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On Classes of Matrices with Variants of the Diagonal Dominance Property

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Abstract

We study the relations between several classes of matrices with variants of the diagonal dominance property, and identify those classes which form pairs of incomparable classes. For an incomparable pair (X_1, X_2) of classes of matrices with variants of the diagonal dominance property, we also study the problem of providing sufficient conditions for the matrices in X_i to be in X_j with $\{i, j\} = \{1, 2\}$. The article is a continuation of a series of articles on the topic and related topics by the author; see [1] [2] [3] [4].

Keywords

Doubly Diagonally Dominant, Generalized Diagonally Dominant, (S_1, S_2) Separation Induced Diagonally Dominant, Row-Column Diagonally Dominant with Index α

1. Introduction and Notation

The theory of matrices with variants of the diagonal dominance property has attracted the attention of researchers in matrix analysis and its applications. Desplanques [5] established the invertibility of every strictly diagonally dominant complex matrix; see Definition 2.1. (Lévy [6] established the result earlier for real matrices). The pioneering work of Lévy and Desplanques motivated researchers to study matrices with variants of the diagonal dominance property. For more results on the subject; see, for example, [1] and [3]-[25]. As usual, we denote the algebra of all $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers by $\mathbb{C}^{n\times n}$. For every $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and every $i \in \{1, \cdots, n\}$, we define the row sum $r_i(A)$ and column sum $c_i(A)$ by

$$r_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{and} \quad c_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ji}|. \tag{1.1}$$

In (1.1), it is understood that $r_1(A)=c_1(A)=0$ if A is a 1×1 matrix.

The objectives of this paper are to investigate the following two problems:

1) Identify among several classes of matrices with variants of the diagonal dominance property those which form pairs of incomparable classes. If X_1 and X_2 are subclasses of $\mathbb{C}^{n\times n}$, we say that (X_1,X_2) is a pair of *incomparable classes* if $X_1\not\subset X_2$ and $X_2\not\subset X_1$.

2) If (X_1,X_2) is a pair of incomparable classes of matrices in $\mathbb{C}^{n\times n}$ with variants of the diagonal dominance property, provide sufficient conditions for matrices in X_i to be in X_j, where $\{i,j\}=\{1,2\}$. We investigate this problem for most pairs of incomparable classes identified in 1).

The set of positive integers is denoted by $\mathbb{N}$, and for every $n\in\mathbb{N}$, we denote the set $\{1,\cdots,n\}$ by $\langle n\rangle$. The empty set is denoted by $\varnothing$. We denote the cardinality of a nonempty finite set S by $\operatorname{card}S$. The set of all $n\times m$ complex matrices is denoted by $\mathbb{C}^{n\times m}$. The set $\mathbb{C}^{n\times 1}$ is simply written as $\mathbb{C}^n$. If $x\in\mathbb{C}^n$ and x_i is the entry of x in the *ith* row, $i=1,\cdots,n$, we write x as $(x_1,\cdots,x_n)^t$. We denote by 0 the zero matrix, and when there is a need to emphasize its size, we will use the symbol $0_{n\times m}$ to denote the $n\times m$ zero matrix in $\mathbb{C}^{n\times m}$. The multiplicative group of $n\times n$ invertible matrices is denoted by $\mathbf{GL}(n)$, and its identity is written as I_n. Let $A=(a_{ij})\in\mathbb{C}^{n\times m}$. The entry a_{ij} of A is sometimes written as $(A)_{ij}$. The transpose of A is denoted by A^t. If $a_{ij}>0$ ($a_{ij}\ge 0$) for all $i\in\langle n\rangle$ and $j\in\langle m\rangle$, we write $A>0$ ($A\ge 0$). The matrix $|A|$ is the matrix in $\mathbb{C}^{n\times m}$ defined by $(|A|)_{ij}=|a_{ij}|$ for all $i\in\langle n\rangle$ and $j\in\langle m\rangle$. If $\Upsilon\in\mathbb{C}^{n\times n}$ is diagonal and γ_i is the *ith* diagonal entry of Υ, $i=1,\cdots,n$, we denote Υ by $\operatorname{diag}(\gamma_1,\cdots,\gamma_n)$. If $B=(b_{ij})\in\mathbb{C}^{n\times n}$, the diagonal matrix $\operatorname{diag}(b_{11},\cdots,b_{nn})$ is denoted by $D(B)$.

