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Moore-Penrose Inverse and Semilinear Equations

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Abstract

In this paper, we study the existence of solutions for the semilinear equation $Ax + f(x) = b$, where A is a $m \times n$, $m \geq n$, $b \in I\!R^n$ and $f : I\!R^m \to I\!R^n$ is a nonlinear continuous function. Assuming that the Moore-Penrose inverse $A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$ exists (A^{T} denotes the transposed matrix of A) which is true whenever the determinant of the $n \times n$ matrix AA^{T} is different than zero, and the following condition on the nonlinear term f satisfied $\lim_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|} = 0$. We prove that the semilinear equation has solutions for all $b \in I\!R^n$. Moreover, these solutions can be found from the following fixed point relation $x_b = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\left(b - f\left(x_b\right)\right)$.

Keywords

Semilinear Equations, Moore-Penrose Inverse, Rothe's Fixed Point Theorem

1. Introduction

This work is devoted to study the existence of solutions for the following semilinear equation

$$Ax + f(x) = b, \; b \in I\!R^n, \; x \in I\!R^m \tag{1.1}$$

where A is a $m \times n$ matrix, $m \geq n$, $b \in I\!R^n$ and $f : I\!R^m \to I\!R^n$ is a nonlinear continuous function.

Definition 1.1. The Equation (1.1) is said to be solvable if for all $b \in I\!R^n$ there exists $x \in I\!R^m$ such that

$$Ax + f(x) = b.$$

Proposition 1.1. *The Equation* (1.1) *is solvable if, and only if, the operator* $A+f(\cdot):\mathbb{R}^m \to \mathbb{R}^n$ *is surjective.*

The corresponding linear equation $Ax=b$ has been studied in [1] where a generalization of Cramer's Rule is given applying the Moore-Penrose inverse $A^{+}=A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$ that can be used when $\left(AA^{\mathrm{T}}\right)^{-1}$ exists, and a result from [2]. More information about the Moore-Penrose inverse can be found in [3] and [4].

In this paper, using Moore-Penrose inverse A^{+} and the Rothe's Fixed Theorem [5] [6] [7], we shall prove the following theorem:

Theorem 1.1. *If* $A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$ *exists and* f *is continuous and satisfies the condition*

$$\lim_{\|x\|\to\infty}\frac{\|f(x)\|}{\|x\|}=0, \tag{1.2}$$

then Equation (1.1) *is solvable.*

Moreover, for each $b\in\mathbb{R}^n$ *there exists* $x_b\in\mathbb{R}^m$ *such that*

$$Ax_b+f(x_b)=b,$$

where $x_b=A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\left(b-f(x_b)\right)$.

The following theorem will be used to prove our main result.

Theorem 1.2. (*Rothe's Fixed Theorem* [4] [5] [6]) *Let* E *be a Banach space. Let* $B\subset E$ *be a closed convex subset such that the zero of* E *is contained in the interior of* B. *Let* $\Phi:B\to E$ *be a continuous mapping with* $\Phi(B)$ *relatively compact in* E *and* $\Phi(\partial B)\subset B$. *Then there is a point* $x^{*}\in B$ *such that* $\Phi(x^{*})=x^{*}$.

2. Proof of the Main Theorems

In this section we shall prove the main results of this paper, Theorem 1.1, formulated in the introduction of this paper, which concern with the solvability of the semilinear Equation (1.1).

Proof of Theorem 1.1. Using the Moore-Penrose inverse we define the operator $\mathcal{K}:\mathbb{R}^m\to\mathbb{R}^m$ by

$$\mathcal{K}(x)=A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\left(b-f(x)\right),$$

and from condition (1.2) we obtain that

$$\lim_{\|x\|\to\infty}\frac{\|\mathcal{K}(x)\|}{\|x\|}=0. \tag{2.3}$$

Claim. The operator $\mathcal{K}$ has a fixed point. In fact, for a fixed $0<\rho<1$, there exists $R>0$ big enough such that

$$\|\mathcal{K}(x)\|\le\rho\|x\|,\ \|x\|=R.$$

Hence, if we denote by $B(0,R)$ the ball of center zero and radius $R>0$, we get that $\mathcal{K}(\partial B(0,R))\subset B(0,R)$. Since $\mathcal{K}$ is compact and maps the sphere $\partial B(0,R)$ into the interior of the ball $B(0,R)$, we can apply Rothe's fixed point Theorem 1.2 to ensure the existence of a fixed point $x_b\in B(0,R)\subset\mathbb{R}^m$ such that

$$x_b = \mathcal{K}(x_b). \tag{2.4}$$

Then,

$$x_b = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\left(b - f\left(x_b\right)\right).$$

Then

$$Ax_b = b - f\left(x_b\right) \Leftrightarrow Ax_b + f\left(x_b\right) = b.$$

This complete the proof. □

From Banach Fixed Point Theorem it is easy to prove the following theorem that we will use to prove the next result of this paper.

