Advances in Linear Algebra & Matrix Theory, 2017, 7, 84-107
http://www.scirp.org/journal/alamt
ISSN Online: 2165-3348
ISSN Print: 2165-333X

# Tight Monomials with $t$-Value ≤ 9 for Quantum Group of Type $D_4$

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**How to cite this paper:** Hu, Y.W., Hu, J.F. and Wu, Q.R. (2017) Tight Monomials with $t$-Value ≤ 9 for Quantum Group of Type $D_4$. *Advances in Linear Algebra & Matrix Theory*, **7**, 84-107.
https://doi.org/10.4236/alamt.2017.74009

**Received:** August 31, 2017
**Accepted:** December 9, 2017
**Published:** December 12, 2017



## Abstract

All monomials with $t$-value ≤9 in Canonical basis of quantum group for type $D_4$ are determined in this paper.

## Keywords

Quantum Group, Canonical Basis, Tight Monomial

## 1. Introduction

Quantum group, also called quantized enveloping algebra, was introduced independently by (Drinfel'd, V. G., 1985) [1] and (Jimbo, M., 1985) [2]. It plays an important role in the study of Lie groups, Lie algebras, algebraic groups, Hopf algebras, etc. The positive part of a quantum group has a kind of important basis, *i.e.*, Canonical basis $\mathbf{B}$ introduced by (Lusztig, G., 1990) [3], which plays an important role in the theory of quantum groups and their representations. Some efforts on $\mathbf{B}$ have been done. (Lusztig, G., 1990) [3] introduced algebraic definition of $\mathbf{B}$ for the quantum groups in the simply laced cases, and gave explicitly the longest monomials in $\mathbf{B}$ for type $A_1, A_2$. Afterwards, (Lusztig, G., 1992) [4] extended algebraic definition of $\mathbf{B}$ to the non-simply laced cases and gave 2 longest monomials in $\mathbf{B}$ for type $B_2$. Then, (Lusztig, G., 1993) [5] associated a quadratic form to every monomial, and proved that, given certain linear conditions, the monomial is tight (respectively, semi-tight) provided that this quadratic form satisfies a certain positivity condition (respectively, nonnegativity condition). He showed that the positivity condition always holds in type $A_3$ and computed 8 longest tight monomials for type $A_3$, and he asked when we have (semi-) tightness in type $A_n$. Based on Lusztig's work, (Xi, N. H., 1999 [6]; Xi, N. H., 1999 [7]) found explicitly all 14 elements in $\mathbf{B}$ for type $A_3$ and all 6 elements $\mathbf{B}$ for type $B_2$. For type $A_4$, (Hu, Y. W., Ye, J. C., Yue, X. Q., 2003 [8]; Hu, Y. W., Ye, J. C., 2005 [9]; Li, X. C., Hu, Y. W., 2012 [10]) determined all 62 longest monomials, all 144 polynomials with one-dimensional support, 112 polynomials with two-dimensional support in $\mathbf{B}$. (Marsh, R., 1998) [11] carried out thorough investigation for type $A_n$. He showed that the positivity condition is always satisfied in type $A_4$ for a certain orientation of the Dynkin diagram, presented a semi-tight longest monomial for type $A_5$, and exhibited a special longest monomial for type $A_r$ (for any $r \ge 6$) with a quadratic form that does not even satisfy the conditions for semi-tightness, for any orientation of the Dynkin diagram (although it may turn out that the corresponding monomial is still tight). (Bedard, R., 2004) [12] proved that all longest monomials of type $D_4$ are semi-tight. (Reineke, M., 2001) [13] associated a new quadratic form to every monomial, and gave a sufficient and necessary condition for the monomial to be tight for the simply laced cases. (Deng, B. M., Du, J., 2010) [14] proved that the Reineke's criterion works also for any quantized enveloping algebra associated with a symmetrizable Cartan matrix, and they gave all monomials in $\mathbf{B}$ for type $B_2$, in which 2 longest monomials are the same as Lusztig and Xi's results. By use of this criterion, (Wang, X. M., 2010) [15] listed all tight monomials for type $A_3$ and $G_2$, in which 8 longest monomials for type $A_3$ are same as Lusztig and Xi's results. (Hu,Y. W., Li, G. W., Wang, J., 2015) [16] determined all monomials with $t$-value $\le$ 6 in $\mathbf{B}$ for type $A_5$, and (Hu, Y. W., Geng, Y. J., 2015) [17] determined all monomials $t$- value $\le 6$ in $\mathbf{B}$ for type $B_3$.

This paper computed all monomials with $t$-value $\le 9$ in $\mathbf{B}$ for type $D_4$.

## 2. Preliminaries

Let $C = \left(c_{ij}\right)_{i,j\in\Gamma_0}$ be a Cartan matrix of finite type such that $c_{ii} = 2, c_{ij} \le 0$ for any $i \ne j$, $D = \operatorname{diag}\left(d_i\right)_{i\in\Gamma_0}$ be a diagonal matrix with integer entries making the matrix $DC$ symmetric. Let $\mathfrak{g} = \mathfrak{g}\left(C\right)$ be the complex semisimple Lie algebra associated with $C$, and $\mathbf{U} = \mathbf{U}_v\left(\mathfrak{g}\right)$ (here $v$ is an indeterminate) be the corresponding quantized enveloping algebra, whose positive part $\mathbf{U}^+ = \left\langle E_i \middle| i \in \Gamma_0 \right\rangle$ is the $\mathbb{Q}\left(v\right)$-subalgebra of $\mathbf{U}$, subject to the relations $\sum_{r+s=1-c_{ij}} \left(-1\right)^s E_i^{(s)} E_j E_i^{(r)} = 0, \forall i, j \in \Gamma_0$, where $E_i^{(s)} = E_i^s \big/ [s]_i^!$, $[s]_i^! = [1]_i \cdots [s]_i$, $[a]_i = \left(v^{ad_i} - v^{-ad_i}\right) \big/ \left(v^{d_i} - v^{-d_i}\right)$. Let $\mathcal{A} = \mathbb{Z}\left[v, v^{-1}\right]$, $U^+ = \left\langle E_i^{(s)} \middle| i \in \Gamma_0, s \in \mathbb{N} \right\rangle$ be the $\mathcal{A}$-subalgebra of $\mathbf{U}^+$. Corresponding to every reduced expression $\mathbf{i} = \left(i_1, \cdots, i_v\right)$ of the longest element $w_0 = s_{i_1} \cdots s_{i_v}$ of the Weyl group $W = \left\langle s_i \right\rangle$ of $\mathfrak{g}$, one constructs a PBW basis $B_{\mathbf{i}}$ of $\mathbf{U}^+$. Lusztig [4] proved that the $\mathbb{Z}\left[v^{-1}\right]$-submodule $\mathcal{L}_{\mathbf{i}} = \left\langle B_{\mathbf{i}} \right\rangle$ of $\mathbf{U}^+$ is independent of the choice of $\mathbf{i}$, write it $\mathcal{L}$; the image of $B_{\mathbf{i}}$ under the canonical projection $\pi : \mathcal{L} \to \mathcal{L} / v^{-1}\mathcal{L}$ is independent of the choice of $\mathbf{i}$, write it $B$; for any element $b \in B$ there is a unique element $\mathbf{b} \in \mathcal{L}$ which is fixed by the bar map of $\mathbf{U}^+$ defined by

$v \to v^{-1}$ and satisfies $\pi(\mathbf{b}) = b$. The set $\mathbf{B} = \{\mathbf{b} | b \in B\}$ forms a $\mathbb{Z}[v^{-1}]$-basis of $\mathcal{L}$, an $\mathcal{A}$-basis of $U^+$ and a $\mathbb{Q}(v)$-basis of $\mathbf{U}^+$, Lusztig calls $\mathbf{B}$ *Canonical basis* of quantum group.

According to Lusztig, a *monomial* in $\mathbf{U}^+$ is an element of the form

$$E_{i_1}^{(a_1)} \cdots E_{i_t}^{(a_t)} \tag{1}$$

where $i_1, \cdots, i_t \in \Gamma_0, a_1, \cdots, a_t \in \mathbb{N}$. When $t = v$ and $s_{i_1} \cdots s_{i_v} = w_0$ is the longest element of Weyl group, the monomial (1) is called the longest monomial. We say that (1) is tight (or semi-tight) if it belongs to $\mathbf{B}$ (or is a linear combination of elements in $\mathbf{B}$ with constant coefficients).

Let $Q = (Q_0, Q_1)$ be a finite quiver with vertex set $Q_0$ and arrow set $Q_1$. Write $\rho \in Q_1$ as $t_\rho \xrightarrow{\rho} h_\rho$, where $h_\rho$ and $t_\rho$ denote the head and the tail of $\rho$ respectively. An automorphism $\sigma$ of $Q$ is a permutation on the vertices of $Q$ and on the arrows of $Q$ such that $\sigma(h_\rho) = h_{\sigma(\rho)}$ and $\sigma(t_\rho) = t_{\sigma(\rho)}$ for any $\rho \in Q_1$. Denote the quiver with automorphism $\sigma$ as $(Q, \sigma)$. Attach to the pair $(Q, \sigma)$ a valued quiver $\Gamma = \Gamma(Q, \sigma) = (\Gamma_0, \Gamma_1)$ as follows. Its vertex set $\Gamma_0$ and arrow set $\Gamma_1$ are simply the sets of $\sigma$-orbits in $Q_0$ and $Q_1$, respectively. The valuation of $\Gamma$ is given by $d_i$ = #{vertices in the $\sigma$-orbit of $i$}, $\forall i \in \Gamma_0$; $m_\rho$ = #{arrows in the $\sigma$-orbit of $\rho$}, $\forall \rho \in \Gamma_1$. The Euler form of $\Gamma$ is defined to be the bilinear form $\langle , \rangle : \mathbb{Z}[\Gamma_0] \times \mathbb{Z}[\Gamma_0] \to \mathbb{Z}$ given by $\langle X, Y \rangle = \sum_{i \in \Gamma_0} d_i x_i y_i - \sum_{\rho \in \Gamma_1} m_\rho x_{t_\rho} y_{h_\rho}$, where $X = \sum_{i \in \Gamma_0} x_i i$, $Y = \sum_{i \in \Gamma_0} y_i i \in \mathbb{Z}[\Gamma_0]$, so $X \cdot Y = \langle X, Y \rangle + \langle Y, X \rangle$ is the symmetric Euler form. The valued quiver $\Gamma$ defines a Cartan matrix $C_\Gamma = C_{Q,\sigma} = (c_{ij})_{i,j \in \Gamma_0}$, where

$$c_{ij} = \begin{cases} 2 - 2 \sum\limits_{\substack{\rho \in \Gamma_1 \\ h_\rho = t_\rho = i}} \dfrac{m_\rho}{d_i}, i = j; \\ - \sum\limits_{\substack{\rho \in \Gamma_1 \\ \{h_\rho, t_\rho\} = \{i, j\}}} \dfrac{m_\rho}{d_i}, i \neq j. \end{cases}$$

For $t \in \mathbb{N}$, let $\mathbf{i} = (i_1, \cdots, i_t) \in \Gamma_0^t$, $\mathbf{a} = (a_1, \cdots, a_t) \in \mathbb{N}^t$, and write $E_{\mathbf{i}}^{(\mathbf{a})} = E_{i_1}^{(a_1)} \cdots E_{i_t}^{(a_t)} \in U^+$. Define $\mathcal{M}_{\mathbf{i},\mathbf{a}} = \{A = (a_{rm})_{tt} | a_{rm} \in \mathbb{N}, \mathrm{ro}(A) = \mathrm{co}(A) = \mathbf{a}, a_{rm} = 0, \forall i_r \neq i_m\}$, where $\mathrm{ro}(A) = \left(\sum_{m=1}^{t} a_{1m}, \cdots, \sum_{m=1}^{t} a_{tm}\right)$, $\mathrm{co}(A) = \left(\sum_{r=1}^{t} a_{r1}, \cdots, \sum_{r=1}^{t} a_{rt}\right)$. Obviously, $D_{\mathbf{a}} = \mathrm{diag}(a_1, \cdots, a_t) \in \mathcal{M}_{\mathbf{i},\mathbf{a}}$.

Lusztig gave the following criterion for a monomial to be tight or semi-tight.