The set of eigenvalues of $A\in\mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$, and the spectral radius of A is written as $\rho(A)$. So, $\rho(A)=\max\{|\lambda|:\lambda\in\sigma(A)\}$. Similar matrices in $\mathbb{C}^{n\times n}$ have the same eigenvalues. Among similar matrices, those which are similar through diagonal matrices, are of particular interest. If $A,B\in\mathbb{C}^{n\times n}$, we say that A is diagonally similar to B if there exists a diagonal matrix $X\in\mathbf{GL}(n)$ such that $A=X^{-1}BX$. The set of all matrices, which are diagonally similar to $A\in\mathbb{C}^{n\times n}$, is denoted by $\mathcal{DS}(A)$. If $\mathbb{E}\subset\mathbb{C}^{n\times n}$, we define the diagonal similarity orbit $\mathcal{DS}(\mathbb{E})$ of $\mathbb{E}$ by

$$\mathcal{DS}(\mathbb{E})=\left\{X^{-1}AX : A\in\mathbb{E}, X\in\mathbf{GL}(n) \text{ and } X \text{ is diagonal}\right\}. \tag{1.2}$$

Submatrices play a role in the development of the topics studied in the paper. Let $A\in\mathbb{C}^{n\times n}$, $n\ge 2$, and let S and T be nonempty subsets of $\langle n\rangle$. The submatrix of A that lies in the rows and columns of A indexed by S and T, respectively, is denoted by $A(S,T)$. If $S=T$, we write $A(S,S)$ simply as $A(S)$; see p. 17 of [26]. For every nonempty subset S of $\langle n\rangle$ and each $i\in\langle n\rangle$, it is instructive to evaluate the ℓ_1-norm of the off-diagonal entries among the *ith* row (column), which belong to the columns (rows) of A defined by the set S. Formally, we define $r_i^S(A)$ and $c_i^S(A)$ by

$$r_i^S(A):=\sum_{j\in S,\, j\ne i}|a_{ij}| \quad\text{and}\quad c_i^S(A):=\sum_{j\in S,\, j\ne i}|a_{ji}|. \tag{1.3}$$

It is clear that $r_i^S(A) \le r_i(A)$ and $c_i^S(A) \le c_i(A)$. The sums in (1.3) are used in association with the notion of separation of $\langle n \rangle$.

Definition 1.1 Let $n \in \mathbb{N} \setminus \{1\}$. If *S* is a nonempty proper subset of $\langle n \rangle$, we call the pair $(S, \langle n \rangle \setminus S)$ a *separation* of $\langle n \rangle$.

Remark 1.1 Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $n \ge 2$. Let *S* be a subset of $\langle n \rangle$ with $\operatorname{card} S \ge 2$. If $S = \{s_1, \cdots, s_p\}$, where $s_1 < \cdots < s_p$, then $a_{s_i s_i} = (A(S))_{ii}$, $r_{s_i}^S(A) = r_i(A(S))$ and $c_{s_i}^S(A) = c_i(A(S))$ for all $i \in \langle p \rangle$.

The paper is organized as follows. In Section 2, we list the classes of matrices with variants of the diagonal dominance property, which we consider in the paper. Section 3 outlines some of the preliminary facts about the classes defined in Section 2. The section provides a motivation for the results in the remaining sections of the paper. In Section 4, we study in depth the relation between doubly diagonally dominant matrices and (S_1, S_2) separation-induced doubly diagonally dominant matrices. We analyze in Section 5 the relation between the class of generalized diagonally dominant matrices and the class of (S_1, S_2) separation-induced doubly diagonally dominant matrices. We also show that the former class forms with the class of doubly diagonally dominant matrices a pair of incomparable classes. In Section 6, we study the relations between the row-column diagonally dominant matrices with index α and the other classes we considered in Section 2.

2. Matrices with Variants of the Diagonal Dominance Property

We outline in this section the classes of matrices we consider in the rest of the paper. Irreducible matrices play an important role in the development of the theory. A matrix $A \in \mathbb{C}^{n \times n}$ is called *irreducible* if it not reducible. A matrix $B \in \mathbb{C}^{n \times n}$ is called *reducible* if either $n = 1$ and $B = 0$; or $n \ge 2$ and *B* is similar by way of permutation to a strictly upper triangular block matrix; see Definition 6.2.21 in [26]. We denote the set of all irreducible matrices in $\mathbb{C}^{n \times n}$ by $\mathbf{IR}(n)$.

