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Iterative Methods for Solving the Nonlinear Matrix Equation $X - A^* X^p A - B^* X^{-q} B = I \left(0 < p,q < 1\right)$

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Abstract

Consider the nonlinear matrix equation $X - A^* X^p A - B^* X^{-q} B = I \left(0 < p,q < 1\right)$. By using the fixed point theorem for mixed monotone operator in a normal cone, we prove that the equation with $0 < p,q < 1$ always has the unique positive definite solution. Two different iterative methods are given, including the basic fixed point iterative method and the multi-step stationary iterative method. Numerical examples show that the iterative methods are feasible and effective.

Keywords

Nonlinear Matrix Equation, Positive Definite Solution, Iterative Method, Normal Cone

1. Introduction

We consider the matrix equation

$$X - A^* X^p A - B^* X^{-q} B = I, 0 < p,q < 1 \quad (1.1)$$

where I is an $n \times n$ identity matrix, A, B are arbitrary $mn \times n$ matrices. A^*, B^* are used to denote the conjugate transpose of the matrix A and B, separately.

Equation (1.1) often arises in dynamic programming, control theory, stochastic filtering, statistics, and so on, see [1]. Equation (1.1) has been investigated in some special cases [1]-[10]. Reurings [2] showed that the matrix equation $X - A^* X^p A = I \left(0 < p < 1\right)$ and $X - A^* X^{-q} A = I \left(0 < q < 1\right)$ always has a unique positive definite solution. Recently, Gao [3] provides a new proof for the uniqueness of the positive definite solution of the matrix equation

$X - A^* X^q A = I\left(0 < q < 1\right)$ via the fixed point theorem. The method is shown much easier than the way of [1]. Based on this, we study the combination of these two kinds of equations (*i.e.* Equation (1.1)).

In this paper, we prove that the Equation (1.1) always has the unique positive definite solution by using the fixed point theorem for mixed monotone operator in the normal cone $P(n)$. Furthermore, two different iterative methods are given, including the basic fixed point iteration method and multi-step stationary iterative method. In addition, numerical examples show that the iterative methods are feasible and effective.

In this paper, we use $H(n)$ to denote $n \times n$ Hermitian matrices, $\mathcal{P}(n)$ to denote $n \times n$ positive definite matrices, and $\overline{\mathcal{P}}(n)$ to denote $n \times n$ positive semidefinite matrices. For $X, Y \in \mathcal{P}(n)$, we write $X \ge Y$ $(X > Y)$ if $X - Y$ is positive semidefinite(definite).

2. Uniqueness of Positive Definite Solution

In this section, we will prove that the equation $X - A^* X^p A - B^* X^{-q} B = I\left(0 < p, q < 1\right)$ always has the unique positive definite solution. In the process, we need to use the fixed point theorem for mixed monotone operator in a normal cone. So we first introduce the relevant theory through several definitions and lemmas. In this paper, we always use P to denote a solid cone of a real Banach space E. P^0 to denote the interior points set of P. A cone is said to be a solid cone if $P^0 \ne \phi$.

Definition 2.1. [11]. A cone $P \subset E$ is said to be normal if there exists a constant $N > 0$ such that $0 \le x \le y$ implies $\|x\| \le N\|y\|$. That is the norm $\|\cdot\|$ is semimonotone.

Definition 2.2. [11]. The operator $\Gamma : D \times D \to E, D \subset P$ is said to be a mixed monotone operator if $\Gamma(x, y)$ is increasing in x and decreasing in y, that is,

$$\Gamma(x_1, y_1) \ge \Gamma(x_2, y_2), \forall x_1, x_2, y_1, y_2 \in D \text{ with } x_1 \le x_2 \text{ and } y_2 \le y_1.$$

An element x^* is called a fixed point of Γ if $\Gamma(x^*, x^*) = x^*$.

