_0}(A).$$

Since A is a generalized irreducible α-matrix, we have

$$|a_{i_0 i_0}| \ge \left(r_{i_0}^S(A)\right)^{\alpha}\left(c_{i_0}^S(A)\right)^{1-\alpha} + r_{i_0}^{\bar{S}}(A) = r_{i_0}^{\bar{S}}(A)$$

hence,

$$|a_{i_0 i_0}| = r_{i_0}^{\bar{S}}(A), \quad i_0 \in S \tag{9}$$

Furthermore, by (6) and (9), we get that

$$|a_{i_0 i_0}||x_{i_0}| = \sum_{j\in S}|a_{i_0 j}||x_j| = r_{i_0}^{\bar{S}}(A)$$

which implies that there is $j_0 \in \bar{S}$ such that $a_{i_0 j_0} \ne 0$ and $|x_{i_0}| = |x_{j_0}| \ne 0$.

Because A is irreducible. Let $S_1 = \left(S \setminus \{i_1\}\right) \cup \{j_0\}$, for $i_1 \in S, i_1 \ne i_0$. Note that

$$r_{i_0}^{S_1}(A) \ge |a_{i_0 j_0}| > 0$$

then we only consider S_1 instead of S.

For every $i \in S$, $r_{i_0}^S(A) > 0$, By canceling $\left(r_i^S(A)\right)^{\alpha}$ on both sides of (8)and raising both sides of (8) to the power $\dfrac{1}{1-\alpha}$, we have

$$\sum_{i\in S}\left(c_i^S(A)\right)|x_i|^{\frac{1}{1-\alpha}} \le \left(\sum_{j\in S, j\ne i}|a_{ij}||x_j|^{\frac{1}{1-\alpha}}\right) i \in S$$

where strict inequality holds above for at least one $i \in S$. Summing on all i in S in the above inequalities gives

$$\sum_{i\in S}\left(c_i^S(A)\right)|x_i|^{\frac{1}{1-\alpha}} < \sum_{i\in S}\left(\sum_{j\in S, j\ne i}|a_{ij}||x_j|^{\frac{1}{1-\alpha}}\right)$$

equivalently

$$\sum_{i\in S}\left(c_i^S(A)\right)\left|x_i\right|^{\frac{1}{1-\alpha}}<\sum_{i\in S}\left(\sum_{j\in S,j\neq i}\left|a_{ij}\right|\left|x_j\right|^{\frac{1}{1-\alpha}}\right)=\sum_{j\in S}\left(c_i^S(A)\right)\left|x_j\right|^{\frac{1}{1-\alpha}}.$$

This is a contradiction. Therefore, A is nonsingular.

Moreover, similar to the proof of Theorem 7, we can easily prove that A is an H-matrix.

From Theorem 10, we easily get the corresponding eigenvalue localization set as below.

Corollary 1. For any $A=(a_{ij})\in C^{n\times n}$, and any $\alpha\in[0,1]$, then

$$\sigma(A)\subseteq\bigcap_{k\in N/S}\bigcup_{|S|=k}\left(\left(\bigcup_{i\in S_1}\Gamma_i^{\alpha,k,S_1}\right)\cup\left(\bigcup_{i\in S_2}\Gamma_i^{\alpha,k,S_2}\right)\right)$$

where

$$\Gamma_i^{\alpha,k,S_1}=\left\{z\in C:\left|z-a_{ii}\right|\leq\left(r_i^S(A)\right)^{\alpha}\left(c_i^{S_1}(A)\right)^{1-\alpha}+r_i^{\bar{S}}(A)\right\};$$

$$\Gamma_i^{\alpha,k,S_2}=\left\{z\in C:\left|z-a_{ii}\right|<\left(r_i^S(A)\right)^{\alpha}\left(c_i^{S_2}(A)\right)^{1-\alpha}+r_i^{\bar{S}}(A)\right\}.$$

and $S_2=S\setminus S_1$ with S_1 is the set of indices for which strict inequality holds in (5).

3. Applications

Combining the nonsingularity of generalized (irreducible) α-matrices with the generalized arithmetic-geometric mean inequality:

$$\alpha a+(1-\alpha)b\geq a^{\alpha}b^{1-\alpha}$$

where $a,b\geq0$ and $\alpha\in[0,1]$.