Definition 2.1 Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Define the sets $\mathcal{J}(A)$ and $\mathcal{L}(A)$ by

$$\mathcal{J}(A) = \{k \in \langle n \rangle : |a_{kk}| > r_k(A)\} \text{ and } \mathcal{L}(A) = \{k \in \langle n \rangle : |a_{kk}| < r_k(A)\}. \quad (2.1)$$

1) The matrix *A* is called *diagonally dominant* if $|a_{ii}| \ge r_i(A)$ for all $i \in \langle n \rangle$. If $\mathcal{J}(A) = \langle n \rangle$, we say that *A* is *strictly diagonally dominant*. We call *A irreducibly diagonally dominant if* $\mathcal{J}(A) \ne \varnothing$ *and A is both diagonally dominant and irreducible. We say that A is generalized diagonally dominant* if there exists a nonsingular diagonal matrix $\Upsilon \in \mathbb{C}^{n \times n}$ such that $A\Upsilon$ is diagonally dominant. We call *A strictly generalized diagonally dominant* (also known as *nonsingular H-matrix*; see [11]) if there exists a nonsingular diagonal matrix $\Upsilon \in \mathbb{C}^{n \times n}$ such that $\mathcal{J}(A\Upsilon) = \langle n \rangle$. If there exists a nonsingular diagonal matrix $\Upsilon \in \mathbb{C}^{n \times n}$ such that $A\Upsilon$ is irreducibly diagonally dominant, we say that *A* is *irreducibly generalized diagonally dominant*.

In the following items, we assume $n \ge 2$.

2) We call A *doubly diagonally dominant* if

$$|a_{ii}||a_{jj}| \geq r_i(A) r_j(A) \text{ for all } i, j \in \langle n \rangle \text{ with } i \neq j. \tag{2.2}$$

We say that A is *strictly doubly diagonally dominant* if the inequalities in (2.2) are all strict. If A is doubly diagonally dominant, irreducible and at least one of inequalities (2.2) is strict, we call A *irreducibly doubly diagonally dominant*.

3) Let (S_1, S_2) be a separation of $\langle n \rangle$. We say that A is (S_1, S_2) *separation- induced doubly diagonally dominant* if

$$\left(|a_{ii}| - r_i^{S_1}(A)\right)\left(|a_{jj}| - r_j^{S_2}(A)\right) \geq r_i^{S_2}(A) r_j^{S_1}(A) \tag{2.3}$$

for all $i \in S_1$ and $j \in S_2$. A is called (S_1, S_2) *separation-induced strictly doubly diagonally dominant* if $\mathcal{J}(A)$ is nonempty and the inequalities of (2.3) are strict for all $i \in S_1$ and $j \in S_2$. We say that A is (S_1, S_2) *separation-induced irreducibly doubly diagonally dominant* if A is irreducible, $\mathcal{J}(A)$ is nonempty, A is doubly diagonally dominant with respect to the separation (S_1, S_2) and there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that

$$\left(|a_{s_1 s_1}| - r_{s_1}^{S_1}(A)\right)\left(|a_{s_2 s_2}| - r_{s_2}^{S_2}(A)\right) > r_{s_1}^{S_2}(A) r_{s_2}^{S_1}(A). \tag{2.4}$$

4) Let $\alpha \in (0,1)$. We call A *row-column diagonally dominant with index* α if

$$|a_{ii}| \geq \left(r_i(A)\right)^{\alpha} \left(c_i(A)\right)^{1-\alpha} \text{ for all } i \in \langle n \rangle. \tag{2.5}$$

If all the inequalities in (2.5) are strict, we say that A is *strictly row-column diagonally dominant with index* α. A is called *irreducibly row-column diagonally dominant with index* α if A is irreducible, row-column diagonally dominant with index α and there exists $k \in \langle n \rangle$ such that

$$|a_{kk}| > \left(r_k(A)\right)^{\alpha} \left(c_k(A)\right)^{1-\alpha}. \tag{2.6}$$

Let $n \in \mathbb{N}$. To simplify the terminology, we introduce the following abbreviated notations:

$$\begin{aligned} \mathbf{D}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is diagonally dominant}\right\}, \\ \mathbf{SD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is strictly diagonally dominant}\right\}, \\ \mathbf{IRD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is irreducibly diagonally dominant}\right\}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathbf{GD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is generalized diagonally dominant}\right\}, \\ \mathbf{SGD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is strictly generalized diagonally dominant}\right\}, \\ \mathbf{IRGD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is irreducibly generalized diagonally dominant}\right\}. \end{aligned} \tag{2.8}$$

In the following terminology, we assume $n \geq 2$:

$$\begin{aligned} \mathbf{DD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is doubly diagonally dominant}\right\}, \\ \mathbf{SDD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is strictly doubly diagonally dominant}\right\}, \\ \mathbf{IRDD}(n) &= \left\{A \in \mathbb{C}^{n\times n} : A \text{ is irreducibly doubly diagonally dominant}\right\}. \end{aligned} \tag{2.9}$$