Theorem 2.1. *Let* W *be a Hilbert space and* $H : W \to W$ *is a Lipschitz function with a Lipschitz constant* $0 < h < 1$ *and consider* $F(w) = w + Hw$. *Then* F *is an homeomorphism whose inverse is a Lipschitz function with a Lipschitz constant* $(1-h)^{-1}$.

Theorem 2.2. *If the Moore-Penrose* $A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$ *exists and the following condition holds*

$$\left\| f\left(x_2\right) - f\left(x_1\right)\right\| \le L\left\|x_2 - x_1\right\|, \quad x_1, x_2 \in I\!R^m, \tag{2.5}$$

and

$$\left\|A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\right\| L < \rho < 1, \tag{2.6}$$

then the Equation (1.1) *is solvable and a solution of it is given by*

$$x_b = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\left(I + f \circ \Gamma\right)^{-1}(b), \tag{2.7}$$

where $\Gamma = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$.

Proof. Define the operator $F = A + f : I\!R^m \to I\!R^n$. Then $F \circ \Gamma = I + f \circ \Gamma$ and

$$\left\|\left(f \circ \Gamma\right)\left(b_2\right) - \left(f \circ \Gamma\right)\left(b_1\right)\right\| \le \left\|\Gamma\right\| L \left\|b_2 - b_1\right\|, \ \forall b_1, b_2 \in I\!R^n,$$

and from condition (2.6)

$$\left\|\Gamma\right\| L < \rho < 1. \tag{2.8}$$

Therefore, from Theorem 2.1 and (2.8) we have that $F \circ \Gamma = I + f \circ \Gamma$ is a homeomorphism Lipschitizian with a Lipschitz constant $\dfrac{1}{1-\rho}$.

Then,

$$F \circ \left(\Gamma \circ \left(I + f \circ \Gamma\right)^{-1}\right) = I.$$

Hence, $x_b = \left(\Gamma \circ \left(I + f \circ \Gamma\right)^{-1}\right)(b)$ is a solution of (1). In fact,

$$F\left(x_b\right) = b \Leftrightarrow Ax_b + f\left(x_b\right) = b,$$

and this complete the proof. □

3. Practical Example

Now, we shall apply **Theorem 1.1** to find one solution of the following semilinear system

$$\begin{cases} x_1 + x_2 + \sin(x_1 x_2) = 1 \\ -x_1 + x_2 + x_3 + \cos(x_2 x_3) = 1 \end{cases} \tag{3.9}$$

In this case, the vector of unknown x, the operators A, $f(x)$ and the system second member b are:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \text{ and } f(x) = \begin{pmatrix} \sin(x_1 x_2) \\ \cos(x_2 x_3) \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore, (3.9) can be written in the form of (1.1).

$$Ax + f(x) = b \tag{3.10}$$

Applying Theorem 1.1 a solution of (3.10) can be obtained as a solution of the fixed-point problem:

$$x = A^{\mathrm{T}} \left(AA^{\mathrm{T}}\right)^{-1} \left(b - f(x)\right) \tag{3.11}$$

In this particular example, one has:

$$A^{\mathrm{T}} \left(AA^{\mathrm{T}}\right)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \tag{3.12}$$

To solve this problem numerically, one uses fixed-point iterations directly, *i.e.* one uses the following fixed point method:

$$\begin{cases} lx_{n+1} = A^{\mathrm{T}} \left(AA^{\mathrm{T}}\right)^{-1} \left(b - f(x_n)\right) \\ x_0 = \begin{pmatrix} 20 \\ 10 \\ -1 \end{pmatrix} \end{cases} \tag{3.13}$$

and an error tolerance of 10^{-10}, where the error is defined for each iteration as

$$\text{Error}(n) = \left\| x_n - x_{n-1} \right\|, \quad \text{for } n = 1, 2, \ldots \tag{3.14}$$

In the following figures one shows the convergence process to obtain the approximate solution. Thus, **Figure 1** shows the fixed-point iterations (3.13) for different groups of iterations, *i.e.* in the subfigure "Iteration from 0 to 7" it being showed the seven first fixed-point iteration values and the initial condition x_0, thus in the figure "Iteration from 8 to 15" it being showed the next eight the fixed-point iteration values and so on for the other subfigures. By changing the scale in the subfigures, one observes the accumulation of the point-fixed iteration values in a specific place of space and that is an indicative of fixed-point iterations convergence.