**Theorem 2.1** ([Lusztig, 1993, §6 Theorem) [5]. Let $\mathbf{U}$ be the quantum group of type $A_n, D_n, E_n$, $\mathbf{i} \in \Gamma_0^t, \mathbf{a} \in \mathbb{N}^t$ as above. If the following quadratic form takes only values < 0 on $\mathcal{M}_{\mathbf{i},\mathbf{a}} \setminus \{D_{\mathbf{a}}\}$ (respectively, ≤ 0 on $\mathcal{M}_{\mathbf{i},\mathbf{a}}$), then monomial $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight (respectively, semi-tight).

$$q'_{\mathbf{i},\mathbf{a}}(A) = \sum_{\substack{1 \le m \le t \\ 1 \le p < r \le t}} a_{pm} a_{rm} - \sum_{\substack{1 \le p < r \le t \\ 1 \le l < m \le t}} a_{pm} a_{rl}$$

It should be noticed that the above theorem is sufficient but not necessary, M. Reineke gave a sufficient and necessary condition by symmetrizing Lusztig's quadratic form.

**Theorem 2.2** ([Reineke, 2001, Theorem 3.2]) [13]. Let $\mathbf{U}$ be the quantum group of type $A_n, D_n, E_n$, $\mathbf{i} \in \Gamma_0^t, \mathbf{a} \in \mathbb{N}^t$ as above, the monomial $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight if and only if the following quadratic form takes only values <0 on $\mathcal{M}_{\mathbf{i},\mathbf{a}} \setminus \{D_{\mathbf{a}}\}$

$$q_{\mathbf{i},\mathbf{a}}(A) = \sum_{\substack{1\le m\le t \\ 1\le p<r\le t}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le t \\ 1\le l<m\le t}} (i_l \cdot i_m) a_{pm}a_{rl} + \sum_{\substack{1\le r\le t \\ 1\le l<m\le t}} a_{rm}a_{rl}$$

In fact, $q_{\mathbf{i},\mathbf{a}}(A) = q'_{\mathbf{i},\mathbf{a}}(A) + q'_{\mathbf{i},\mathbf{a}}(A^T)$ (see [Reineke, 2001 [13], Lemma 3.3]).

Deng and Du generalized the tight monomial criterion given by Reineke to any quantum group associated with symmetrizable matrices.

**Theorem 2.3** ([Deng, Du, 2010, Theorem 2.5]) [14]. Let $\mathbf{U}$ be the quantum group associated with any symmetrizable matrices, $\mathbf{i} \in \Gamma_0^t, \mathbf{a} \in \mathbb{N}^t$ as above, the monomial $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight if and only if the following quadratic form takes only values < 0 on $\mathcal{M}_{\mathbf{i},\mathbf{a}} \setminus \{D_{\mathbf{a}}\}$

$$q_{\mathbf{i},\mathbf{a}}(A) = \sum_{\substack{1\le m\le t \\ 1\le p<r\le t}} \langle i_m, i_m \rangle a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le t \\ 1\le l<m\le t}} (i_l \cdot i_m) a_{pm}a_{rl} + \sum_{\substack{1\le r\le t \\ 1\le l<m\le t}} \langle i_r, i_r \rangle a_{rm}a_{rl}$$

By Theorem 2.3, we have the following Corollaries.

**Corollary 2.4.** When $i_1, \cdots, i_t$ are mutually different, monomial $E_{i_1}^{(a_1)} \cdots E_{i_t}^{(a_t)}$ is tight.

**Proof:** In fact, $i_1, \cdots, i_t$ are mutually different, so $\mathcal{M}_{\mathbf{i},\mathbf{a}} = \{D_{\mathbf{a}}\}$.

**Corollary 2.5.** If $E_{i_{p+1}}^{(a_{p+1})} \cdots E_{i_{p+q}}^{(a_{p+q})}$ is tight, then for any mutually different $i_1, \cdots, i_p \notin \{i_{p+1}, \cdots, i_{p+q}\}$ and any mutually different $i_{p+q+1}, \cdots, i_t \notin \{i_1, \cdots, i_p; i_{p+1}, \cdots, i_{p+q}\}$, $t \le l(w_0)$, $E_{\mathbf{i}}^{(\mathbf{a})} = E_{i_1}^{(a_1)} \cdots E_{i_t}^{(a_t)}$ is also tight.

**Proof:** Write $\mathbf{j} = (i_{p+1}, \cdots, i_{p+q})$, $\mathbf{b} = (a_{p+1}, \cdots, a_{p+q})$, $\mathbf{i} = (i_1, \cdots, i_t)$, $\mathbf{a} = (a_1, \cdots, a_t)$, then

$$E_{\mathbf{i}}^{(\mathbf{a})} = E_{i_1}^{(a_1)} \cdots E_{i_t}^{(a_t)}, E_{\mathbf{j}}^{(\mathbf{b})} = E_{i_{p+1}}^{(a_{p+1})} \cdots E_{i_{p+q}}^{(a_{p+q})}.$$

For any $\tilde{A} \in \mathcal{M}_{\mathbf{i},\mathbf{a}}$, we have

$$\tilde{A} = \begin{pmatrix} D_{\mathbf{a}'} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & D_{\mathbf{a}''} \end{pmatrix},$$

where $\mathbf{a}' = (a_1, \cdots, a_p)$, $\mathbf{a}'' = (a_{p+q+1}, \cdots, a_t)$. It is easy to see that $A \in \mathcal{M}_{\mathbf{j},\mathbf{b}}$ and $q(\tilde{A}) = q(A)$. Moreover, $\tilde{A} = D_{\mathbf{a}}$ if and only if $A = D_{\mathbf{b}}$. Since $E_{\mathbf{j}}^{(\mathbf{b})}$ is tight, we get by Theorem 2.3 that $q(\tilde{A}) = q(A) < 0$ for all $\tilde{A} \in \mathcal{M}_{\mathbf{i},\mathbf{a}} \setminus \{D_{\mathbf{a}}\}$, applying Theorem 2.3 again, we conclude that $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight.

The following two theorems are very useful in determining tight monomials.

**Theorem 2.6** ([Deng & Du, 2010, Corollary 2.6, Theorem 6.2) [14]. Let $\mathbf{i} = (i_1, \cdots, i_t) \in \Gamma_0^t$ and $\mathbf{a} = (a_1, \cdots, a_t) \in \mathbb{N}^t$. If $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight, then

(a) For $\forall 1 \le r \le s \le t$, monomial $E_{i_r}^{(a_r)} \cdots E_{i_s}^{(a_s)}$ is also tight;

(b) For $\forall 1 \le r < t$, $i_r \ne i_{r+1}$.

**Theorem 2.7** ([Lusztig, 1990, Proposition 3.3 and Lusztig, 1993, §13]) [3]. Let

$\Phi$ be the non-trivial automorphism of $\mathbf{U}^+$ induced by Dynkin diagram automorphism $\varphi$ of $\mathfrak{g}$, and $\Psi:\mathbf{U}^+ \to (\mathbf{U}^+)^{opp}$ be the unique $\mathbb{Q}(v)$-algebra isomorphism such that $E_j \to E_j$. If $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight, then $\Phi(E_{\mathbf{i}}^{(\mathbf{a})})$ and $\Psi(E_{\mathbf{i}}^{(\mathbf{a})})$ are all tight.

A quadratic form $f:\mathbb{Z}^n \to \mathbb{Z}$ denoted by

$$f(x_1,\cdots,x_n)=\sum_{i=1}^{n} x_i^2+\sum_{i<j} a_{ij}x_i x_j$$

with $a_{ij}\in\mathbb{Z}$ is called a *unit form*.

The symmetric matrix $A_f=(a_{ij})$ (when $i>j$, set $a_{ij}=a_{ji}$) with $a_{ii}=2$, defines a bilinear form $f(X,Y)=XA_fY^{\mathrm{T}}$, where

$$X=(x_1,\cdots,x_n) \text{ and } Y=(y_1,\cdots,y_n).$$

In particular, we have

$$f(X)=\frac{1}{2}f(X,X) \text{ and } f(X,Y)=f(X+Y)-f(X)-f(Y).$$

For a vector $w\in\mathbb{Q}^n$ with non-negative coordinates, we write $w\geq 0$. The vector in $\mathbb{Z}^n$ which has a 1 in the $i^{\text{th}}$ coordinate ($1\leq i\leq n$) and 0's elsewhere is denoted by $r_i$.

Let $f$ be a unit form. We define the set of *positive roots* of $f$ as $\{w\in\mathbb{Z}^n : 0\leq w \text{ and } f(w)=1\}$. The linear transformation $\sigma_i:\mathbb{Z}^n\to\mathbb{Z}^n$ defined by $\sigma_i(w)=w-f(w,r_i)r_i$ is called the *reflection* with respect to $r_i$. The transformation $\sigma_i$ has the property that $\sigma_i^2=\mathrm{id}$ and $f(\sigma_i(w))=f(w)$ for every $w\in\mathbb{Z}^n$.

Let $f$ be a quadratic form, if $0<f(X)$ for every $0\neq X\in\mathbb{N}^n$, then we call $f$ *weakly positive*.

The following *algorithm* and theorem are taken from (Blouin, Dean, Denver, Pershall, 1995) [18]:

Let $f:\mathbb{Z}^n\to\mathbb{Z}(n\geq 3)$ be a unit form. First of all, we define

$$R_1=\{r_i : 1\leq i\leq n\}.$$

Next we want to construct $R_j$ recursively. Assume that we have defined a set of positive roots of $f$ as

$$R_s=\{z_1,\cdots,z_m\} \text{ with } s\geq 1,$$

and that the process has not failed (to be defined subsequently). Then we construct $R_{s+1}$ as follows. Let $z_j\in R_s$. If either:

1) there is some $1\leq i\leq n$ such that $f(z_j,r_i)\leq -2$, or

2) there is some $1\leq i\leq n$ such that $z_j(i)\geq 7$,

then the process is said to *fail*. Assume the process does not fail (so 1) and 2) do not occur for any $z_j\in R_s$). Let $D_s\subseteq R_s$ be the set of those roots $z_j$ with the property that there is some $1\leq i\leq n$ such that $f(z_j,r_i)=-1$. If $D_s=\phi$, then $R_{s+1}:=\phi$ and the process is said to be *successful*. If $D_s\neq\phi$, then

$$R_{s+1}:=\{\sigma_i(z_j) \mid z_j\in D_s \text{ and } r_i \text{ is such that } f(z_j,r_i)=-1\}.$$

**Remark:** If $R_{s+1}\neq\phi$, then we apply the algorithm again to obtain $R_{s+2}$, etc. The roots in $R_{s+1}$ (if $R_{s+1}\neq\phi$) are all greater than the roots in $R_s$. Thus condition 2) guarantees that this procedure is finite and if it is not successful then it will eventually fail.

**Theorem 2.8** ([Blouin *et al.*, 1995, Theorem 4]) [18]. The unit form $f$ is weakly positive if and only if the above process is successful. If the process is successful with $R_{s+1}=\phi$, then $R_1\cup\cdots\cup R_s$ are all the positive roots of $f$.

## 3. Main Results

Let $\mathfrak{g}$ be of type $D_4$ as follows.

Let $\mathbf{i}=(i_1,\cdots,i_t)\in\Gamma_0^t$, $\mathbf{a}=(a_1,\cdots,a_t)\in\mathbb{N}^t$. For convenience, we abbreviate a monomial $E_{i_1}^{(a_1)}\cdots E_{i_t}^{(a_t)}$ with any $(a_1,\cdots,a_t)\in\mathbb{N}^t$ as a word $i_1\cdots i_t$ (1 as 0), an inequality $a_{j_1}+\cdots+a_{j_p}\leq a_{l_1}+\cdots+a_{l_q}$ as $j_1\cdots j_p-l_1\cdots l_q$. For example, a monomial $E_1^{(a_1)}E_2^{(a_2)}E_3^{(a_3)}E_4^{(a_4)}$ is abbreviated to a word 1234, and a monomial $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}\left(a_1+a_3\leq a_2\right)$ to a word 121 (13 − 2), etc.

If $s_{i_1}\cdots s_{i_t}$ is a reduced expression, we call $i_1\cdots i_t$ a reduced word. By Theorem 2.6 (b), we only consider those *reduced tight words* $i_1\cdots i_t$ with $i_r\neq i_{r+1},\forall 1\leq r<t$, in this case, $i_1\cdots i_t$ is called the *word with t-value*. If $i_r\cdot i_{r+1}=0$ for some $1\leq r<t$, we identify the word $i_1\cdots i_{r-1}i_r i_{r+1}i_{r+2}\cdots i_t$ with the word $i_1\cdots i_{r-1}i_{r+1}i_r i_{r+2}\cdots i_t$. Denote the set of all words with *t*-value by $M_t$. The non-trivial Dynkin diagram automorphism $\varphi$ of $\mathfrak{g}$ is $\varphi:1\mapsto 3,3\mapsto 4,4\mapsto 1,2\mapsto 2$. Let us present the so called *M − S word-procedure* from *t*-value to $(t+1)$-value.