If (S_1, S_2) is a separation of $\langle n \rangle$, we introduce the notation

$$\mathbf{DD}(S_1,S_2,n)=\{A\in\mathbb{C}^{n\times n}: A \text{ is } (S_1,S_2) \text{ separation-induced doubly diagonally dominant}\},$$
$$\mathbf{SDD}(S_1,S_2,n)=\{A\in\mathbb{C}^{n\times n}: A \text{ is } (S_1,S_2) \text{ separation-induced strictly doubly diagonally dominant}\}, \quad (2.10)$$
$$\mathbf{IRDD}(S_1,S_2,n)=\{A\in\mathbb{C}^{n\times n}: A \text{ is } (S_1,S_2) \text{ separation-induced irreducibly doubly diagonally dominant}\}.$$

If $\alpha\in(0,1)$, then

$$\mathbf{RCD}(\alpha;n)=\{A\in\mathbb{C}^{n\times n}: A \text{ is row-column diagonally dominant with index } \alpha\},$$
$$\mathbf{SRCD}(\alpha;n)=\{A\in\mathbb{C}^{n\times n}: A \text{ is strictly row-column diagonally dominant with index } \alpha\}, \quad (2.11)$$
$$\mathbf{IRRCD}(\alpha;n)=\{A\in\mathbb{C}^{n\times n}: A \text{ is irreducibly row-column diagonally dominant with index } \alpha\}.$$

3. Preliminaries

Some of the important facts linking the classes introduced in Definition 2.1 are reviewed in this section. The information provide motivations for the results established in the subsequent sections.

Remark 3.1 Let $n\in\mathbb{N}$. Then

1) $\mathbf{D}(n)\subset\mathbf{GD}(n)$, $\mathbf{SD}(n)\subset\mathbf{SGD}(n)\subset\mathbf{GD}(n)$ and $\mathbf{IRD}(n)\cup[\mathbf{SGD}(n)\cap\mathbf{IR}(n)]\subset\mathbf{IRGD}(n)\subset\mathbf{GD}(n)$.

In items (2)-(6), we assume $n\ge 2$:

2) $\mathbf{D}(n)\cup\mathbf{SDD}(n)\subset\mathbf{DD}(n)$, $\mathbf{SD}(n)\subset\mathbf{SDD}(n)$ and $\mathbf{IRD}(n)\subset\mathbf{IRDD}(n)$.

3) $\mathbf{SDD}(n)\cap\mathbf{IR}(n)\subset\mathbf{IRDD}(n)\subset\mathbf{DD}(n)\cap\mathbf{IR}(n)$.

4) If (S_1,S_2) is a separation of $\langle n\rangle$, then

i) $\mathbf{DD}(S_1,S_2,n)=\mathbf{DD}(S_2,S_1,n)$, and similar equalities hold for $\mathbf{SDD}(S_1,S_2,n)$ and $\mathbf{IRDD}(S_1,S_2,n)$.

ii) $\mathbf{SDD}(S_1,S_2,n)\subset\mathbf{DD}(S_1,S_2,n)$.

iii) $\mathbf{SDD}(S_1,S_2,n)\cap\mathbf{IR}(n)\subset\mathbf{IRDD}(S_1,S_2,n)\subset\mathbf{DD}(S_1,S_2,n)\cap\mathbf{IR}(n)$.

iv) If $A=(a_{ij})\in\mathbb{C}^{n\times n}$, then $A\in\mathbf{IRDD}(S_1,S_2,n)$ if and only if A is irreducible, $A\in\mathbf{DD}(S_1,S_2,n)$ and there exist $s_1\in S_1$ and $s_2\in S_2$ such that (2.4) holds and

$$|a_{s_1s_1}|>r_{s_1}^{S_1}(A). \quad (3.1)$$

v) $\mathbf{D}(n)\subset\mathbf{DD}(S_1,S_2,n)$ and $\mathbf{SD}(n)\subset\mathbf{SDD}(S_1,S_2,n)$.

5) $\mathbf{DD}(\{1\},\{2\},2)=\mathbf{DD}(2)$, $\mathbf{SDD}(\{1\},\{2\},2)=\mathbf{SDD}(2)$ and $\mathbf{IRDD}(\{1\},\{2\},2)=\mathbf{IRDD}(2)$.