As in the previous figure, **Figure 2** shows the convergence error (3.14) of the fixed-point iterations for different groups of iterations. Herein, one can appreciate error convergences to zero quickly.

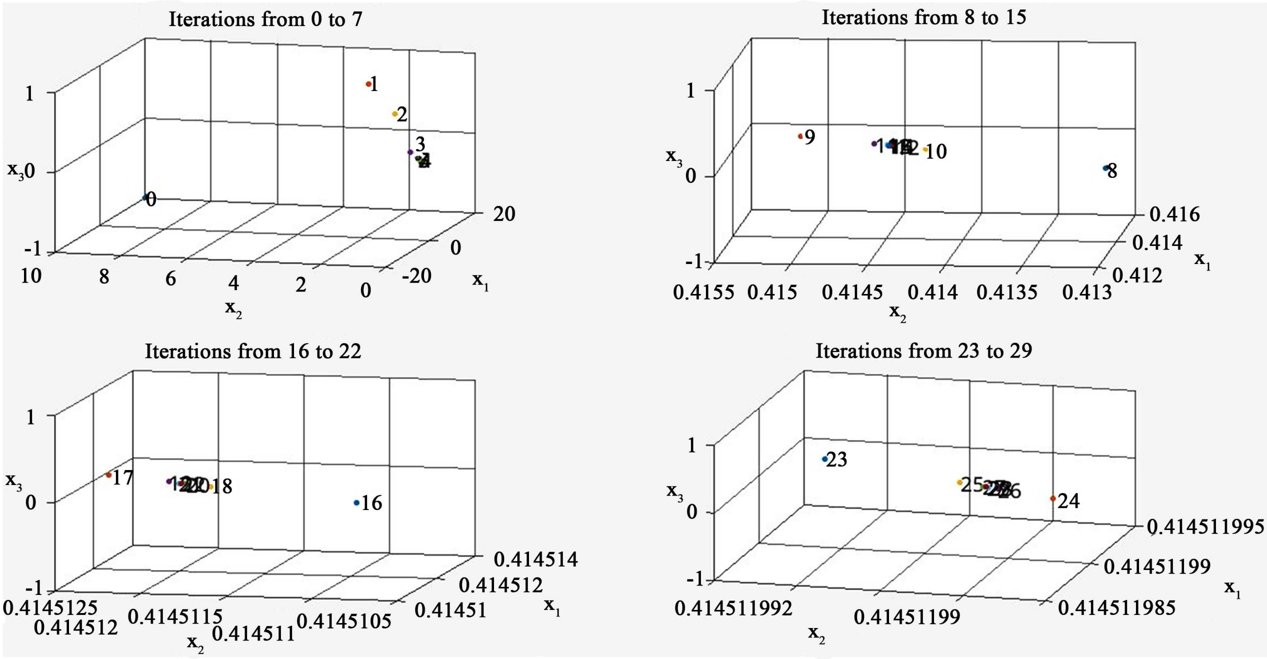

Figure 1. Convergence of fixed-point iterations.