Step 1. Take any $i_1\cdots i_t\in M_t$, adding a number $i_{t+1}\in\{1,2,3,4\}$ different from $i_1$ (or $i_t$) in the front (or behind) of $i_1$ (or $i_t$), deleting the those words with *t*-value, getting all words with $(t+1)$-value from $i_1\cdots i_t$.

Step 2. Repeat step 1 until all words in $M_t$ are considered, deleting the non-tight words with $(t+1)$-value, get $M_{t+1}$.

Step 3. Use $\Phi$ and $\Psi$, we have $S_{t+1}$ satisfying $M_{t+1}=\Phi(S_{t+1})\cup\Psi\Phi(S_{t+1})$.

For example, applying the $M-S$ word-procedure to $M_1=\{1,2,3,4\}$, get $M_2=\{12,13,14,21,23,24,32,34,42\}$. Considering $\Phi,\Psi$,
$24\xleftarrow{\varphi}23\xleftarrow{\varphi}21\xleftarrow{\Psi}12\xrightarrow{\varphi}32\xrightarrow{\varphi}42$ and $13\xrightarrow{\varphi}34\xrightarrow{\varphi}14$, so $S_2=\{12,13\}$.

From now on, write $M_t=\Phi(S_t)\cup\Psi\Phi(S_t),t\geq 2$.

**Theorem 3.1.** For the quantum group for type $D_4$, we have the following results.

1) $t=0$, $M_0=\{0\}$, tight monomial has only one.

2) $t=1$, $M_1=\Phi(S_1)$, tight monomials have 4 families, where $S_1=\{1,2\}$.

3) $t=2$, $M_2$ includes 9 families of tight monomials, where $S_2=\{12,13\}$.

4) $t=3$, $M_3$ includes 19 families of tight monomials, where $S_3=S_3^1\cup S_3^2$,

$S_3^1=\{123,132,134\}$; $S_3^2=\{121,212(13-2)\}$.

5) $t=4$, $M_4$ includes 35 families of tight monomials, where

$S_4 = S_4^1 \cup S_4^2 \cup S_4^3$,

$S_4^1 = \{1234, 1342\}$; $S_4^2 = \{1213, 1214, 2123, 2124(13-2)\}$;

$S_4^3 = \{2132(14-23)\}$.

6) $t = 5$, $M_5$ includes 58 families of tight monomials, where $S_5 = S_5^1 \cup S_5^2 \cup S_5^3$,

$S_5^1 = \{12134, 24213(13-2); 31214, 12324(24-3);$
$12342(25-34); 21342(15-234)\}$;

$S_5^2 = \{12312, 12412(14-2, 25-34); 13213(14-3, 25-3);$
$12321, 12421(15-24-3)\}$;

$S_5^3 = \{21232(13-2, 35-4)\}$.

7) $t = 6$, $M_6$ includes 93 families of tight monomials, where $S_6 = S_6^1 \cup S_6^2 \cup S_6^3 \cup S_6^4$,

$S_6^1 = \{123124, 124123(14-2, 25-34); 123214, 124213(15-24-3);$
$132142, 312342(14-3, 36-45); 132134(14-3, 25-3);$
$123421(16-25-34); 123142(14-2, 26-345)\}$;

$S_6^2 = \{132132(14-3, 25-3, 36-45)\}$;

$S_6^3 = \{212342(13-2, 36-45); 123242, 124232(24-3, 46-5)\}$;

$S_6^4 = \{123121, 124121(14-2, 46-5, 25-34);$
$123212, 124212(46-5, 15-24-3)\}$.

8) $t = 7$, $M_7$ includes 133 families of tight monomials, where $S_7 = S_7^1 \cup S_7^2 \cup S_7^3 \cup S_7^4$,

$S_7^1 = \{1234213, 1243214(16-25-34, 37-5);$
$1321342(14-3, 25-3, 37-456); 1342134(15-4, 26-4, 37-4);$
$1234123(15-2, 26-345, 37-6); 1324213(16-35-4, 27-35);$
$1234234(36-5, 47-5, 25-34); 1342132(15-4, 26-4, 47-56)\}$;

$S_7^2 = \{1231242, 1241232(14-2, 57-6, 25-34);$
$1231241, 1241231(25-34, 14-2, 47-5);$
$1324212, 3124232(57-6, 16-35-4);$
$1232142, 1242132(15-24-3, 47-56);$
$1232124, 1242123(46-5, 15-24-3);$
$1232421(17-246, 24-3, 46-5); 1234212(57-6, 16-25-34);$
$1231421(14-2, 47-6, 26-345); 2132142(25-4, 14-23, 47-56)\}$;

$S_7^3 = \{2123242(13-2, 35-4, 57-6)\}$;

$S_7^4 = \{2123212, 2124212(13-2, 26-35-4, 57-6)\}$.

9) $t = 8$, $M_8$ includes 185 families of tight monomials, where

$S_8 = S_8^1 \cup S_8^2 \cup S_8^3 \cup S_8^4$,

$S_8^1 = \{12134234(13-2, 26-345, 47-6, 58-6);$

$12342134(16-25-34, 48-5, 37-5);$

$13421342(15-4, 48-567, 26-4, 37-4)\};$

$S_8^2 = \{12132142, 12142132(13-2, 36-5, 25-34, 58-67);$

$12142321, 12132421(13-2, 38-57-6, 25-34)\};$

$S_8^3 = \{12132423, 12142324(13-2, 25-34, 48-57-6);$

$12134213, 12143214(13-2, 37-6, 26-345, 48-6);$

$12321423, 32123421(15-24-3, 38-47-56);$

$12324213, 12423214(17-246, 24-3, 38-46-5);$

$12342132, 12432142(16-25-34, 58-67, 37-5);$

$13214232, 31234212(14-3, 27-36-45, 68-7);$

$13214213, 31234231(14-3, 47-6, 28-36-45);$

$13213242(14-3, 36-45, 68-7, 25-3);$

$13242132(16-35-4, 58-67, 27-35);$

$21321342(25-4, 14-23, 36-4, 48-567)\};$

$S_8^4 = \{12321242, 12421232(15-24-3, 46-5, 68-7);$

$12324212, 12423212(17-246, 24-3, 46-5, 68-7);$

$12324232, 12423242(24-3, 68-7, 37-46-5);$

$21232142, 21242132(13-2, 58-67, 26-35-4);$

$21234212(13-2, 68-7, 27-36-45)\}.$

10) $t = 9$, $M_9$ includes 265 families of tight monomials, where

$S_9 = S_9^1 \cup S_9^2 \cup S_9^3 \cup S_9^4 \cup S_9^5$,

$S_9^1 = \{121321423, 121421324(13-2, 36-5, 25-34, 49-58-67);$

$121342132, 121432142(13-2, 37-6, 26-345, 48-6, 69-78);$

$121324213, 121423214(13-2, 38-57-6, 49-57, 25-34);$

$132142132, 312342312(14-3, 47-6, 69-78, 28-36-45);$

$124213214, 123214213(15-24-3, 39-47-56, 58-7);$

$124213213, 123214214(15-24-3, 47-56, 58-7, 69-7);$

$132134213(14-3, 25-3, 48-7, 59-7, 37-456)\};$

$S_9^2 = \{123214234, 124213243(15-24-3, 69-7, 38-47-56);$

$121342134(13-2, 37-6, 48-6, 59-6, 26-345);$

$123421234(16-25-34, 38-57-6, 49-57);$

$123421324(16-25-34, 37-5, 49-58-67);$

$123421342(16-25-34,59-678,37-5,48-5);$
$132134214(14-3,48-7,37-456,25-3,69-7);$
$134213242(15-4,79-8,26-4,38-47-56);$
$213421342(26-5,15-234,59-678,37-5,48-5)\};$

$S_9^3=\{121324232,121423242(13-2,25-34,48-57-6,79-8);$
$123212423,124212324(15-24-3,46-5,68-7,39-468);$
$123214232,124213242(15-24-3,38-47-56,79-8);$
$123242132,124232142(17-246,24-3,69-78,38-46-5);$
$123242321,124232421(19-2468,24-3,68-7,37-46-5);$
$123421232,123421242(16-25-34,38-57-6,79-8);$
$132421232,312423212(16-35-4,57-6,79-8,28-357);$
$123242123(17-246,24-3,46-5,39-468,68-7);$
$212342132,212432142(13-2,27-36-45,48-6,69-78);$
$213242132(27-46-5,14-23,69-78,38-46)\};$

$S_9^4=\{121324212,121423212(13-2,25-34,38-57-6,79-8);$
$121321421(13-2,36-5,69-8,25-34,58-67);$
$123212421(15-24-3,59-68-7,46-5)\};$

$S_9^5=\{212321242,212421232(13-2,26-35-4,57-6,79-8);$
$212324212(13-2,28-357,35-4,57-6,79-8)\}.$

## 4. Proof of Theorem 3.1

Consider the following quiver $Q=(Q_0,Q_1)$ of type $D_4$ where $Q_1=\{1\xrightarrow{\rho_1}2,3\xrightarrow{\rho_2}2,4\xrightarrow{\rho_3}2\}$, $Q_0=\{1,2,3,4\}$. Let $\sigma$ be the identity automorphism of $Q$ such that $\sigma(i)=i,\sigma(\rho_i)=\rho_i,i=1,2,3,4$, then the valued quiver of $(Q,\sigma)$ is $\Gamma=\Gamma(Q,\sigma)=(\Gamma_0,\Gamma_1)=(Q_0,Q_1)$, the valuation is given by $d_i=1,m_{\rho_i}=1,i=1,2,3,4$. For $X=x_1 1+x_2 2+x_3 3+x_4 4$, $Y=y_1 1+y_2 2+y_3 3+y_4 4\in\mathbb{Z}[\Gamma_0]$, Euler form $\langle,\rangle$ on $\Gamma$ is

$$\langle X,Y\rangle_\Gamma=\sum_{i=1}^{4}d_i x_i y_i-\sum_{i=1}^{3}m_{\rho_i}x_{t_{\rho_i}}y_{h_{\rho_i}}=x_1y_1+x_2y_2+x_3y_3+x_4y_4-x_1y_2-x_3y_2-x_4y_2,$$

symmetric Euler form $\cdot$ on $\Gamma$ is

$$X\cdot Y=\langle X,Y\rangle_\Gamma+\langle Y,X\rangle_\Gamma=2x_1y_1+2x_2y_2+2x_3y_3+2x_4y_4-x_1y_2-x_2y_1-x_3y_2-x_2y_3-x_4y_2-x_2y_4.$$

By simple computation, we have $\langle i,i\rangle=1,i=1,2,3,4$; $\langle 1,2\rangle=\langle 3,2\rangle=\langle 4,2\rangle=-1$, the other $\langle i,j\rangle=0,i\neq j$. So $i\cdot i=2,i=1,2,3,4$; $1\cdot 2=2\cdot 3=2\cdot 4=-1$, $1\cdot 3=1\cdot 4=3\cdot 4=0$.

Let us prove Theorem 3.1. For any $t\in\mathbb{N}$, let $\mathbf{i}=(i_1,\cdots,i_t)\in\Gamma_0^t$, $\mathbf{a}=(a_1,\cdots,a_t)\in\mathbb{N}^t$.

Case 1. $t\le 2$. By Corollary 2.4, words with $t\le 2$ are all tight, so (1)~(3) hold.

Case 2. $t=3$. Applying the $M-S$ word-procedure to $M_2$, we get $S_3$. Corollary 2.4 $\Rightarrow S_3^1$. Consider $S_3^2$, for $\mathbf{i}\in\{(1,2,1),(2,1,2)\}$, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{M=M_x | x\in\mathbb{N}\}$, where

$$M=\begin{pmatrix} a_1-x & 0 & x \\ 0 & a_2 & 0 \\ x & 0 & a_3-x \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle(a_1x-x^2)+2\langle i_3,i_3\rangle(a_3x-x^2)+(i_1\cdot i_3)x^2+((i_1\cdot i_2)+(i_2\cdot i_3))a_2x \\ &= -2x^2+2(a_1+a_3-a_2)x. \end{aligned}$$

Obviously, $q(M)\le 0 \Leftrightarrow a_1+a_3\le a_2$. And $q(M)=0\Leftrightarrow a_1+a_3\le a_2$, $x=0$, so two words in $S_3^2$ are all tight by Theorem 2.3, (4) holds.