6) If $\alpha\in(0,1)$, then

i) $\mathbf{SRCD}(\alpha;n)\subset\mathbf{RCD}(\alpha;n)$.

ii) $\mathbf{SRCD}(\alpha;n)\cap\mathbf{IR}(n)\subset\mathbf{IRRCD}(\alpha;n)\subset\mathbf{RCD}(\alpha;n)\cap\mathbf{IR}(n)$.

iii) $\mathbf{RCD}(\alpha;2)\subset\mathbf{DD}(2)$.

iv) $\mathbf{SRCD}(\alpha;2)\subset\mathbf{SDD}(2)$.

v) $\mathbf{IRRCD}(\alpha;2)\subset\mathbf{SDD}(2)\cap\mathbf{IR}(2)=\mathbf{IRDD}(2)$.

7) Let $n\ge 3$. The classes $\mathbf{DD}(S_1,S_2,n)$, $\mathbf{SDD}(S_1,S_2,n)$ and $\mathbf{IRDD}(S_1,S_2,n)$ depend on the separation (S_1,S_2) of $\langle n\rangle$. For example, the irreducible matrix A defined by

$$A=\begin{pmatrix}1 & 0.1 & 0.2\\ 0.1 & 1 & 0.1\\ 1 & 2.8 & 3\end{pmatrix}$$

satisfies $A\in\mathbf{SDD}(\{3\},\{1,2\},3)$ but $A\notin\mathbf{DD}(\{1\},\{2,3\},3)$. Then from (ii) of item (4), we deduce that $A\in\mathbf{DD}(\{3\},\{1,2\},3)$ and $A\notin\mathbf{SDD}(\{1\},\{2,3\},3)$, and from A being irreducible and (iii) of item (4), we see that
$A\in\mathbf{IRDD}(\{3\},\{1,2\},3)$ and $A\notin\mathbf{IRDD}(\{1\},\{2,3\},3)$.

The following fact is less obvious than the inclusions in (v) of item (4) of Remark 3.1.

Lemma 3.1 *Let* $n\in\mathbb{N}\setminus\{1\}$, *and let* (S_1,S_2) *be a separation of* $\langle n\rangle$. *Then* $\mathbf{IRD}(n)\subset\mathbf{IRDD}(S_1,S_2,n)$.

Proof. Let $A=(a_{ij})\in\mathbf{IRD}(n)$. It follows from $A\in\mathbf{D}(n)$ and (v) of item (4) of Remark 3.1 that $A\in\mathbf{DD}(S_1,S_2,n)$. Then from $\mathcal{J}(A)\neq\varnothing$ and $A\in\mathbf{IR}(n)$, we deduce that in order to show that $A\in\mathbf{IRDD}(S_1,S_2,n)$, it remains to show the existence of $s_1\in S_1$ and $s_2\in S_2$ such that (2.4) is satisfied. From $\mathcal{J}(A)\neq\varnothing$ and (S_1,S_2) being a separation of $\langle n\rangle$, there exists $k\in S_1\cup S_2$ such that $|a_{kk}|>r_k(A)$. Assume that $k\in S_1$. The other case is proven similarly. From $A\in\mathbf{IR}(n)$ and (S_1,S_2) being a separation of $\langle n\rangle$, there exists $l\in S_2$ such that $r_l^{S_1}(A)>0$. Thus from $A\in\mathbf{D}(n)$,
$k\in S_1\cap\mathcal{J}(A)$, we get

$$\left(|a_{kk}|-r_k^{S_1}(A)\right)\left(|a_{ll}|-r_l^{S_2}(A)\right)>r_k^{S_2}(A)\,r_l^{S_1}(A).$$

Hence, with taking $s_1=k$ and $s_2=l$, we obtain (2.4).

Using (1.2), the following lemma provides characterizations of the classes $\mathbf{GD}(n)$, $\mathbf{SGD}(n)$ and $\mathbf{IRGD}(n)$. The lemma somehow justifies the use of the word "*generalized*" in the titles for the 3 classes. We omit the proof.

Lemma 3.2 *Let* $n\in\mathbb{N}$. *Then* $\mathbf{GD}(n)=\mathcal{DS}(\mathbf{D}(n))$, $\mathbf{SGD}(n)=\mathcal{DS}(\mathbf{SD}(n))$ *and* $\mathbf{IRGD}(n)=\mathcal{DS}(\mathbf{IRD}(n))$.

Additional facts about the classes in Definition 2.1 are outlined in the following lemma.

Lemma 3.3 *Let* $n\in\mathbb{N}\setminus\{1\}$. *Then*:

1) $\mathbf{SD}(n)\cup\mathbf{IRD}(n)\subset\mathbf{GL}(n)$ and $\mathbf{SGD}(n)\cup\mathbf{IRGD}(n)\subset\mathbf{GL}(n)$, and the two inclusions are proper.

2) $\mathbf{SDD}(n)\subset\mathbf{GL}(n)$.