Lemma 2.1. [12]. Let P be a normal and solid cone of a real Banach space E and $\Gamma : P^0 \times P^0 \to P^0$ be a mixed monotone operator. Assume that for all $0 < t < 1$, there exists $0 < \beta < 1$ such that

$$\Gamma\left(tx, \frac{1}{t} y\right) \ge t^\beta \Gamma(x, y)$$

holds for all $x, y \in P^0$. Then Γ has exactly one fixed point x^* in P^0. And for any $x_0, y_0 \in P^0$, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*,$$

where

$$x_n = \Gamma(x_{n-1}, y_{n-1}), y_n = \Gamma(y_{n-1}, x_{n-1}), \quad n = 0,1,2,\cdots$$

Lemma 2.2. [11]. A cone P is normal if and only if $x_n \le z_n \le y_n$, $x_n \to x$ and

$y_n \to x$ imply $z_n \to x$.

Lemma 2.3. [13]. If $A \geq B > 0$ and $0 \leq \gamma \leq 1$, then $A^\gamma \geq B^\gamma$.

We define the spectral norm $\|\cdot\|$ in $H(n)$, then $H(n)$ is a real Banach space. We all know that $\bar{P}(n)$ is a cone in $H(n)$ and the interior points set is $P(n)$. Since the spectral norm is monotone, then we have that the set $\bar{P}(n)$ is normal cone from Definition 2.1. We define F by $F(X,Y) = I + A^* X^p A + B^* Y^{-q} B$, then we have the following theorem.

Theorem 2.1. Eqution (1.1) always has a unique positive definite solution *X*. And for any $X_0, Y_0 \in P^0$, we have

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n = X^*,$$

where

$$X_n = F(X_{n-1}, Y_{n-1}), Y_n = F(Y_{n-1}, X_{n-1}), \quad n = 0,1,2,\cdots$$

Proof. Consider that the solution of Equation (1.1) is a fixed point of *F*. Now we will prove that the operator F satisfies the conditions of Lemma 2.1.

For $\forall X, Y, Z, W \in P(n)$ with $X \geq Y, W \geq Z$, by Lemma 2.3, we have

1) $F(X,Y) = I + A^* X^p A + B^* Y^{-q} B \in P(n)$, that is $F: P(n) \times P(n) \to P(n)$;

2)
$$\begin{aligned} F(X,Z) &= I + A^* X^p A + B^* Z^{-q} B \\ &\geq I + A^* Y^p A + B^* W^{-q} B \\ &= F(Y,W), \end{aligned}$$

which means that the operator F is mixed monotone;

3) Let $\beta = \max\{p, q\}$, then we have $0 < \beta < 1$. For $\forall t \in (0,1)$, we get

$$\begin{aligned} F\left(tX, \frac{1}{t}Y\right) &= I + A^* t^p X^p A + B^* \left(\frac{1}{t}\right)^{-q} Y^{-q} B \\ &= I + A^* t^p X^p A + B^* t^q Y^{-q} B \\ &\geq t^\beta I + t^\beta A^* X^p A + t^\beta B^* Y^{-q} B \\ &= t^\beta F(X,Y). \end{aligned}$$

According to Lemma 2.1, we get that the operator F has a unique fixed point X in $P(n)$, which is the unique positive definite solution of Equation (1.1).

Furthermore, for any $X_0, Y_0 \in P^0$, we have

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n = X^*,$$

where

$$X_n = F(X_{n-1}, Y_{n-1}), Y_n = F(Y_{n-1}, X_{n-1}), \quad n = 0,1,2,\cdots$$

□

3. Iterative Methods

In this section, we propose two different iterative methods for solving Equation (1.1), including the basic fixed point iterative method and multi-step stationary iterative method.

According to Theorem 2.1, sequences $X_n = F(X_{n-1}, Y_{n-1}), Y_n = F(Y_{n-1}, X_{n-1}), n = 0,1,2,\cdots$ converge to the unique positive definite solution of Equation (1.1). Let $X_0 = Y_0$, we have $X_n = Y_n = F(X_{n-1}, X_{n-1}) = I + A^* X_{n-1}^p A + B^* X_{n-1}^{-q} B$. Consider the above iteration, which is the basic fixed point iterative method. The following are the algorithm.

Take $X_0 = I$

$$X_{n+1} = I + A^* X_n^p A + B^* X_n^{-q} B, \quad n = 0,1,2,\cdots \tag{3.1}$$

We then consider a kind of multi-step stationary iterative method.