We obtain two other subclasses of H-matrices, consequently, two new eigenvalue localization set.

Theorem 11. If for a matrix $A=(a_{ij})\in C^{n\times n}$, there exists $\alpha\in[0,1]$ and $k\in N$ such that for each subset $S\subseteq N$ of cardinality k

$$\left|a_{ii}\right|>\alpha r_i^S(A)+(1-\alpha)c_i^S(A)+r_i^{\bar{S}}(A) \tag{10}$$

holds, then A, which is called a generalized sum α-matrix, is nonsingular, moreover it is an H-matrix.

Proof. By the generalized arithmetic-geometric mean inequality, we have

$$\left|a_{ii}\right|>\alpha r_i^S(A)+(1-\alpha)c_i^S(A)+r_i^{\bar{S}}(A)\geq\left(r_i^S(A)\right)^{\alpha}\left(c_i^S(A)\right)^{1-\alpha}+r_i^{\bar{S}}(A)$$

This implies that A is generalized α-matrix. Hence A is nonsingular. Furthermore, similar to the proof of Theorem 7, we can obtain easily that A is an H-matrix.

From Theorem 11, we also get a corresponding eigenvalue localization set.

Corollary 2. For any $A=(a_{ij})\in C^{n\times n}$, and any $\alpha\in[0,1]$, then

$$\sigma(A)\subseteq\bigcap_{k\in N/S}\bigcup_{|S|=k}\bigcup_{i\in S}\gamma_i^{\alpha,k,S}$$

where

$$\gamma_i^{\alpha,k,S} = \left\{ z \in C : \left| z - a_{ii} \right| \le \alpha r_i^S(A) + (1-\alpha) c_i^S(A) + r_i^{\bar{S}}(A) \right\}$$

According to Theorem 10 and the generalized arithmetic-geometric mean inequality, we can obtain easily the following subclass of *H*-matrices and the corresponding eigenvalue localization set.

Theorem 12. If for an irreducible matrix $A = (a_{ij}) \in C^{n\times n}$, there exists $\alpha \in [0,1]$ and $k \in N$ such that for each subset $S \subset N$ of cardinality *k*.

$$\left| a_{ii} \right| \ge \alpha r_i^S(A) + (1-\alpha) c_i^S(A) + r_i^{\bar{S}}(A) \tag{11}$$

holds, with at least one inequality in (11) being strict, then A is nonsingular, moreover it is an *H*-matrix.

Corollary 3. For any $A = (a_{ij}) \in C^{n\times n}$, and any $\alpha \in [0,1]$, then

$$\sigma(A) \subseteq \bigcap_{k\in N/S} \bigcup_{|S|=k} \left(\left(\bigcup_{i\in S_1} \gamma_i^{\alpha,k,S_1} \right) \cup \left(\bigcup_{i\in S_2} \gamma_i^{\alpha,k,S_2} \right) \right)$$

where

$$\gamma_i^{\alpha,k,S_1} = \left\{ z \in C : \left| z - a_{ii} \right| \le \alpha r_i^S(A) + (1-\alpha) c_i^S(A) + r_i^{\bar{S}}(A) \right\}$$

$$\gamma_i^{\alpha,k,S_2} = \left\{ z \in C : \left| z - a_{ii} \right| < \alpha r_i^S(A) + (1-\alpha) c_i^S(A) + r_i^{\bar{S}}(A) \right\}$$

and $S_2 = S \setminus S_1$ with S_1 is the set of indices for which strict inequality holds in (11).

4. Simplifications of Eigenvalue Localization Sets

The eigenvalue localization sets in Theorem 9 and Corollary 2 are not of much practical use because of the restriction of *α*. To solve this problem, we in this section use the method provided in [5] [6], and obtain more convenient forms of the two eigenvalue localization sets. First, the sufficient and necessary conditions of generalized *α*-matrices and generalized sum *α*-matrices are given.