3) $\mathbf{IRDD}(2)\subset\mathbf{GL}(2)$ and $\mathbf{IRDD}(n)\not\subset\mathbf{GL}(n)$ for all $n\geq 3$.

4) $\mathbf{IRD}(n)=\mathbf{IRDD}(n)\cap\mathbf{D}(n)$.

5) $\mathbf{DD}(n)\subset\bigcup_{i=1}^{n}\mathbf{DD}(\{i\},\langle n\rangle\setminus\{i\},n)$.

6) $\mathbf{SDD}(n)\subset\bigcup_{i=1}^{n}\mathbf{SDD}(\{i\},\langle n\rangle\setminus\{i\},n)$.

7) If (S_1,S_2) is a separation of $\langle n\rangle$, then

$$\mathbf{SDD}(S_1,S_2,n)\subset\mathbf{SGD}(n)\tag{3.2}$$

and

$$\mathbf{IRDD}(S_1,S_2,n)\subset\mathbf{IRGD}(n)\subset\mathbf{SGD}(n).\tag{3.3}$$

8) $\mathbf{SDD}(n)\subset\mathbf{SGD}(n)$.

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9) If $\alpha \in (0,1)$, then

$$\mathbf{SRCD}(\alpha;n) \cup \mathbf{IRRCD}(\alpha;n) \subset \mathbf{GL}(n). \tag{3.4}$$

Remark 3.2 We make the following observations in regard to Lemma 3.3.

1) In item (1), the inclusion $\mathbf{IRD}(n) \subset \mathbf{GL}(n)$ was established by Taussky ([23], Theorem II).

2) The inclusion of item (2) is proper; it was first proved by Ostrowski [19].

3) If $n \ge 3$, the fact $\mathbf{IRDD}(n) \not\subset \mathbf{GL}(n)$ in item 3) was illustrated through examples in [21] and [25].

4) Items (5) and (6) follow through a careful reading of the proof of Proposition 1 of [18]. If $A \in \mathbf{D}(n)$ then $A \in \bigcap_{i=1}^{n} \mathbf{DD}(\{i\}, \langle n\rangle \setminus \{i\}, n)$. If $A \in \mathbf{DD}(n) \setminus \mathbf{D}(n)$, then $A \in \mathbf{DD}(\{l\}, \langle n\rangle \setminus \{l\}, n)$, where l is the unique integer in $\langle n\rangle$ satisfying $\mathcal{L}(A) = \{l\}$ (see (2.1)). If $A \in \mathbf{SD}(n)$ then $A \in \bigcap_{i=1}^{n} \mathbf{SDD}(\{i\}, \langle n\rangle \setminus \{i\}, n)$. If $A \in \mathbf{SDD}(n) \setminus \mathbf{SD}(n)$, then $A \in \mathbf{SDD}(\{m\}, \langle n\rangle \setminus \{m\}, n)$, where m is the unique integer in $\langle n\rangle$ satisfying $\mathcal{J}(A) = \langle n\rangle \setminus \{m\}$.

5) In contrast to items (5) and (6), we observe that

$$\mathbf{IRDD}(n) \not\subset \bigcup_{i=1}^{n} \mathbf{IRDD}(\{i\}, \langle n\rangle \setminus \{i\}, n).$$

For example, let $A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}$. Then $A \in \mathbf{IRDD}(3)$, but $A \notin \bigcup_{i=1}^{3} \mathbf{IRDD}(\{i\}, \langle 3\rangle \setminus \{i\}, 3)$. Note that for the separation $(\{2\},\{1,3\})$ of $\langle 3\rangle$, we have:

$$\left(|a_{22}| - r_2^{\{2\}}(A)\right)\left(|a_{11}| - r_1^{\{1,3\}}(A)\right) = (4-0)(1-1) = 0,$$

$$r_2^{\{1,3\}}(A) r_1^{\{2\}}(A) = \left(|a_{21}| + |a_{23}|\right)|a_{12}| = (2+0)(1) = 2.$$

So,

$$\left(|a_{22}| - r_2^{\{2\}}(A)\right)\left(|a_{11}| - r_1^{\{1,3\}}(A)\right) = (4-0)(1-1) = 0 < 2 = r_2^{\{1,3\}}(A) r_1^{\{2\}}(A).$$

Also, for the separation $(\{3\},\{1,2\})$ of $\langle 3\rangle$, we have $|a_{33}|\left(|a_{11}| - r_1^{\{1,2\}}(A)\right) = 0 < 2 = r_3(A)|a_{13}|$. For the separation $(\{1\},\{2,3\})$ of $\langle 3\rangle$, the strict inequality (2.4) is not satisfied, since

$$\left(|a_{11}| - r_1^{\{1\}}(A)\right)\left(|a_{jj}| - r_j^{\{2,3\}}(A)\right) = r_1^{\{2,3\}}(A) r_j^{\{1\}}(A) = 4$$

for $j = 2,3$.