Theorem 3.1. For arbitrary initial matrix $X_1, X_2 \in P(n)$, the matrix sequence

$$X_{n+1} = I + A^* X_{n-1}^p A + B^* X_n^{-q} B, \quad n = 2,3,\cdots \tag{3.2}$$

converges to the unique positive definite solution X of Equation (1.1).

Proof. Since X is the unique positive definite solution of Equation (1.1). For X, X_1 and X_2, there exist a positive number $0 < \alpha \le 1$ satisfy:

$$\alpha X \le X_1, X_2 \le \alpha^{-1} X. \tag{3.3}$$

For any positive integer k, there exists a unique nonnegative integer K such that $k = 2K+1$ or $k = 2K+2$. We will use mathematical induction to prove the following inequality

$$\alpha^{\beta^K} X \le X_k \le \left(\alpha^{-1}\right)^{\beta^K} X, k = 1,2,\cdots \tag{3.4}$$

where $\beta = \max\{p, q\}$. When $K = 0$, that is $k = 1,2$, the inequality (3.4) holds following (3.3). When $K = n-1$, that is $k = 2n-1, 2n$, Assume (3.4) is true. That is

$$\alpha^{\beta^{n-1}} X \le X_{2n-1}, X_{2n} \le \left(\alpha^{-1}\right)^{\beta^{n-1}} X. \tag{3.5}$$

Now we need to prove (3.4) is true for $K = n$, at this time $k = 2n+1, 2n+2$. From (3.5) and $\beta = \max\{p, q\}$, we get that

$$\alpha^{\beta^n} X^p \le \alpha^{\beta^{n-1} p} X^p \le X_{2n-1}^p \le \left(\alpha^{-1}\right)^{\beta^{n-1} p} X^p \le \left(\alpha^{-1}\right)^{\beta^n} X^p,$$

$$\begin{aligned}\alpha^{\beta^n} X^{-q} \le \alpha^{\beta^{n-1} p} X^{-q} &= \left(\alpha^{-1}\right)^{\beta^{n-1}(-q)} X^{-q} \le X_{2n}^{-q} \\ &\le \alpha^{\beta^{n-1}(-q)} X^{(-q)} = \left(\alpha^{-1}\right)^{\beta^{n-1} q} X^{-q} \le \left(\alpha^{-1}\right)^{\beta^n} X^{-q},\end{aligned}$$

From $0 < \alpha \le 1$, it follows that $0 < \alpha^{\beta^n} \le 1$, $\left(\alpha^{-1}\right)^{\beta^n} \ge 1$. Then

$$\begin{aligned} X_{2n+1} &= I + A^* X_{2n-1}^p A + B^* X_{2n}^{-q} B \\ &\ge \alpha^{\beta^n} I + A^* \alpha^{\beta^n} X^p A + B^* \alpha^{\beta^n} X^{-q} B \\ &= \alpha^{\beta^n} \left[I + A^* X^p A + B^* X^{-q} B \right] \\ &= \alpha^{\beta^n} X \end{aligned}$$

and

$$\begin{aligned} X_{2n+1} &= I + A^* X_{2n-1}^p A + B^* X_{2n}^{-q} B \\ &\le \left(\alpha^{-1}\right)^{\beta^n} I + A^* \left(\alpha^{-1}\right)^{\beta^n} X^p A + B^* \left(\alpha^{-1}\right)^{\beta^n} X^{-q} B \\ &= \left(\alpha^{-1}\right)^{\beta^n} \left[I + A^* X^p A + B^* X^{-q} B \right] \\ &= \left(\alpha^{-1}\right)^{\beta^n} X. \end{aligned}$$

Therefore

$$\alpha^{\beta^n} X \le X_{2n+1} \le \left(\alpha^{-1}\right)^{\beta^n} X. \tag{3.6}$$

From $0 < q \le 1$, (3.6) implies

$$\alpha^{\beta^n} X^{-q} \le \alpha^{\beta^n q} X^{-q} \le X_{2n+1}^{-q} \le \left(\alpha^{-1}\right)^{\beta^n q} X^{-q} \le \left(\alpha^{-1}\right)^{\beta^n} X^{-q}.$$