For a matrix $A = (a_{ij}) \in C^{n\times n}$ with $n \ge 2$, and for $S \subseteq N$ of cardinality $k \in N$, we partition the set of indices S into three sets:

$$R = \left\{ i \in S : r_i^S(A) > c_i^S(A) \right\}$$

$$C = \left\{ i \in S : r_i^S(A) < c_i^S(A) \right\}$$

$$L = \left\{ i \in S : r_i^S(A) = c_i^S(A) \right\}$$

where $r_i^S(A) = c_i^S(A) = 0$.

Consequently, $R = C = 0$ if *k* = 1. Obviously, $S = R \cup C \cup L$.

Lemma 13. A matrix $A = (a_{ij}) \in C^{n\times n}$ with $n \ge 2$, is a generalized *α*-matrix if and only if there exists $k \in N$, such that for each subset $S \subseteq N$ of cardinality *k* the following two conditions hold:

1) $\left| a_{ii} \right| > \min\left\{ r_i^S(A), c_i^S(A) \right\} + r_i^{\bar{S}}(A), i \in S$;

2) $\log_{\frac{r_i^S(A)}{c_i^S(A)}} \frac{|a_{ii}| - r_i^S(A)}{c_i^S(A)} > \log_{\frac{r_j^S(A)}{c_j^S(A)}} \frac{c_j^S(A)}{|a_{ii}| - r_i^S(A)}$,

for each $i \in R$, for which $c_i^S(A) \neq 0$, and for each $j \in C$, for which $r_j^S(A) \neq 0$.

Proof. The case $k = 1$: The class of generalized α-matrices reduces to strictly diagonally dominant matrices. And note that the condition (1) changes to

$$|a_{ii}| > r_i^{\bar{S}}(A) = r_i(A), i \in S.$$

This also holds for each $S \subseteq N$ of cardinality $k = 1$, that is, for any $i \in N$, $|a_{ii}| > r_i(A)$, which implies that A is strictly diagonally dominant.

The case $k = n$: The class of generalized α-matrices reduces to α_2-matrices. On the other hand, the condition (1) changes to

$$|a_{ii}| > \min\{r_i^S(A), c_i^S(A)\} = \min\{r_i(A), c_i(A)\}.$$

And the condition (2) changes to

$$\log_{\frac{r_i^S(A)}{c_i^S(A)}} \frac{|a_{ii}|}{c_i(A)} > \log_{\frac{c_j^S(A)}{r_j^S(A)}} \frac{c_j(A)}{|a_{ii}|}, \quad i \in S.$$

Hence by Theorem 5 in [5], A in this case is an α_2-matrix.

The case $1 < k < n$: Similar to the proof of Theorem 5 in [5], the conclusion in this case follows easily.

Similar to the proof of Lemma 13, for generalized sum α-matrices we also obtain easily its sufficient and necessary condition by Theorem 4 in [5].

Lemma 14. A matrix $A = (a_{ij}) \in C^{n \times n}$ with $n \geq 2$, is a generalized sum α-matrix if and only if there exists $k \in N$ such that for each subset $S \subseteq N$ of cardinality k the following two conditions hold:

1) $|a_{ii}| > \min\{r_i^S(A), c_i^S(A)\} + r_i^{\bar{S}}(A), i \in S$;

2) $\frac{|a_{ii}| - r_i^S(A) - c_i^S(A)}{r_i^S(A) - c_i^S(A)} > \frac{c_i^S(A) - \left(|a_{ii}| - r_i^{\bar{S}}(A)\right)}{c_i^S(A) - r_i^S(A)}$

for each $i \in R$ and each $j \in C$.

We now establish two eigenvalue localization sets by Lemmas 13 and 14, which are the equivalent forms of the sets in Theorem 9 and Corollary 2 respectively.