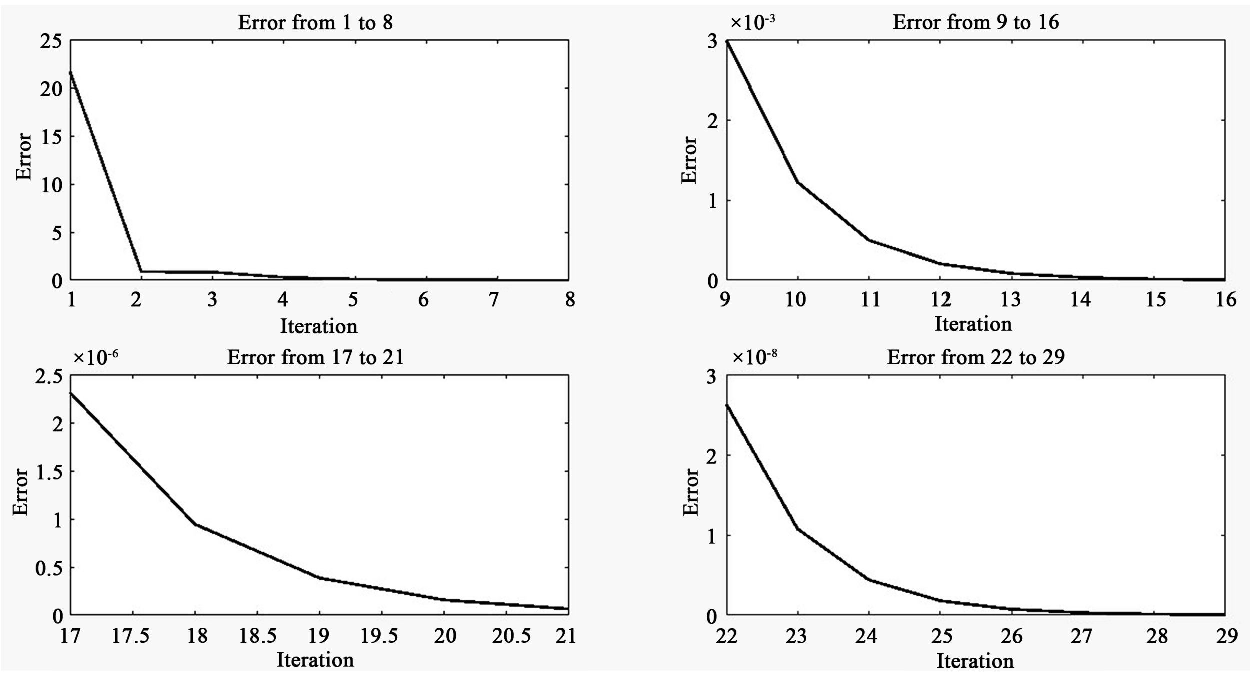

Figure 2. Error for each iteration.

The approximated value obtained for x solution of (3.13) is:

$$\begin{pmatrix} 414.511990290326e-003 \\ 414.511990290326e-003 \\ 0.00000000000000e+000 \end{pmatrix}$$

Here in, one presents the value Table 1 of fixed-point iteration.

Table 1. Fixed-point iteration values.

i	W_1	W_2	W_3
0	20.0000000000000e+000	10.0000000000000e+000	−1.00000000000000e+000
1	17.1128840687711e−003	1.85618441314522e+000	919.535764538226e−003
2	−83.6859033311723e−003	1.05192657611689e+000	567.806239724032e−003
3	457.390133251325e−003	630.527636166526e−003	86.5687514576005e−003
4	357.047377906710e−003	358.536714067085e−003	744.668080187583e−006
5	436.167364616208e−003	436.167400258264e−003	17.8210279311308e−009
6	405.451739883687e−003	405.451739883687e−003	0.00000000000000e+000
7	418.174158181806e−003	418.174158181806e−003	0.00000000000000e+000
8	413.010123030581e−003	413.010123030581e−003	0.00000000000000e+000
9	415.124320120457e−003	415.124320120457e−003	0.00000000000000e+000
10	414.261735970446e−003	414.261735970446e−003	0.00000000000000e+000
11	414.614167222791e−003	414.614167222791e−003	0.00000000000000e+000
12	414.470255524096e−003	414.470255524096e−003	0.00000000000000e+000
13	414.529034297305e−003	414.529034297305e−003	0.00000000000000e+000
14	414.505029226087e−003	414.505029226087e−003	0.00000000000000e+000
15	414.514833210466e−003	414.514833210466e−003	0.00000000000000e+000
16	414.510829199791e−003	414.510829199791e−003	0.00000000000000e+000
17	414.512464474425e−003	414.512464474425e−003	0.00000000000000e+000
18	414.511796615081e−003	414.511796615081e−003	0.00000000000000e+000
19	414.512069374520e−003	414.512069374520e−003	0.00000000000000e+000
20	414.511957977294e−003	414.511957977294e−003	0.00000000000000e+000
21	414.512003472857e−003	414.512003472857e−003	0.00000000000000e+000
22	414.511984892088e−003	414.511984892088e−003	0.00000000000000e+000
23	414.511992480630e−003	414.511992480630e−003	0.00000000000000e+000
24	414.511989381406e−003	414.511989381406e−003	0.00000000000000e+000
25	414.511990647155e−003	414.511990647155e−003	0.00000000000000e+000
26	414.511990130213e−003	414.511990130213e−003	0.00000000000000e+000
27	414.511990341336e−003	414.511990341336e−003	0.00000000000000e+000
28	414.511990255112e−003	414.511990255112e−003	0.00000000000000e+000
29	414.511990290326e−003	414.511990290326e−003	0.00000000000000e+000

Supported

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