From (3.4), we have

$$\alpha^{\beta^n} X^p \le \alpha^{\beta^{n-1} p} X^p \le X_{2n}^p \le \left(\alpha^{-1}\right)^{\beta^{n-1} p} X^p \le \left(\alpha^{-1}\right)^{\beta^n} X^p.$$

Then

$$\begin{aligned} X_{2n+2} &= I + A^* X_{2n}^p A + B^* X_{2n+1}^{-q} B \\ &\ge \alpha^{\beta^n} I + A^* \alpha^{\beta^n} X^p A + B^* \alpha^{\beta^n} X^{-q} B \\ &= \alpha^{\beta^n} \left[I + A^* X^p A + B^* X^{-q} B \right] \\ &= \alpha^{\beta^n} X \end{aligned}$$

and

$$\begin{aligned} X_{2n+2} &= I + A^* X_{2n}^p A + B^* X_{2n+1}^{-q} B \\ &\le \left(\alpha^{-1}\right)^{\beta^n} I + A^* \left(\alpha^{-1}\right)^{\beta^n} X^p A + B^* \left(\alpha^{-1}\right)^{\beta^n} X^{-q} B \\ &= \left(\alpha^{-1}\right)^{\beta^n} \left[I + A^* X^p A + B^* X^{-q} B \right] \\ &= \left(\alpha^{-1}\right)^{\beta^n} X. \end{aligned}$$

Therefore

$$\alpha^{\beta^n} X \le X_{2n+2} \le \left(\alpha^{-1}\right)^{\beta^n} X.$$

Now we have proved that (3.4) is true for $K = n$, at this time $k = 2n+1, 2n+2$.

Hence we get that the inequality (3.4) holds for any positive integer *k*. Let $k \to \infty$, then $K \to \infty$, we have

$$\alpha^{\beta^K} \to 1, \left(\alpha^{-1}\right)^{\beta^K} \to 1.$$

Therefore,

$$X_k \to X.$$

4. Numerical Examples

We now use numerical examples to illustrate our results. All computations were performed using MATLAB, version 7.01. we denote $\varepsilon(X)=\left\|X-I-A^{*}X^{p}A-B^{*}X^{-q}B\right\|_{\infty}$, and use the stopping criterion $\varepsilon(X)<1.0\times10^{-10}$.

Example 4.1. Consider Equation (1.1) with $p=\frac{1}{3},q=\frac{1}{2}$ and

$$A=\begin{pmatrix}0.1953 & 0.2310\\ 0.1835 & 0.1796\end{pmatrix},\quad B=\begin{pmatrix}0.9501 & 0.6068\\ 0.2311 & 0.4860\end{pmatrix}.$$

We use the basic fixed point iterative method (3.1) to solve Equation (1.1). Let $X_0=I$, after 14 iterations we get the following result:

$$X\approx X_{14}=\begin{pmatrix}1.7834 & 0.5628\\ 0.5628 & 1.5167\end{pmatrix}$$

and $\varepsilon(X_{14})=\left\|X_{14}-I-A^{*}X_{14}^{p}A-B^{*}X_{14}^{-q}B\right\|_{\infty}=4.0992\times10^{-11}.$

Considering the multi-step stationary iterative method (3.2) with $X_1=X_2=I$, after 19 iterations we get the following result:

$$X\approx X_{19}=\begin{pmatrix}1.7834 & 0.5628\\ 0.5628 & 1.5167\end{pmatrix}$$

and $\varepsilon(X_{19})=\left\|X_{19}-I-A^{*}X_{19}^{p}A-B^{*}X_{19}^{-q}B\right\|_{\infty}=3.1824\times10^{-11}.$

Example 4.2. Consider Eq.(1.1) with $p=\frac{1}{2},q=\frac{1}{3}$ and

$$A=\begin{pmatrix}0.2913 & 0.5185 & 0.6154\\ 0.2621 & 0.4214 & 0.7919\\ 0.0565 & 0.0447 & 0.8218\end{pmatrix},\quad B=\begin{pmatrix}0.9382 & 0.5355 & 0.3936\\ 0.4763 & 0.0969 & 0.9579\\ 0.9057 & 0.0013 & 0.8929\end{pmatrix},$$