Corollary 4. For any $A = (a_{ij}) \in C^{n \times n}$, then

$$\sigma(A) \subseteq \bar{\Gamma}^{k,S}(A) \cup \hat{\Gamma}^{k,S}(A),$$

where

$$\bar{\Gamma}^{k,S}(A) = \bigcap_{k \in N/S} \bigcup_{|S|=k} \bigcup_{i \in S} \left\{z \in C : |z - a_{ii}| \leq \min\left(r_i^S(A), c_i^S(A)\right) + r_i^{\bar{S}}(A)\right\};$$

$$\hat{\Gamma}^{k,S}(A) = \bigcap_{k \in N \setminus S} \bigcup_{|S|=k} \bigcup_{\substack{i \in R \subseteq S, c_i^S(A) \neq 0 \\ j \in C \subseteq S, r_i^S(A) \neq 0}} \hat{\Gamma}_{ij}^{k,S}(A);$$

and

$$\hat{\Gamma}_{ij}^{k,S}(A)=\left\{z\in C:\frac{|z-a_{ii}|-r_i^{\bar{S}}(A)}{c_i^S(A)}\left(\frac{|z-a_{jj}|-r_j^{\bar{S}}(A)}{c_j^S(A)}\right)^{\log_{\frac{c_i^S(A)}{r_i^S(A)}}\frac{r_i^S(A)}{c_i^S(A)}}\le 1\right\}.$$

Proof. For any $\lambda\in\sigma(A)$, $\lambda I-A$ is singular. Note that the moduli of every off-diagonal entry of $\lambda I-A$ is the same as *A*. Hence, for each $S\subseteq N$, the sets $R\subseteq N$ and $C\subseteq N$ for the matrix $\lambda I-A$ remain the same. If $\lambda\neq\bar{\Gamma}^{k,S}(A)\cup\hat{\Gamma}^{k,S}(A)$, then $\lambda I-A$ satisfies the conditions (1) and (2) of Lemma 13, hence $\lambda I-A$ is a generalized α-matrix, which implies that $\lambda I-A$ is nonsingular. This is a contradiction. Hence, $\lambda=\bar{\Gamma}^{k,S}(A)\cup\hat{\Gamma}^{k,S}(A)$.

Combining with Lemma 14 and similar to the proof of Corollary 4, we have the following result.

Corollary 5. For any $A=(a_{ij})\in C^{n\times n}$, then

$$\sigma(A)\subseteq\bar{\Gamma}^{k,S}(A)\cup\hat{\gamma}^{k,S}(A),$$

where $\bar{\Gamma}^{k,S}(A)$ is defined as Corollary 4,

$$\hat{\gamma}^{k,S}(A)=\bigcap_{k\in N/S}\bigcup_{|S|=k}\bigcup_{\substack{i\in R\subseteq S\\ j\in C\subseteq S}}\hat{\gamma}_{ij}^{k,S}(A).$$

and

$$\begin{aligned}\hat{\gamma}^{k,S}(A)=\Big\{z\in C:&\left(|z-a_{ii}|-r_i^{\bar{S}}(A)\right)\left(c_j^S(A)-r_j^S(A)\right)\\&+\left(|z-a_{jj}|-r_j^{\bar{S}}(A)\right)\left(r_i^S(A)-c_i^S(A)\right)\\&\le c_j^S(A)r_i^S(A)-c_i^S(A)r_j^S(A)\Big\}\end{aligned}$$

Remark 1. Obviously, the forms of the sets in Corollaries 4 and 5, which are without the restriction of α, are easier to be determined than those in Theorem 9 and Corollary 2. In addition, similar to the proof of Lemma 3.5 in [6], we can prove that the set in Corollary 4 is tighter than that in Corollary 5, *i.e.*,

$$\left(\bar{\Gamma}^{k,S}(A)\cup\hat{\Gamma}^{k,S}(A)\right)\subseteq\left(\bar{\Gamma}^{k,S}(A)\cup\hat{\gamma}^{k,S}(A)\right)$$

However, $\bar{\Gamma}^{k,S}(A)\cup\hat{\Gamma}^{k,S}(A)$ is determined more difficultly than $\bar{\Gamma}^{k,S}(A)\cup\hat{\gamma}^{k,S}(A)$. because it is difficult to compute exactly $\log_{\frac{c_j^S(A)}{r_j^S(A)}}\frac{r_i^S(A)}{c_i^S(A)}$ in some cases.

5. Conclusion

In this paper, we present a new class of matrices-generalized irreducible α-matrices, and prove that a generalized irreducible α-matrix is an *H*-matrix. Furthermore, using the generalized arithmetic-geometric mean inequality, we obtain two new classes of *H*-matrices. As applications of the obtained results, three regions including all the eigenvalues of a matrix are given.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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