6) Gao and Wang ([12], Theorem 1) established (3.2). For every integer $n \ge 5$, the inclusion is proper. We consider the following two cases:

Case 1: $n = 5$. Define the matrix A_5 by

$$A_5 = (a_{ij}) = \begin{pmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 \\ 0 & 2 & 2 & 6 & 2 \\ 0 & 0 & 4 & 4 & 8 \end{pmatrix}. \tag{3.5}$$

Then, with

$$\Upsilon_5 = \operatorname{diag}(1, 1.1, 1.15, 1.2, 1.25), \tag{3.6}$$

we have $A_5\Upsilon_5 \in \mathbf{SD}(5)$. Thus $A_5 \in \mathbf{SGD}(5)$. However, it can be shown that for every separation (S_1, S_2) of $\langle 5\rangle$, there exists a pair $(i, j) \in S_1 \times S_2$ such that

$$\left(|a_{ii}| - r_i^{S_1}(A_5)\right)\left(|a_{jj}| - r_j^{S_2}(A_5)\right) = r_i^{S_2}(A_5)\, r_j^{S_1}(A_5).$$

Hence the matrix A_5 defined by (3.5) satisfies $A_5 \notin \mathbf{SDD}(S_1, S_2, 5)$ for any separation (S_1, S_2) of $\langle 5\rangle$.

Case 2: $n > 5$. Define A_n by

$$A_n = (a_{ij}) = A_5 \oplus I_{n-5}, \tag{3.7}$$

where A_5 is the matrix defined by (3.5). It then follows from (3.5), (3.6) and case 1 that the diagonal matrix $\Upsilon_n = \Upsilon_5 \oplus I_{n-5}$ satisfies $A_n\Upsilon_n \in \mathbf{SD}(n)$. Thus $A_n \in \mathbf{SGD}(n)$. Let (S_1, S_2) be a separation of $\langle n\rangle$. From (3.5) and (3.7), it is clear that

$$\langle 5\rangle \subset S_1 \Rightarrow |a_{55}| - r_5^{S_1}(A_n) = r_5^{S_2}(A_n) = 0$$

and

$$\langle 5\rangle \subset S_2 \Rightarrow |a_{55}| - r_5^{S_2}(A_n) = r_5^{S_1}(A_n) = 0.$$

So, to complete the proof that $A_n \notin \mathbf{SDD}(S_1, S_2, n)$, it remains to consider the case:

$$S_1 \cap \langle 5\rangle \neq \varnothing \text{ and } S_2 \cap \langle 5\rangle \neq \varnothing.$$

It then follows that $(S_1 \cap \langle 5\rangle, S_2 \cap \langle 5\rangle)$ is a separation of $\langle 5\rangle$. Hence from (3.7), we deduce that

$$r_m^{S_1}(A_n) = r_m^{S_1\cap\langle 5\rangle}(A_n) = r_m^{S_1\cap\langle 5\rangle}(A_5) \text{ and } r_m^{S_2}(A_n) = r_m^{S_2\cap\langle 5\rangle}(A_n) = r_m^{S_2\cap\langle 5\rangle}(A_5)$$

for all $m \in \langle 5\rangle$. Then from case 1 and the fact that $(S_1 \cap \langle 5\rangle, S_2 \cap \langle 5\rangle)$ is a separation of $\langle 5\rangle$, we see that there exist $i \in S_1 \cap \langle 5\rangle$ and $j \in S_2 \cap \langle 5\rangle$ such that

$$\left(|a_{ii}| - r_i^{S_1}(A_n)\right)\left(|a_{jj}| - r_j^{S_2}(A_n)\right) = r_i^{S_2}(A_n)\, r_j^{S_1}(A_n).$$

The integer 5 is the smallest integer we were able to find with which the inclusion of (3.2) is proper.

7) Item (8) follows from item (6) and (3.2).

8) Let $n \in \mathbb{N} \setminus \{1\}$. In contrast to the inclusion in item (8), we observe that

$$\mathbf{DD}(n) \not\subset \mathbf{GD}(n). \tag{3.8}$$

For example, let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be defined by $a_{12} = 1$ and $a_{ij} = 0$ for all $(i, j) \in (\langle n\rangle \times \langle n\rangle) \setminus \{(1,2)\}$. Then $A \in \mathbf{DD}(n) \setminus \mathbf{GD}(n)$. It can be shown that $\mathbf{GD}(2) \subset \mathbf{DD}(2)$. We will use Theorems 4.1 and 5.2 to show that $\mathbf{GD}(n) \not\subset \mathbf{DD}(n)$ for all $n \geq 3$; see Remark 5.2.