Case 3. $t=4$. Applying the $M-S$ word-procedure to $M_3$, we get $S_4\cup\{1212\}$. Corollary 2.4 $\Rightarrow S_4^1$. Corollary 2.5 and $S_3^2\Rightarrow S_4^2$. Consider words 2132, 1212. For word 2132, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{M=M_x | x\in\mathbb{N}\}$, where

$$M=\begin{pmatrix} a_1-x & 0 & 0 & x \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ x & 0 & 0 & a_4-x \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle(a_1x+a_4x-2x^2)+(i_1\cdot i_1)x^2+2(i_1\cdot i_2)a_2x+2(i_1\cdot i_3)a_3x \\ &= -2x^2+2(a_1+a_4-a_2-a_3)x. \end{aligned}$$

Obviously, $q(M)\le 0\Leftrightarrow a_1+a_4\le a_2+a_3$. And $q(M)=0\Leftrightarrow a_1+a_4\le a_2+a_3$, $x=0$, so word 2132(14-23) is tight by Theorem 2.3.

For word 1212, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{M=M_{x,y} | (x,y)\in\mathbb{N}^2\}$, where

$$M=\begin{pmatrix} a_1-x & 0 & x & 0 \\ 0 & a_2-y & 0 & y \\ x & 0 & a_3-x & 0 \\ 0 & y & 0 & a_4-y \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle(a_1x+a_3x-2x^2)+2\langle i_2,i_2\rangle(a_2y+a_4y-2y^2) \\ &\quad +(i_1\cdot i_1)x^2+(i_2\cdot i_2)y^2+2(i_1\cdot i_2)(a_2x+a_3y-xy) \\ &= -x^2-y^2-(x-y)^2+2(a_1+a_3-a_2)x+2(a_2+a_4-a_3)y. \end{aligned}$$

Obviously, $q(M)\le 0\Leftrightarrow a_1+a_3\le a_2, a_2+a_4\le a_3$. And $q(M)=0\Leftrightarrow a_1+a_3\le a_2$, $a_2+a_4\le a_3$, $x=y=0$, but $a_1+a_3\le a_2$, $a_2+a_4\le a_3\Rightarrow a_1+a_4\le 0$, this is a contradiction, so word 1212 is not tight for any $a=(a_1,a_2,a_3,a_4)\in\mathbb{N}^4$, (5) holds.

Case 4. $t=5$. Applying the $M-S$ word-procedure to $M_4$, deleting words including subwords in $\Phi(1212)\cup\Psi\Phi(1212)$, we get $S_5$. By Corollary 2.5 and $S_3^2, S_4^3$, words in $S_5^1$ besides 21342 are all tight. For word 21342, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{M=M_x | x\in\mathbb{N}\}$, where

$$M=\begin{pmatrix} a_1-x & 0 & 0 & 0 & x \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ x & 0 & 0 & 0 & a_5-x \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle\left(a_1x+a_5x-2x^2\right)+\left(i_1\cdot i_1\right)x^2+2\left(i_1\cdot i_2\right)a_2x \\ &\quad +2\left(i_1\cdot i_3\right)a_3x+2\left(i_1\cdot i_4\right)a_4x \\ &= -2x^2+2(a_1+a_5-a_2-a_3-a_4)x. \end{aligned}$$

Obviously, $q(M)\le 0 \Leftrightarrow a_1+a_5\le a_2+a_3+a_4$. And $q(M)=0 \Leftrightarrow a_1+a_5\le a_2+a_3+a_4$, $x=0$, so word 21342(15-234) is tight by Theorem 2.3.

Consider $S_5^2$. For words 12312, 12412, 13213, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{M=M_{x,y}\,\middle|\,(x,y)\in\mathbb{N}^2\right\}$, where

$$M=\begin{pmatrix} a_1-x & 0 & 0 & x & 0 \\ 0 & a_2-y & 0 & 0 & y \\ 0 & 0 & a_3 & 0 & 0 \\ x & 0 & 0 & a_4-x & 0 \\ 0 & y & 0 & 0 & a_5-y \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle\left(a_1x+a_4x-2x^2\right)+2\langle i_2,i_2\rangle\left(a_2y+a_5y-2y^2\right)+\left(i_1\cdot i_1\right)x^2 \\ &\quad +\left(i_2\cdot i_2\right)y^2+2\left(i_1\cdot i_3\right)a_3x+2\left(i_2\cdot i_3\right)a_3y+2\left(i_1\cdot i_2\right)\left(a_2x+a_4y-xy\right) \\ &= \begin{cases} -(x-y)^2-x^2-y^2+2\left(a_1+a_4-a_2\right)x+2\left(a_2+a_5-a_3-a_4\right)y, & 12312, 12412, \\ -2x^2-2y^2+2\left(a_1+a_4-a_3\right)x+2\left(a_2+a_5-a_3\right)y, & 13213. \end{cases} \end{aligned}$$

Obviously, $q(M)\le 0 \Leftrightarrow \begin{cases} a_1+a_4\le a_2, a_2+a_5\le a_3+a_4, & 12312, 12412, \\ a_1+a_4\le a_3, a_2+a_5\le a_3, & 13213 \end{cases}$. And $q(M)=0 \Leftrightarrow \begin{cases} a_1+a_4\le a_2, a_2+a_5\le a_3+a_4, & 12312, 12412, \\ a_1+a_4\le a_3, a_2+a_5\le a_3, & 13213 \end{cases}$ and $x=y=0$, so words 12312, 12412(14-2, 25-34), 13213(14-3, 25-3) are tight by Theorem 2.3.

For words 12321, 12421, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{M=M_{x,y}\,\middle|\,(x,y)\in\mathbb{N}^2\right\}$, where

$$M=\begin{pmatrix} a_1-x & 0 & 0 & 0 & x \\ 0 & a_2-y & 0 & y & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & y & 0 & a_4-y & 0 \\ x & 0 & 0 & 0 & a_5-x \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle\left(a_1x+a_5x-2x^2\right)+2\langle i_2,i_2\rangle\left(a_2y+a_4y-2y^2\right)+\left(i_1\cdot i_1\right)x^2 \\ &\quad +\left(i_2\cdot i_2\right)y^2+2\left(i_1\cdot i_2\right)\left(a_2x+a_4x\right)+2\left(i_1\cdot i_3\right)a_3x+2\left(i_2\cdot i_3\right)a_3y \\ &= -2x^2-2y^2+2\left(a_1+a_5-a_2-a_4\right)x+2\left(a_2+a_4-a_3\right)y \end{aligned}$$

Obviously, $q(M)\le 0 \Leftrightarrow a_1+a_5\le a_2+a_4\le a_3$. And $q(M)=0 \Leftrightarrow a_1+a_5\le a_2+a_4\le a_3$ and $x=y=0$, so words 12321, 12421(14-25-3) are tight by Theorem 2.3.

Now let us consider $S_5^3$, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}} = \left\{ M = M_{x,x_1,x_2,x_3} \middle| (x, x_1, x_2, x_3) \in \mathbb{N}^4 \right\}$, where

$$M = \begin{pmatrix} a_1 - x - x_1 & 0 & x & 0 & x_1 \\ 0 & a_2 & 0 & 0 & 0 \\ x_2 & 0 & a_3 - x_2 - x_3 & 0 & x_3 \\ 0 & 0 & 0 & a_4 & 0 \\ x + x_1 - x_2 & 0 & x_2 + x_3 - x & 0 & a_5 - x_1 - x_3 \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1, i_1 \rangle \left( a_1 x + a_1 x_1 + a_3 x_2 + a_3 x_3 + a_5 x_1 + a_5 x_3 - 2x^2 - 2x_1^2 - 2x_2^2 \right. \\ &\quad \left. - 2x_3^2 - 2x_1 x + 2x_2 x + x_3 x + x_1 x_2 - x_1 x_3 - 2x_2 x_3 \right) + (i_1 \cdot i_1)\left( a_3 x \right. \\ &\quad \left. + 2a_3 x_1 - a_3 x_2 + x^2 + x_1^2 + x_2^2 + x_3^2 + x_1 x - x_2 x - x_3 x - x_1 x_2 + x_2 x_3 \right) \\ &\quad + 2(i_1 \cdot i_2)(a_2 x + a_2 x_1) + 2(i_1 \cdot i_4)(a_4 x_1 + a_4 x_3) \\ &= -(x - x_2)^2 - x^2 - 2x_1^2 - x_2^2 - 2x_3^2 - 2x_1 x - 2x_1 x_3 - 2x_2 x_3 \\ &\quad + 2(a_1 + a_3 - a_2)x + 2(a_1 + 2a_3 + a_5 - a_2 - a_4)x_1 + 2(a_3 + a_5 - a_4)x_3. \end{aligned}$$

Obviously, $q(M) \le 0 \Leftrightarrow a_1 + a_3 \le a_2$, $a_3 + a_5 \le a_4$. And $q(M) = 0 \Leftrightarrow a_1 + a_3 \le a_2$, $a_3 + a_5 \le a_4$, $x = x_1 = x_2 = x_3 = 0$, so word 21232(13-2, 35-4) is tight. So (6) holds.

Case 5. $t = 6$. Applying the $M - S$ word-procedure to $M_5$, deleting words including subwords in $\Phi(1212) \cup \Psi\Phi(1212)$, we get $S_6 \cup \{123123\}$.

Firstly, as $S_5^2, S_5^3$, we can prove that words in $S_6^1, S_6^3$ are all tight.

Secondly, consider words 123123, 132132, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}} = \left\{ M = M_{x,y,z} \middle| (x, y, z) \in \mathbb{N}^3 \right\}$, where

$$M = \begin{pmatrix} a_1 - x & 0 & 0 & x & 0 & 0 \\ 0 & a_2 - y & 0 & 0 & y & 0 \\ 0 & 0 & a_3 - z & 0 & 0 & z \\ x & 0 & 0 & a_4 - x & 0 & 0 \\ 0 & y & 0 & 0 & a_5 - y & 0 \\ 0 & 0 & z & 0 & 0 & a_6 - z \end{pmatrix}$$

and

$$\begin{aligned} q(M) &= 2\langle i_1, i_1 \rangle \left( a_1 x + a_4 x - 2x^2 \right) + 2\langle i_2, i_2 \rangle \left( a_2 y + a_5 y - 2y^2 \right) \\ &\quad + 2\langle i_3, i_3 \rangle \left( a_3 z + a_6 z - 2z^2 \right) + (i_1 \cdot i_1)x^2 + (i_2 \cdot i_2)y^2 + (i_3 \cdot i_3)z^2 \\ &\quad + 2(i_1 \cdot i_2)(a_2 x + a_4 y - xy) + 2(i_1 \cdot i_3)(a_3 x + a_4 z - xz) \\ &\quad + 2(i_2 \cdot i_3)(a_3 y + a_5 z - yz) \\ &= \begin{cases} -(x - y)^2 - (z - y)^2 - x^2 - z^2 + 2(a_1 + a_4 - a_2)x \\ + 2(a_2 + a_5 - a_3 - a_4)y + 2(a_3 + a_6 - a_5)z, \ 123123, \\ -(x - z)^2 - (z - y)^2 - x^2 - y^2 + 2(a_1 + a_4 - a_3)x \\ + 2(a_2 + a_5 - a_3)y + 2(a_3 + a_6 - a_4 - a_5)z, \ 132132. \end{cases} \end{aligned}$$

Obviously, $q(M)\le 0 \Leftrightarrow \begin{cases} a_1+a_4\le a_2, a_2+a_5\le a_3+a_4, a_3+a_6\le a_5, 123123, \\ a_1+a_4\le a_3, a_2+a_5\le a_3, a_3+a_6\le a_4+a_5, 132132. \end{cases}$

And $q(M)=0 \Leftrightarrow \begin{cases} a_1+a_4\le a_2, a_2+a_5\le a_3+a_4, a_3+a_6\le a_5, 123123, \\ a_1+a_4\le a_3, a_2+a_5\le a_3, a_3+a_6\le a_4+a_5, 132132, \end{cases}$ and $x=y=z=0$, but $a_1+a_4\le a_2$, $a_2+a_5\le a_3+a_4$, $a_3+a_6\le a_5 \Rightarrow a_1+a_6\le 0$, this is a contradiction, so word 123123 is not tight for any $(a_1,\cdots,a_6)\in\mathbb{N}^6$, and word 132132(14-3, 25-3, 36-45) is tight by Theorem 2.3.