Considering the basic fixed point iterative method (3.1) with $X_0=I$, after 13 iterations we get the following result:

$$X\approx X_{13}=\begin{pmatrix}2.3605 & 0.8354 & 1.8154\\ 0.8354 & 2.0042 & 1.6716\\ 1.8154 & 1.6716 & 6.3423\end{pmatrix}$$

and $\varepsilon(X_{13})=\left\|X_{13}-I-A^{*}X_{13}^{p}A-B^{*}X_{13}^{-q}B\right\|_{\infty}=9.7863\times10^{-11}.$

Considering the multi-step stationary iterative method (3.2) with $X_1=I,X_2=2I$, after 35 iterations we get the following result:

$$X\approx X_{35}=\begin{pmatrix}2.3605 & 0.8354 & 1.8154\\ 0.8354 & 2.0042 & 1.6716\\ 1.8154 & 1.6716 & 6.3423\end{pmatrix}$$

and $\varepsilon(X_{35})=\left\|X_{35}-I-A^{*}X_{35}^{p}A-B^{*}X_{35}^{-q}B\right\|_{\infty}=5.8041\times10^{-11}.$

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References

[1] Ferrante, A. and Levy, B.C. (1996) Hermitian Solutions of the Equation $X = Q + NX^{-1}N^*$. *Linear Algebra and its Applications*, **247**, 359-373. https://doi.org/10.1016/0024-3795(95)00121-2

[2] Reurings, M.C.B. (2003) Symmetric Matrix Equations. Universal Press, The Netherlands.

[3] Gao, D.J. and Zhang, Y.H. (2007) On the Hermitian Positive Definite Solutions of the Matrix Equation $X - A^*X^qA = Q(q > 0)$. *Mathematica Numerica Sinica*, **29**, 73-80.

[4] Hasanov, V.I. (2005) Positive Definite Solutions of the Matrix Equations $X \pm A^*X^{-q}A = Q$. *Linear Algebra and Its Applications*, **404**, 166-182. https://doi.org/10.1016/j.laa.2005.02.024

[5] Jia, G.J. and Gao, D.J. (2011) Perturbation Estimates for the Nonlinear Matrix Equation $X - A^*X^qA = Q(0 < q < 1)$. *Journal of Applied Mathematics and Computing*, **35**, 295-304. https://doi.org/10.1007/s12190-009-0357-z

[6] Li, J. and Zhang, Y.H. (2009) Perturbation Analysis of the Matrix Equation $X - A^*X^{-q}A = Q$. *Linear Algebra and Its Applications*, **431**, 1489-1501. https://doi.org/10.1016/j.laa.2009.05.013

[7] Xiao, H. and Wang, J.T. (2009) On the Matrix Equation $X - A^*X^{-p}A = Q(p > 1)$. *Chinese Journal of Engineering Mathematics*, **26**, 305-309.

[8] Duan, X.F. and Liao, A.P. and Tang, B. (2008) On the Nonlinear Matrix Equation $X - \sum_{i=1}^{m} A_i^*X^{\delta_i}A_i = Q$. *Linear Algebra and Its Applications*, **429**, 110-121. https://doi.org/10.1016/j.laa.2008.02.014

[9] Lim, Y. (2009) Solving the Nonlinear Matrix Equation $X = Q + \sum_{i=1}^{m} M_iX^{\delta_i}M_i^*$ via a Contraction Principle. *Linear Algebra and its Applications*, **430**, 1380-1383. https://doi.org/10.1016/j.laa.2008.10.034

[10] Duan, X.F. and Liao, A.P. (2009) On Hermitian Positive Definite Solution of the Matrix Equation $X - \sum_{i=1}^{m} A_i^*X^rA_i = Q$. *Journal of Computational and Applied Mathematics*, **229**, 27-36. https://doi.org/10.1016/j.cam.2008.10.018

[11] Guo, D.J. (2001) Nonlinear Functional Analysis. Shandong Sci. and Tech. Press, Jinan.

[12] Guo, D.J. and Lakshmikantham, V. (1988) Nonlinear Problems in Abstract Cones. ACA-DEMIC Press, London.

[13] Zhan, X. (2002) Matrix Inequalities. Springer-Verlag, Berlin. https://doi.org/10.1007/b83956