Lastly, let us consider $S_6^4$. Take word 123121 as an example, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{M=M_{x,x_1,x_2,x_3,y}\middle|(x,x_1,x_2,x_3,y)\in\mathbb{N}^5\right\}$, where

$$M=\begin{pmatrix} a_1-x-x_1 & 0 & 0 & x & 0 & x_1 \\ 0 & a_2-y & 0 & 0 & y & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ x_2 & 0 & 0 & a_4-x_2-x_3 & 0 & x_3 \\ 0 & y & 0 & 0 & a_5-y & 0 \\ x+x_1-x_2 & 0 & 0 & x_2+x_3-x & 0 & a_6-x_1-x_3 \end{pmatrix}$$

and

$$\begin{aligned} q(M) &= 2\langle i_1,i_1\rangle\big(a_1x+a_1x_1+a_4x_2+a_4x_3+a_6x_1+a_6x_3-2x^2-2x_1^2-2x_2^2-2x_3^2 \\ &\quad -2x_1x+2x_2x+x_3x+x_1x_2-x_1x_3-2x_2x_3\big)+2\langle i_2,i_2\rangle\big(a_2y+a_5y-2y^2\big) \\ &\quad +(i_1\cdot i_1)\big(a_4x+2a_4x_1-a_4x_2+x^2+x_1^2+x_2^2+x_3^2+x_1x-x_2x-x_3x-x_1x_2+x_2x_3\big) \\ &\quad +(i_2\cdot i_2)y^2+2(i_1\cdot i_2)\big(a_2x+a_2x_1+a_5x_1+a_5x_3+a_4y-x_2y-x_3y\big) \\ &\quad +2(i_1\cdot i_3)\big(a_3x+a_3x_1\big)+2(i_2\cdot i_3)a_3y \\ &= -(x-x_2)^2-(x_2-y))^2-(x_3-y)^2-x^2-2x_1^2-x_3^2-2x_1x-2x_1x_3-2x_2x_3 \\ &\quad +2(a_1+a_4-a_2)x+2(a_1+2a_4+a_6-a_2-a_5)x_1+2(a_4+a_6-a_5)x_3 \\ &\quad +2(a_2+a_5-a_3-a_4)y. \end{aligned}$$

Obviously, $q(M)\le 0\Leftrightarrow a_1+a_4\le a_2$, $a_4+a_6\le a_5$, $a_2+a_5\le a_3+a_4$. And $q(M)=0\Leftrightarrow a_1+a_4\le a_2$, $a_4+a_6\le a_5$, $a_2+a_5\le a_3+a_4$, $x=x_1=x_2=x_3=y=0$, so word 123121(14-2, 46-5, 25-34) is tight by Theorem 2.3. So (7) holds.

Case 6. $t=7$. Applying the $M-S$ word-procedure to $M_6$, deleting words including subwords in $\Phi(\{1212,123123\})\cup\Psi\Phi(\{1212,123123\})$, we get $S_7\cup\{2132132,1232123\}$.

As $S_6^2,S_6^4$, we can prove $S_7^1,S_7^2$. Consider word 2123242 in $S_7^3$, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{M=M_{x,x_1,\cdots,x_8}\middle|(x,x_1,\cdots,x_8)\in\mathbb{N}^9\right\}$, where

$$M=\begin{pmatrix} b_{11} & 0 & x & 0 & x_1 & 0 & x_2 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ x_3 & 0 & b_{33} & 0 & x_4 & 0 & x_5 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ x_6 & 0 & x_7 & 0 & b_{55} & 0 & x_8 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ b_{71} & 0 & b_{73} & 0 & b_{75} & 0 & b_{77} \end{pmatrix},$$

$$b_{11}=a_1-x-x_1-x_2,\ b_{33}=a_3-x_3-x_4-x_5,\ b_{55}=a_5-x_6-x_7-x_8,$$
$$b_{77}=a_7-x_2-x_5-x_8,\ b_{71}=x+x_1+x_2-x_3-x_6,$$
$$b_{73}=x_3+x_4+x_5-x-x_7,\ b_{75}=x_6+x_7+x_8-x_1-x_4,$$

$$\begin{aligned}q(M)=&2\langle i_1,i_1\rangle(a_1x+a_1x_1+a_1x_2+a_3x_3+a_3x_4+a_3x_5+a_5x_6+a_5x_7+a_5x_8\\&+a_7x_2+a_7x_5+a_7x_8-2x^2-2x_1^2-2x_2^2-2x_3^2-2x_4^2-2x_5^2-2x_6^2-2x_7^2-2x_8^2\\&-2x_1x-2x_2x+2x_3x+x_4x+x_5x+x_6x-x_7x-2x_1x_2+x_1x_3-x_1x_4+2x_1x_6\\&+x_1x_7+x_1x_8+x_2x_3-x_2x_5+x_2x_6-x_2x_8-2x_3x_4-2x_3x_5-x_3x_6+x_3x_7\\&-2x_4x_5+x_4x_6+2x_4x_7+x_4x_8+x_5x_7-x_5x_8-2x_6x_7-2x_6x_8-2x_7x_8)\\&+(i_1\cdot i_1)(a_3x+2a_3x_1+2a_3x_2-a_3x_3+a_5x_1+2a_5x_2-a_5x_6+a_5x_4+2a_5x_5-a_5x_7\\&+x^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2+x_1x+x_2x-x_3x-x_4x-x_5x+x_7x\\&+x_1x_2-x_1x_3-x_1x_5-x_1x_6-x_1x_8-x_2x_3-x_2x_6+x_3x_4+x_3x_5-x_3x_7+x_4x_5-x_4x_6\\&-x_4x_7-x_4x_8-x_5x_6-x_5x_7+x_6x_7+x_6x_8+x_7x_8)+2(i_1\cdot i_2)(a_2x+a_2x_1+a_2x_2)\\&+2(i_1\cdot i_4)(a_4x_1+a_4x_2+a_4x_4+a_4x_5)+2(i_1\cdot i_6)(a_6x_2+a_6x_5+a_6x_8)\\=&-2f(x,x_1,\cdots,x_8)+2(a_1+a_3-a_2)x+2(a_1+2a_3+a_5-a_2-a_4)x_1\\&+2(a_1+2a_3+2a_5+a_7-a_2-a_4-a_6)x_2+2(a_3+a_5-a_4)x_4\\&+2(a_3+2a_5+a_7-a_4-a_6)x_5+2(a_5+a_7-a_6)x_8,\end{aligned}$$

where $f(x,x_1,\cdots,x_8)=\frac{1}{2}XA_fX^{\mathrm{T}}$, $X=(x,x_1,\cdots,x_8)$ symmetric matrix $A_f$ is as follows

$$A_f=\begin{pmatrix}2&1&1&-1&0&0&-1&0&0\\1&2&1&0&1&1&-1&-1&0\\1&1&2&0&0&1&0&0&1\\-1&0&0&2&1&1&1&0&0\\0&1&0&1&2&1&0&-1&0\\0&1&1&1&1&2&1&0&1\\-1&-1&0&1&0&1&2&1&1\\0&-1&0&0&-1&0&1&2&1\\0&0&1&0&0&1&1&1&2\end{pmatrix}$$

Using the above algorithm in §2, we have

$$R_1=\{r_1,\cdots,r_9\};$$
$$R_2=\{r_1+r_4,r_1+r_7,r_2+r_7,r_2+r_8,r_5+r_8\};$$
$$R_3=\phi.$$

By Theorem 2.8, the unit form $f(x,x_1,\cdots,x_8)$ is weakly positive, *i.e.*, $f(x,x_1,\cdots,x_8)\geq 0$ for any $(x,x_1,\cdots,x_8)\in\mathbb{N}^9$, and $f(x,x_1,\cdots,x_8)=0\Leftrightarrow x=x_1=\cdots=x_8=0$. So, $q(M)\leq 0\Leftrightarrow a_1+a_3\leq a_2$, $a_5+a_7\leq a_6$, $a_3+a_5\leq a_4$. And $q(M)=0\Leftrightarrow a_1+a_3\leq a_2$, $a_5+a_7\leq a_6$, $a_3+a_5\leq a_4$, $x=x_1=\cdots=x_8=0$, so word 2123242(13-2, 57-6, 35-4) is tight by Theorem 2.3.

At last, let us see $S_7^4$, for words 2123212, 2124212 in $S_7^4$, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{M=M_{x,x_1,\cdots,x_8,y}\mid(x,x_1,\cdots,x_8,y)\in\mathbb{N}^{10}\}$, where

$$M=\begin{pmatrix} b_{11} & 0 & x & 0 & x_1 & 0 & x_2 \\ 0 & a_2-y & 0 & 0 & 0 & y & 0 \\ x_3 & 0 & b_{33} & 0 & x_4 & 0 & x_5 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ x_6 & 0 & x_7 & 0 & b_{55} & 0 & x_8 \\ 0 & y & 0 & 0 & 0 & a_6-y & 0 \\ b_{71} & 0 & b_{73} & 0 & b_{75} & 0 & b_{77} \end{pmatrix}$$

$$\begin{aligned} &b_{11}=a_1-x-x_1-x_2,\ b_{33}=a_3-x_3-x_4-x_5,\ b_{55}=a_5-x_6-x_7-x_8,\\ &b_{77}=a_7-x_2-x_5-x_8,\ b_{71}=x+x_1+x_2-x_3-x_6,\\ &b_{73}=x_3+x_4+x_5-x-x_7,\ b_{75}=x_6+x_7+x_8-x_1-x_4, \end{aligned}$$

$$\begin{aligned} q(M)=&\,2\langle i_1,i_1\rangle\big(a_1x+a_1x_1+a_1x_2+a_3x_3+a_3x_4+a_3x_5+a_5x_6+a_5x_7+a_5x_8\\ &+a_7x_2+a_7x_5+a_7x_8-2x^2-2x_1^2-2x_2^2-2x_3^2-2x_4^2-2x_5^2-2x_6^2-2x_7^2\\ &-2x_8^2-2x_2x+2x_3x+x_4x+x_5x+x_6x-x_7x-2x_1x_2+x_1x_3-x_1x_4+2x_1x_6\\ &+x_1x_7-2x_1x+x_1x_8+x_2x_3-x_2x_5+x_2x_6-x_2x_8-2x_3x_4-2x_3x_5-x_3x_6\\ &+x_3x_7-2x_4x_5+x_4x_6+2x_4x_7+x_4x_8+x_5x_7-x_5x_8-2x_6x_7-2x_6x_8-2x_7x_8\big)\\ &+2\langle i_2,i_2\rangle\big(a_2y+a_6y-2y^2\big)+(i_1\cdot i_1)\big(a_3x+2a_3x_1+2a_3x_2-a_3x_3+a_5x_1\\ &+2a_5x_2-a_5x_6+a_5x_4+2a_5x_5-a_5x_7+x^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2\\ &+x_7^2+x_8^2+x_1x+x_2x-x_3x-x_4x-x_5x+x_7x+x_1x_2-x_1x_3-x_1x_5-x_1x_6\\ &-x_1x_8-x_2x_3-x_2x_6+x_3x_4+x_3x_5-x_3x_7+x_4x_5-x_4x_6-x_4x_7-x_4x_8\\ &-x_5x_6-x_5x_7+x_6x_7+x_6x_8+x_7x_8\big)+(i_2\cdot i_2)y^2+2(i_1\cdot i_2)\big(a_2x+a_2x_1\\ &+a_2x_2+a_6x_2+a_3y+a_5y+a_6x_5+a_6x_8-x_3y-x_5y-x_6y-x_8y\big)\\ &+2(i_1\cdot i_4)\big(a_4x_1+a_4x_2+a_4x_4+a_4x_5\big)\\ =&-2f(x,x_1,\cdots,x_8,y)+2(a_1+a_3-a_2)x+2(a_1+2a_3+a_5-a_2-a_4)x_1\\ &+2(a_1+2a_3+2a_5+a_7-a_2-a_4-a_6)x_2+2(a_3+a_5-a_4)x_4\\ &+2(a_3+2a_5+a_7-a_4-a_6)x_5+2(a_5+a_7-a_6)x_8+2(a_2+a_6-a_3-a_5)y, \end{aligned}$$

where $f(x,x_1,\cdots,x_8,y)=\frac{1}{2}XA_fX^{\mathrm{T}}$, $X=(x,x_1,\cdots,x_8,y)$, symmetric matrix $A_f$ is as follows

$$A_f=\begin{pmatrix} 2 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 2 \end{pmatrix}$$

Using the above algorithm in §2, we have

$$R_1=\{r_1,\cdots,r_{10}\};$$

$$R_2=\{r_1+r_4,r_1+r_7,r_2+r_7,r_2+r_8,r_4+r_{10},r_5+r_8,r_6+r_{10},r_7+r_{10},r_9+r_{10}\};$$

$$R_3 = \{r_1 + r_4 + r_{10}, r_1 + r_7 + r_{10}, r_2 + r_7 + r_{10}, r_4 + r_9 + r_{10}\};$$
$$R_4 = \{r_1 + r_4 + r_7 + r_{10}, r_1 + r_4 + r_9 + r_{10}\};$$
$$R_5 = \phi.$$

By Theorem 2.8, the unit form $f(x, x_1, \cdots, x_9, y)$ is weakly positive, *i.e.*, $f(x, x_1, \cdots, x_9, y) \geq 0$ for any $(x, x_1, \cdots, x_9, y) \in \mathbb{N}^{10}$, and $f(x, x_1, \cdots, x_9, y) = 0 \Leftrightarrow x = x_1 = \cdots = x_8 = y = 0$. So, $q(M) \leq 0 \Leftrightarrow a_1 + a_3 \leq a_2$, $a_5 + a_7 \leq a_6$, $a_2 + a_6 \leq a_3 + a_5 \leq a_4$. And $q(M) = 0 \Leftrightarrow a_1 + a_3 \leq a_2$, $a_5 + a_7 \leq a_6$, $a_2 + a_6 \leq a_3 + a_5 \leq a_4$ and $x = x_1 = \cdots = x_8 = y = 0$, so words 2123212, 2124212(13-2, 57-6, 26-35-4) are all tight by Theorem 2.3. So (8) holds.

Case 7. $t = 8$. Applying the $M - S$ word-procedure to $M_7$, deleting words including subwords in

$\Phi(\{1212, 123123, 1232123, 2132132\}) \cup \Psi\Phi(\{1212, 123123, 1232123, 2132132\})$,

we get $S_8$.

As $S_7^4$, we can prove that words in $S_8^4$ are all tight.

For $S_8^1$, only consider word 12134234, we have

$\mathcal{M}_{\mathbf{i},\mathbf{a}} = \{M = M_{x,y,z,u} \mid (x, y, z, u) \in \mathbb{N}^4\}$, where

$$M = \begin{pmatrix} a_1 - x & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 - y & 0 & 0 & 0 & y & 0 & 0 \\ x & 0 & a_3 - x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 - z & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & a_5 - u & 0 & 0 & u \\ 0 & y & 0 & 0 & 0 & a_6 - y & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 & a_7 - z & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & a_8 - u \end{pmatrix},$$

and

$$\begin{aligned} q(M) &= 2\langle i_1, i_1\rangle(a_1x + a_3x - 2x^2) + 2\langle i_2, i_2\rangle(a_2y + a_6y - 2y^2) \\ &\quad + 2\langle i_4, i_4\rangle(a_4z + a_7z - 2z^2) + 2\langle i_5, i_5\rangle(a_5u + a_8u - 2u^2) + (i_1 \cdot i_1)x^2 \\ &\quad + (i_2 \cdot i_2)y^2 + (i_4 \cdot i_4)z^2 + (i_5 \cdot i_5)u^2 + 2(i_1 \cdot i_2)(a_2x + a_3y - xy) \\ &\quad + 2(i_2 \cdot i_4)(a_4y + a_6z - yz) + 2(i_2 \cdot i_5)(a_5y + a_6u - yu) \\ &= -2f(x, y, z, u) + 2(a_1 + a_3 - a_2)x + 2(a_2 + a_6 - a_3 - a_4 - a_5)y \\ &\quad + 2(a_4 + a_7 - a_6)z + 2(a_5 + a_8 - a_6)u, \end{aligned}$$

where $f(x, y, z, u) = \frac{1}{2} X A_f X^{\mathrm{T}}, X = (x, y, z, u)$, symmetric matrix $A_f$ is as follows

$$A_f = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

Using the above algorithm in §2, we have

$$R_1 = \{r_1, r_2, r_3, r_4\};$$
$$R_2 = \{r_1 + r_2, r_2 + r_3, r_2 + r_4\};$$

$$R_3=\{r_1+r_2+r_3,r_1+r_2+r_4,r_2+r_3+r_4\};$$
$$R_4=\{r_1+r_2+r_3+r_4\};$$
$$R_5=\{r_1+2r_2+r_3+r_4\};$$
$$R_6=\phi.$$

By Theorem 2.8, the unit form $f(x,y,z,u)$ is weakly positive, *i.e.*, $f(x,y,z,u)\ge 0$ for any $(x,y,z,u)\in\mathbb{N}^4$, and $f(x,y,z,u)=0\Leftrightarrow x=y=z=u=0$. So, $q(M)\le 0\Leftrightarrow a_1+a_3\le a_2$, $a_4+a_7\le a_6$, $a_5+a_8\le a_6$, $a_2+a_6\le a_3+a_4+a_5$. And $q(M)=0\Leftrightarrow a_1+a_3\le a_2$, $a_4+a_7\le a_6$, $a_5+a_8\le a_6$, $a_2+a_6\le a_3+a_4+a_5$ and $x=y=z=u=0$, word 12134234(13-2, 47-6, 58-6, 26-345) is tight by Theorem 2.3.

Consider words 12132142,12142132 in $S_8^2$, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{M=M_{x,x_1,x_2,x_3,y,y_1,y_2,y_3}\,\middle|\,(x,x_1,x_2,x_3,y,y_1,y_2,y_3)\in\mathbb{N}^8\right\}$, where

$$M=\begin{pmatrix} b_{11} & 0 & x & 0 & 0 & x_1 & 0 & 0\\ 0 & b_{22} & 0 & 0 & y & 0 & 0 & y_1\\ x_2 & 0 & b_{33} & 0 & 0 & x_3 & 0 & 0\\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0\\ 0 & y_2 & 0 & 0 & b_{55} & 0 & 0 & y_3\\ b_{61} & 0 & b_{63} & 0 & 0 & b_{66} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0\\ 0 & b_{82} & 0 & 0 & b_{85} & 0 & 0 & b_{88}\end{pmatrix}$$

$$b_{11}=a_1-x-x_1,\ b_{22}=a_2-y-y_1,\ b_{33}=a_3-x_2-x_3,\ b_{55}=a_5-y_2-y_3,$$
$$b_{61}=x+x_1-x_2,\ b_{63}=x_2+x_3-x,\ b_{66}=a_6-x_1-x_3,$$
$$b_{82}=y+y_1-y_2,\ b_{85}=y_2+y_3-y,\ b_{88}=a_8-y_1-y_3.$$

So, we have

$$\begin{aligned}q(M)&=2\langle i_1,i_1\rangle\big(a_1x+a_1x_1+a_3x_2+a_3x_3+a_6x_1+a_6x_3-2x^2-2x_1^2-2x_2^2-2x_3^2\\&\quad-2x_1x+2x_2x+x_3x+x_1x_2-x_1x_3-2x_2x_3\big)+2\langle i_2,i_2\rangle\big(a_2y+a_2y_1+a_5y_2+a_5y_3\\&\quad+a_8y_1+a_8y_3-2y^2-2y_1^2-2y_2^2-2y_3^2-2y_1y+2y_2y+y_3y+y_1y_2-y_1y_3-2y_2y_3\big)\\&\quad+(i_1\cdot i_1)\big(a_3x+2a_3x_1-a_3x_2+x^2+x_1^2+x_2^2+x_3^2+x_1x-x_2x-x_3x-x_1x_2+x_2x_3\big)\\&\quad+(i_2\cdot i_2)\big(a_5y+2a_5y_1-a_5y_2+y^2+y_1^2+y_2^2+y_3^2+y_1y-y_2y-y_3y-y_1y_2+y_2y_3\big)\\&\quad+2(i_1\cdot i_2)(a_2x+a_2x_1+a_5x_1+a_5x_3+a_3y+a_3y_1+a_6y_1+a_6y_3-xy-xy_1-x_1y_1\\&\quad-x_2y_2+xy_2-x_3y_2-x_1y_3-x_3y_3)+2(i_2\cdot i_4)(a_4y+a_4y_1)+2(i_2\cdot i_7)(a_7y_1+a_7y_3)\\&=-2f(x,x_1,x_2,x_3,y,y_1,y_2,y_3)+2(a_1+a_3-a_2)x+2(a_1+2a_3+a_6-a_2-a_5)x_1\\&\quad+2(a_3+a_6-a_5)x_3+2(a_2+a_5-a_3-a_4)y+2(a_2+2a_5+a_8-a_3-a_4-a_6\\&\quad-a_7)y_1+2(a_5+a_8-a_6-a_7)y_3,\end{aligned}$$

where $f(x,x_1,x_2,x_3,y,y_1,y_2,y_3)=\frac{1}{2}XA_fX^{\mathrm{T}}$, $X=(x,x_1,x_2,x_3,y,y_1,y_2,y_3)$, symmetric matrix $A_f$ is as follows

$$A_f=\begin{pmatrix}2&1&-1&0&-1&-1&1&0\\1&2&0&1&0&-1&0&-1\\-1&0&2&1&0&0&-1&0\\0&1&1&2&0&0&-1&-1\\-1&0&0&0&2&1&-1&0\\-1&-1&0&0&1&2&0&1\\1&0&-1&-1&-1&0&2&1\\0&-1&0&-1&0&1&1&2\end{pmatrix}$$

Using the above algorithm in §2, we have

$$R_1=\{r_1,\cdots,r_8\};$$
$$R_2=\{r_1+r_3,r_1+r_5,r_1+r_6,r_2+r_6,r_2+r_8,r_3+r_7,r_4+r_7,r_4+r_8,r_5+r_7\};$$
$$R_3=\{r_1+r_3+r_5,r_1+r_3+r_6,r_3+r_5+r_7,r_4+r_5+r_7\};$$
$$R_4=\{r_1+r_3+r_5+r_7\};$$
$$R_5=\phi.$$

By Theorem 2.8, the unit form $f(x,x_1,x_2,x_3,y,y_1,y_2,y_3)$ is weakly positive, *i.e.*, $f(x,x_1,x_2,x_3,y,y_1,y_2,y_3)\ge 0$ for any $(x,x_1,x_2,x_3,y,y_1,y_2,y_3)\in\mathbb{N}^8$, and $f(x,x_1,x_2,x_3,y,y_1,y_2,y_3)=0\Leftrightarrow x=x_1=x_2=x_3=y=y_1=y_2=y_3=0$. So, $q(M)\le 0\Leftrightarrow a_1+a_3\le a_2$, $a_3+a_6\le a_5$, $a_5+a_8\le a_6+a_7$, $a_2+a_5\le a_3+a_4$. And $q(M)=0\Leftrightarrow a_1+a_3\le a_2$, $a_3+a_6\le a_5$, $a_5+a_8\le a_6+a_7$, $a_2+a_5\le a_3+a_4$ and $x=x_1=x_2=x_3=y=y_1=y_2=y_3=0$, words 12132142, 12142132(13-2, 36-5, 58-67, 25-34) are all tight by Theorem 2.3.

For $S_8^3$, only consider word 12132423, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{M=M_{x,y,y_1,y_2,y_3,z}\,\middle|\,(x,y,y_1,y_2,y_3,z)\in\mathbb{N}^6\right\}$, where

$$M=\begin{pmatrix}a_1-x&0&x&0&0&0&0&0\\0&a_2-y-y_1&0&0&y&0&y_1&0\\x&0&a_3-x&0&0&0&0&0\\0&0&0&a_4-z&0&0&0&z\\0&y_2&0&0&a_5-y_2-y_3&0&y_3&0\\0&0&0&0&0&a_6&0&0\\0&y+y_1-y_2&0&0&y_2+y_3-y&0&a_7-y_1-y_3&0\\0&0&0&z&0&0&0&a_8-z\end{pmatrix},$$

and

$$\begin{aligned}q(M)&=2\langle i_1,i_1\rangle(a_1x+a_3x-2x^2)+2\langle i_2,i_2\rangle(a_2y+a_2y_1+a_5y_2+a_5y_3+a_7y_1+a_7y_3\\&\quad-2y^2-2y_1^2-2y_2^2-2y_3^2-2y_1y+2y_2y+y_3y+y_1y_2-y_1y_3-2y_2y_3)+2\langle i_4,i_4\rangle\\&\quad\times(a_4z+a_8z-2z^2)+(i_1\cdot i_1)x^2+(i_2\cdot i_2)(a_5y+2a_5y_1-a_5y_2+y^2+y_1^2+y_2^2+y_3^2\\&\quad+y_1y-y_2y-y_3y-y_1y_2+y_2y_3)+(i_4\cdot i_4)z^2+2(i_1\cdot i_2)(a_2x+a_3y+a_3y_1-xy\\&\quad-xy_1)+2(i_2\cdot i_4)(a_4y+a_4y_1+a_5z+a_7z-yz-y_1z)+2(i_2\cdot i_6)(a_6y_1+a_6y_3)\\&=-2f(x,y,y_1,y_2,y_3,z)+2(a_1+a_3-a_2)x+2(a_2+a_5-a_3-a_4)y+2(a_2+2a_5\\&\quad+a_7-a_3-a_4-a_6)y_1+2(a_5+a_7-a_6)y_3+2(a_4+a_8-a_5-a_7)z,\end{aligned}$$

where $f(x,y,y_1,y_2,y_3,z)=\frac{1}{2}XA_fX^{\mathrm{T}}, X=(x,y,y_1,y_2,y_3,z)$, symmetric matrix

$A_f$ is as follows

$$A_f = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -1 & 0 & -1 \\ -1 & 1 & 2 & 0 & 1 & -1 \\ 0 & -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Using the above algorithm in §2, we have

$R_1 = \{r_1, \cdots, r_6\}$;

$R_2 = \{r_1 + r_2, r_1 + r_3, r_2 + r_4, r_2 + r_6, r_3 + r_6\}$;

$R_3 = \{r_1 + r_2 + r_4, r_1 + r_2 + r_6, r_1 + r_3 + r_6, r_2 + r_4 + r_6\}$;

$R_4 = \{r_1 + r_2 + r_4 + r_6, r_1 + r_2 + r_3 + r_6\}$;

$R_5 = \{r_1 + 2r_2 + r_4 + r_6, r_1 + r_2 + 2r_3 + r_6, r_1 + r_2 + r_3 + r_4 + r_6\}$;

$R_6 = \{2r_1 + r_2 + 2r_3 + r_6, r_1 + r_2 + 2r_3 + r_4 + r_6, r_1 + r_2 + 2r_3 + 2r_6\}$;

$R_7 = \{2r_1 + r_2 + 2r_3 + r_4 + r_6, 2r_1 + r_2 + 2r_3 + 2r_6, r_1 + r_2 + 2r_3 + r_4 + 2r_6\}$;

$R_8 = \{2r_1 + r_2 + 2r_3 + r_4 + 2r_6\}$;

$R_9 = \{2r_1 + 2r_2 + 2r_3 + r_4 + 2r_6\}$;

$R_{10} = \phi$.

By Theorem 2.8, the unit form $f(x, y, y_1, y_2, y_3, z)$ is weakly positive, *i.e.*, $f(x, y, y_1, y_2, y_3, z) \ge 0$ for any $(x, y, y_1, y_2, y_3, z) \in \mathbb{N}^6$, and $f(x, y, y_1, y_2, y_3, z) = 0 \Leftrightarrow x = y = y_1 = y_2 = y_3 = z = 0$. So, $q(M) \le 0 \Leftrightarrow a_1 + a_3 \le a_2$, $a_2 + a_5 \le a_3 + a_4$, $a_4 + a_8 \le a_5 + a_7 \le a_6$. And $q(M) = 0 \Leftrightarrow a_1 + a_3 \le a_2$, $a_2 + a_5 \le a_3 + a_4$, $a_4 + a_8 \le a_5 + a_7 \le a_6$ and $x = y = y_1 = y_2 = y_3 = z = 0$, word 12132423(13-2, 25-34, 48-57-6) is tight by Theorem 2.3. So (9) holds.

Case 8. $t = 9$. Applying the $M - S$ word-procedure to $M_8$, deleting words including subwords in

$\Phi(\{1212, 123123, 1232123, 2132132\}) \cup \Psi\Phi(\{1212, 123123, 1232123, 2132132\})$,

we get $S_9 \cup \Phi(\{121342342\}) \cup \Psi\Phi(\{121342342\})$.

As $S_8^2, S_8^3, S_8^4$, words in $S_9^1, S_9^2, S_9^3$ are all tight.

Consider $S_9^4$, it is found that there is four 2, three 1 (or 2), one 3, and one 4 in every word, only consider word 121324212, we have

$\mathcal{M}_{\mathbf{i},\mathbf{a}} = \{M = M_{x_1,\cdots x_4, y_1,\cdots,y_9} \mid (x_1, \cdots x_4, y_1, \cdots, y_9) \in \mathbb{N}^{13}\}$, where

$$M = \begin{pmatrix} b_{11} & 0 & x_1 & 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & b_{22} & 0 & 0 & y_1 & 0 & y_2 & 0 & y_3 \\ x_3 & 0 & b_{33} & 0 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_4 & 0 & 0 & b_{55} & 0 & y_5 & 0 & y_6 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 \\ 0 & y_7 & 0 & 0 & y_8 & 0 & b_{77} & 0 & y_9 \\ b_{81} & 0 & b_{83} & 0 & 0 & 0 & 0 & b_{88} & 0 \\ 0 & b_{92} & 0 & 0 & b_{95} & 0 & b_{97} & 0 & b_{99} \end{pmatrix}$$

$$
\begin{aligned}
&b_{11}=a_1-x_1-x_2,\ b_{22}=a_2-y_1-y_2-y_3,\ b_{33}=a_3-x_3-x_4,\\
&b_{55}=a_5-y_4-y_5-y_6,\ b_{77}=a_7-y_7-y_8-y_9,\ b_{81}=x_1+x_2-x_3,\\
&b_{83}=x_3+x_4-x_1,\ b_{88}=a_8-x_2-x_4,\ b_{92}=y_1+y_2+y_3-y_4-y_7,\\
&b_{95}=y_4+y_5+y_6-y_1-y_8,\ b_{97}=y_7+y_8+y_9-y_2-y_5,\\
&b_{99}=a_9-y_3-y_6-y_9,
\end{aligned}
$$

and

$$
\begin{aligned}
q(M)=&\,2\langle i_1,i_1\rangle\big(a_1x_1+a_1x_2+a_3x_3+a_3x_4+a_8x_2+a_8x_4-2x_1^2-2x_2^2-2x_3^2-2x_4^2\\
&-2x_1x_2+2x_1x_3+x_1x_4+x_2x_3-x_2x_4-2x_3x_4\big)+2\langle i_2,i_2\rangle\big(a_2y_1+a_2y_2+a_2y_3+a_5y_4\\
&+a_5y_5+a_5y_6+a_7y_7+a_7y_8+a_7y_9+a_9y_3+a_9y_6+a_9y_9-2y_1^2-2y_2^2-2y_3^2-2y_4^2\\
&-2y_5^2-2y_6^2-2y_7^2-2y_8^2-2y_9^2-2y_1y_2-2y_1y_3+2y_1y_4+y_1y_5+y_1y_6+y_1y_7\\
&-y_1y_8-2y_2y_3+y_2y_4-y_2y_5+2y_2y_7+y_2y_8+y_2y_9+y_3y_4-y_3y_6+y_3y_7-y_3y_9\\
&-2y_4y_5-2y_4y_6-y_4y_7+y_4y_8-2y_5y_6+y_5y_7+2y_5y_8+y_5y_9+y_6y_8-y_6y_9\\
&-2y_7y_8-2y_7y_9-2y_8y_9\big)+(i_1\cdot i_1)\big(a_3x_1+2a_3x_2-a_3x_3+x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2\\
&+x_3x_4-x_1x_3-x_1x_4-x_2x_3\big)+(i_2\cdot i_2)\big(a_5y_1+2a_5y_2+2a_5y_3-a_5y_4+a_7y_2+2a_7y_3\\
&-a_7y_7+a_7y_5+2a_7y_6-a_7y_8+y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2+y_7^2+y_8^2+y_9^2+y_1y_2\\
&+y_1y_3-y_1y_4-y_1y_5-y_1y_6+y_1y_8+y_2y_3-y_2y_4-y_2y_6-y_2y_7-y_2y_9-y_3y_4\\
&-y_3y_7+y_4y_5+y_4y_6-y_4y_8+y_5y_6-y_5y_7-y_5y_8-y_5y_9-y_6y_7-y_6y_8+y_7y_8\\
&+y_7y_9+y_8y_9\big)+2(i_1\cdot i_2)\big(a_2x_1+a_2x_2+a_5x_2+a_5x_4+a_7x_2+a_7x_4+a_3y_1+a_3y_2\\
&+a_3y_3+a_8y_3+a_8y_6+a_8y_9-x_1y_1-x_1y_2-x_1y_3+x_1y_4+x_1y_7-x_2y_3-x_2y_6\\
&-x_2y_9-x_3y_4-x_3y_7-x_4y_4-x_4y_6-x_4y_7-x_4y_9\big)+2(i_2\cdot i_4)\big(a_4y_1+a_4y_2+a_4y_3\big)\\
&+2(i_2\cdot i_6)\big(a_6y_2+a_6y_3+a_6y_5+a_6y_6\big)\\
=&-2f(x_1,\cdots,x_4,y_1,\cdots,y_9)+2(a_1+a_3-a_2)x_1+2(a_1+2a_3+a_8-a_2-a_5-a_7)x_2\\
&+2(a_3+a_8-a_5-a_7)x_4+2(a_2+a_5-a_3-a_4)y_1+2(a_2+2a_5+a_7-a_3-a_4\\
&-a_6)y_2+2(a_2+2a_5+2a_7+a_9-a_3-a_4-a_6-a_8)y_3+2(a_5+a_7-a_6)y_5\\
&+2(a_5+2a_7+a_9-a_6-a_8)y_6+2(a_7+a_9-a_8)y_9,
\end{aligned}
$$

where $f(x_1,\cdots,x_4,y_1,\cdots,y_9)=\frac{1}{2}XA_fX^{\mathrm{T}}$, $X=(x_1,\cdots,x_4,y_1,\cdots,y_9)$ symmetric matrix $A_f$ is as follows

$$
A_f=\begin{pmatrix}
2&1&-1&0&-1&-1&-1&1&0&0&1&0&0\\
1&2&0&1&0&0&-1&0&0&-1&0&0&-1\\
-1&0&2&1&0&0&0&-1&0&0&-1&0&0\\
0&1&1&2&0&0&0&-1&0&-1&-1&0&-1\\
-1&0&0&0&2&1&1&-1&0&0&-1&0&0\\
-1&0&0&0&1&2&1&0&1&1&-1&-1&0\\
-1&-1&0&0&1&1&2&0&0&1&0&0&1\\
1&0&-1&-1&-1&0&0&2&1&1&1&0&0\\
0&0&0&0&0&1&0&1&2&1&0&-1&0\\
0&-1&0&-1&0&1&1&1&1&2&1&0&1\\
1&0&-1&-1&-1&-1&0&1&0&1&2&1&1\\
0&0&0&0&0&-1&0&0&-1&0&1&2&1\\
0&-1&0&-1&0&0&1&0&0&1&1&1&2
\end{pmatrix}
$$

Using the above algorithm in §2, we have

$$R_1=\{r_1,\cdots,r_{13}\};$$

$$R_2=\{r_1+r_3,r_1+r_5,r_1+r_6,r_1+r_7,r_2+r_7,r_2+r_{10},r_2+r_{13},r_3+r_8,r_3+r_{11},$$
$$r_4+r_8,r_4+r_{10},r_4+r_{11},r_4+r_{13},r_5+r_8,r_5+r_{11},r_6+r_{11},r_6+r_{12},r_9+r_{12}\};$$

$$R_3=\{r_1+r_3+r_5,r_1+r_3+r_6,r_1+r_3+r_7,r_1+r_6+r_{12},r_3+r_5+r_8,r_3+r_5+r_{11},$$
$$r_3+r_6+r_{11},r_4+r_5+r_8,r_4+r_8+r_{13},r_4+r_5+r_{11},r_4+r_6+r_{11}\};$$

$$R_4=\{r_1+r_3+r_5+r_8,r_1+r_3+r_5+r_{11},r_1+r_3+r_6+r_{11},r_1+r_3+r_6+r_{12},$$
$$r_3+r_5+r_8+r_{11},r_4+r_5+r_8+r_{11},r_4+r_5+r_8+r_{13}\};$$

$$R_5=\{r_1+r_3+r_5+r_6+r_{11},r_3+r_4+r_5+r_8+r_{11}\};$$

$$R_6=\phi.$$

By Theorem 2.8, the unit form $f(x_1,\cdots,x_4,y_1,\cdots,y_9)$ is weakly positive, *i.e.*, for any $(x_1,\cdots,x_4,y_1,\cdots,y_9)\in\mathbb{N}^{13}$, $f(x_1,\cdots,x_4,y_1,\cdots,y_9)\geq 0$, and $f(x_1,\cdots,x_4,y_1,\cdots,y_9)=0\Leftrightarrow x_1=\cdots=x_4=y_1=\cdots=y_9=0$. So, we have $q(M)\leq 0$ if and only if

$$a_1+a_3\leq a_2,a_3+a_8\leq a_5+a_7\leq a_6,a_7+a_9\leq a_8,a_2+a_5\leq a_3+a_4. \quad (2)$$

$q(M)=0$ if and only if (2) hold and $x_1=\cdots=x_4=y_1=\cdots=y_9=0$, so word 121324212(13-2, 38-57-6, 79-8, 25-34) is tight by Theorem 2.3.

At last, let us see $S_9^5$, we find that there is five 2, two 1, one 2, and one 4 in every word, so it suffices to consider word 212321242, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{M=M_{x_1,\cdots x_{16},y}\,|\,(x_1,\cdots,x_{16},y)\in\mathbb{N}^{17}\}$, where

$$M=\begin{pmatrix} b_{11} & 0 & x_1 & 0 & x_2 & 0 & x_3 & 0 & x_4 \\ 0 & a_2-y & 0 & 0 & 0 & y & 0 & 0 & 0 \\ x_5 & 0 & b_{33} & 0 & x_6 & 0 & x_7 & 0 & x_8 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\ x_9 & 0 & x_{10} & 0 & b_{55} & 0 & x_{11} & 0 & x_{12} \\ 0 & y & 0 & 0 & 0 & a_6-y & 0 & 0 & 0 \\ x_{13} & 0 & x_{14} & 0 & x_{15} & 0 & b_{77} & 0 & x_{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 \\ b_{91} & 0 & b_{93} & 0 & b_{95} & 0 & b_{97} & 0 & b_{99} \end{pmatrix}$$

$$b_{11}=a_1-x_1-x_2-x_3-x_4,\ b_{33}=a_3-x_5-x_6-x_7-x_8,$$
$$b_{55}=a_5-x_9-x_{10}-x_{11}-x_{12},\ b_{77}=a_7-x_{13}-x_{14}-x_{15}-x_{16},$$
$$b_{91}=x_1+x_2+x_3+x_4-x_5-x_9-x_{13},\ b_{93}=x_5+x_6+x_7+x_8-x_1-x_{10}-x_{14},$$
$$b_{95}=x_9+x_{10}+x_{11}+x_{12}-x_2-x_6-x_{15},\ b_{97}=x_{13}+x_{14}+x_{15}+x_{16}-x_3-x_7-x_{11},$$
$$b_{99}=a_9-x_4-x_8-x_{12}-x_{16}.$$

So, we have

$$q(M)=2\langle i_1,i_1\rangle(a_1x_1+a_1x_2+a_1x_3+a_1x_4+a_3x_5+a_3x_6+a_3x_7+a_3x_8+a_5x_9+a_5x_{10}$$
$$+a_5x_{11}+a_5x_{12}+a_7x_{13}+a_7x_{14}+a_7x_{15}+a_7x_{16}+a_9x_4+a_9x_8+a_9x_{12}+a_9x_{16}-2x_1^2$$
$$-2x_2^2-2x_3^2-2x_4^2-2x_5^2-2x_6^2-2x_7^2-2x_8^2-2x_9^2-2x_{10}^2-2x_{11}^2-2x_{12}^2-2x_{13}^2$$
$$-2x_{14}^2-2x_{15}^2-2x_{16}^2-2x_1x_2-2x_1x_3-2x_1x_4+2x_1x_5+x_1x_6+x_1x_7+x_1x_8+x_1x_9$$
$$-x_1x_{10}+x_1x_{13}-x_1x_{14}-2x_2x_3-2x_2x_4+x_2x_5-x_2x_6+2x_2x_9+x_2x_{10}+x_2x_{11}$$

$$
\begin{aligned}
&+x_2x_{12}+x_2x_{13}-x_2x_{15}-2x_3x_4+x_3x_5-x_3x_7+x_3x_9-x_3x_{11}+2x_3x_{13}+x_3x_{14}\\
&+x_3x_{15}+x_3x_{16}+x_4x_5-x_4x_8+x_4x_9-x_4x_{12}+x_4x_{13}-x_4x_{16}-2x_5x_6-2x_5x_7\\
&-2x_5x_8-x_5x_9+x_5x_{10}-x_5x_{13}+x_5x_{14}-2x_6x_7-2x_6x_8+x_6x_9+2x_6x_{10}+x_6x_{11}\\
&+x_6x_{12}+x_6x_{14}-x_6x_{15}-2x_7x_8+x_7x_{10}-x_7x_{11}+x_7x_{13}+2x_7x_{14}+x_7x_{15}+x_7x_{16}\\
&+x_8x_{10}-x_8x_{12}+x_8x_{14}-x_8x_{16}-2x_9x_{10}-2x_9x_{11}-2x_9x_{12}-x_9x_{13}+x_9x_{15}-2x_{10}x_{11}\\
&-2x_{10}x_{12}-x_{10}x_{14}+x_{10}x_{15}-2x_{11}x_{12}+x_{11}x_{13}+x_{11}x_{14}+2x_{11}x_{15}+x_{11}x_{16}\\
&+x_{12}x_{15}-x_{12}x_{16}-2x_{13}x_{14}-2x_{13}x_{15}-2x_{13}x_{16}-2x_{14}x_{15}-2x_{14}x_{16}-2x_{15}x_{16}\big)\\
&+2\langle i_2,i_2\rangle\left(a_2y+a_6y-2y^2\right)+2\left(i_1\cdot i_2\right)\left(a_2x_1+a_2x_2+a_2x_3+a_2x_4+a_6x_3\right.\\
&+a_6x_4+a_6x_7+a_6x_8+a_6x_{11}+a_6x_{12}+a_3y+a_5y-x_5y-x_7y-x_8y-x_9y\\
&\left.-x_{11}y-x_{12}y\right)+2\left(i_1\cdot i_4\right)\left(a_4x_2+a_4x_3+a_4x_4+a_4x_6+a_4x_7+a_4x_8\right)\\
&+2\left(i_1\cdot i_8\right)\left(a_8x_4+a_8x_8+a_8x_{12}+a_8x_{16}\right)\\
=&-2f\left(x_1,\cdots,x_{16},y\right)+2\left(a_1+a_3-a_2\right)x_1+2\left(a_1+2a_3+a_5-a_2-a_4\right)x_2\\
&+2\left(a_1+2a_3+2a_5+a_7-a_2-a_4-a_6\right)x_3+2(a_1+2a_3+2a_5+2a_7+a_9\\
&-a_2-a_4-a_6-a_8))x_4+2\left(a_3+a_5-a_4\right)x_6+2\left(a_3+2a_5+a_7-a_4-a_6\right)x_7\\
&+2\left(a_3+2a_5+2a_7+a_9-a_4-a_6-a_8\right)x_8+2\left(a_5+a_7-a_6\right)x_{11}+2\left(a_5\right.\\
&\left.+2a_7+a_9-a_6-a_8\right)x_{12}+2\left(a_7+a_9-a_9\right)x_{16}+2\left(a_2+a_6-a_3-a_5\right)y,
\end{aligned}
$$

where $f\left(x_1,\cdots,x_{16},y\right)=\frac{1}{2}XA_fX^{\mathrm{T}}, X=\left(x_1,\cdots,x_{16},y\right)$, symmetric matrix $A_f$ is as follows

$$
A_f=\begin{pmatrix}
2 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0\\
1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & 0\\
1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
-1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\\
0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & -1\\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1\\
-1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1\\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & -1\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & -1\\
-1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0\\
0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 2 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

Using the above algorithm in §2, we have

$$R_1=\{r_1,\cdots,r_{17}\};$$

$$
\begin{aligned}
R_2=\{&r_1+r_5,r_1+r_9,r_1+r_{13},r_2+r_9,r_2+r_{10},r_2+r_{13},r_2+r_{14},r_3+r_{13},r_3+r_{14},\\
&r_3+r_{15},r_5+r_{17},r_6+r_{10},r_6+r_{14},r_7+r_{14},r_7+r_{15},r_7+r_{17},r_8+r_{17},\\
&r_9+r_{17},r_{11}+r_{15},r_{11}+r_{17},r_{12}+r_{17}\};
\end{aligned}
$$

$$R_3=\{r_1+r_5+r_{17},r_1+r_9+r_{17},r_2+r_9+r_{17},r_2+r_{10}+r_{13},r_3+r_6+r_{14},r_5+r_{11}+r_{17},$$
$$r_5+r_{12}+r_{17},r_7+r_{14}+r_{17},r_7+r_{15}+r_{17},r_8+r_{11}+r_{17},r_{11}+r_{15}+r_{17}\};$$
$$R_4=\{r_1+r_5+r_9+r_{17},r_1+r_5+r_{11}+r_{17},r_1+r_5+r_{12}+r_{17},r_5+r_{11}+r_{15}+r_{17},$$
$$r_7+r_{11}+r_{15}+r_{17},r_8+r_{11}+r_{15}+r_{17}\};$$
$$R_5=\{r_1+r_5+r_{11}+r_{15}+r_{17}\};$$
$$R_6=\phi.$$

By Theorem 2.8, the unit form $f(x_1,\cdots,x_{16},y)$ is weakly positive, *i.e.*, $f(x_1,\cdots,x_{16},y)\geq 0$ for any $(x_1,\cdots,x_{16},y)\in\mathbb{N}^{17}$, and $f(x_1,\cdots,x_{16},y)=0\Leftrightarrow x_1=\cdots=x_{16}=y=0$. So, we have $q(M)\leq 0$ if and only if

$$a_1+a_3\leq a_2,a_5+a_7\leq a_6,a_7+a_9\leq a_8,a_2+a_6\leq a_3+a_5\leq a_4. \tag{3}$$

$q(M)=0$ if and only if (3) hold and $x_1=\cdots=x_{16}=y=0$, so word 212321242(13-2, 57-6, 79-8, 26-35-4) is tight by Theorem 2.3. So (10) holds.

## Funding

Supported by the NSF of China (No. 11471333).

## References

[1] Bedard, R. (2004) On Tight Monomials in Quantized Enveloping Algebras. *Representation Theory*, **8**, 290-327. https://doi.org/10.1090/S1088-4165-04-00199-2

[2] Blouin, D., Dean, A., Denver, K. and Pershall, K. (1995) Algorithms for Quadratic Forms.

[3] Deng, B.M. and Du, J. (2010) Tight Monomials and Monomial Basis Property. *Journal of Algebra*, **324**, 3355-3377.

[4] Drinfel'd, V.G. (1985) Hopf Algebras and the Quantum Yang-Baxter Equation. *Soviet Mathematics Doklady*, **32**, 254-258.

[5] Hu, Y.W. and Geng, Y.J. (2015) Monomials with *t*-value ≤ 6 in Canonical Basis of Quantized Enveloping Algebra of Type $B_3$. *Chinese Quarterly Journal of Mathematics*, **30**, 462-474.

[6] Hu, Y.W., Li, G.W. and Wang, J. (2015) Tight Monomials in Quantum Group for Type $A_5$ with $t\leq 6$. *Advances in Linear Algebra & Matrix Theory*, **5**, 63-75. https://doi.org/10.4236/alamt.2015.53007

[7] Hu, Y.W., Ye, J.C. and Yue, X.Q. (2003) Canonical Basis of Type $A_4$ (I)-Monomial Elements. *Journal of Algebra*, **263**, 228-245.

[8] Hu, Y.W. and Ye, J.C. (2005) Canonical Bases of Type $A_4$ (II)-Polynomial Elements in One Variable. *Communications in Algebra*, **33**, 3855-3877. https://doi.org/10.1080/00927870500261033

[9] Jimbo, M. (1985) A q-Difference Analogue of $U(g)$ and the Yang-Baxter Equation. *Letters in Mathematical Physics*, **10**, 63-69. https://doi.org/10.1007/BF00704588

[10] Li, X.C. and Hu, Y.W. (2012) Ploynomical Elements in Canonical Basis of Type B with Two-Dimensional Support for $A_4$ (I). *Algebra Colloquium*, **19**, 117-136. https://doi.org/10.1142/S1005386712000089

[11] Lusztig, G. (1990) Canonical Bases Arising from Quantized Enveloping Algebras.

*Journal of the American Mathematical Society*, **3**, 447-498. https://doi.org/10.1090/S0894-0347-1990-1035415-6

[12] Lusztig, G. (1992) Introduction to Quantized Enveloping Algebras. *Progress in Mathematics*, **105**, 49-65. https://doi.org/10.1007/978-1-4612-2978-0_3

[13] Lusztig, G. (1993) Tight Monomials in Quantized Enveloping Algebras. In: Joseph, A. and Schnider, S., Eds., *Quantum Deformations of Algebras and Their Representations*, Israel Math. Conf. Proc., Vol. 7, 117-132.

[14] Marsh, R. (1998) More Tight Monomials in Quantized Enveloping Algebras. *Journal of Algebra*, **204**, 711-732. https://doi.org/10.1006/jabr.1997.7370

[15] Reineke, M. (2001) Monomials in Canonical Bases of Quantum Groups and Quadratic Forms. *Journal of Pure and Applied Algebra*, **157**, 301-309.

[16] Wang, X.M. (2010) Tight Monomials for Type $G_2$ and $A_3$. *Communications in Algebra*, **38**, 3597-3615. https://doi.org/10.1080/00927870903200919

[17] Xi, N.H. (1999) Canonical Basis for Type $A_3$. *Communications in Algebra*, **27**, 5703-5710. https://doi.org/10.1080/00927879908826784

[18] Xi, N.H. (1999) Canonical Basis for Type $B_2$. *Journal of Algebra*, **214**, 8-21. https://doi.org/10.1006/jabr.1998.7688