9) Theorem 2 of [12] establishes (3.3) through the two set inclusions. The second inclusion readily follows. In the proof of $\mathbf{IRDD}(S_1, S_2, n) \subset \mathbf{IRGD}(n)$, it is assumed that $A \in \mathbf{IRDD}(S_1, S_2, n)$ and the separation (S_1, S_2) of $\langle n\rangle$

satisfy the additional condition:

$$S_1 \subset \mathcal{J}(A) \text{ or } S_2 \subset \mathcal{J}(A). \tag{3.9}$$

In general, matrices in $\mathbf{IRDD}(S_1,S_2,n)$ need not to satisfy (3.9); for example, $A=\begin{pmatrix} 2 & 1.1 & 1 \\ 0 & 2 & 2 \\ 2.1 & 1 & 4 \end{pmatrix}$ is in $\mathbf{IRDD}(\{1\},\{2,3\},3)$, but $\{1\} \not\subset \mathcal{J}(A)$ and $\{2,3\} \not\subset \mathcal{J}(A)$. It is possible to establish (3.3) without making the assumption (3.9) by slightly modifying the proof of Theorem 2 of [12]. However, we will use Theorem 5.2 to prove the first inclusion of (3.3); see Corollary 5.3.

10) In (3.4), Ostrowski [20] established the inclusion $\mathbf{SRCD}(\alpha;n) \subset \mathbf{GL}(n)$, and Hadjidimos ([15], Theorem 2.1) proved the inclusion $\mathbf{IRRCD}(\alpha;n) \subset \mathbf{GL}(n)$. Item (4) of Theorem 6.6 provides a simple different proof of Hadjidimos's result.

Remark 3.3 1) In light of the facts given in items (5) and (6) of Lemma 3.3, we will analyze in more depth in Section 4 the relation between $\{\mathbf{DD}(S_1,S_2,n), \mathbf{SDD}(S_1,S_2,n), \mathbf{IRDD}(S_1,S_2,n)\}$ and $\{\mathbf{DD}(n), \mathbf{SDD}(n), \mathbf{IRDD}(n)\}$.

2) We will show in Theorem 5.1 that the relation between $\mathbf{DD}(S_1,S_2,n)$ and $\mathbf{GD}(n)$ is in contrast to the relation between $\mathbf{SDD}(S_1,S_2,n)$ and $\mathbf{SGD}(n)$ (given by (3.2)).

To simplify the set up of some statements in Sections 4 and 6, we introduce Definition 3.1.

Definition 3.1 Let $n \in \mathbb{N} \setminus \{1\}$, and let $\mathcal{E}$ be a nonempty subclass of $\mathbb{C}^{n\times n}$. We say that $\mathcal{E}$ is *invariant under the permutation similarity transformation* if for every permutation matrix $E \in \mathbb{C}^{n\times n}$ and every $A \in \mathcal{E}$, we have $EAE^t \in \mathcal{E}$.

Remark 3.4 Let $n \in \mathbb{N} \setminus \{1\}$, $\alpha \in (0,1)$ and let (S_1,S_2) be a separation of $\langle n \rangle$ with $\operatorname{card} S_1 = p$.

1) The classes $\mathbf{D}(n)$, $\mathbf{DD}(n)$, $\mathbf{IR}(n)$, $\mathbf{IRDD}(n)$, $\mathbf{RCD}(\alpha;n)$, $\mathbf{SRCD}(\alpha;n)$ and $\mathbf{IRRCD}(\alpha;n)$ are all invariant under the permutation similarity transformation.

2) There exists a permutation matrix $E \in \mathbb{C}^{n\times n}$ such that the linear transformation $T:\mathbf{DD}(S_1,S_2,n) \to \mathbf{DD}(\langle p \rangle, \langle n \rangle \setminus \langle p \rangle, n)$ defined by $T(A) = EAE^t$ is an isomorphism from $\mathbf{DD}(S_1,S_2,n)$ onto $\mathbf{DD}(\langle p \rangle, \langle n \rangle \setminus \langle p \rangle, n)$.

Similar observations could be stated for the pairs:

$$(\mathbf{SDD}(S_1,S_2,n), \mathbf{SDD}(\langle p \rangle, \langle n \rangle \setminus \langle p \rangle, n)) \text{ and } (\mathbf{IRDD}(S_1,S_2,n), \mathbf{IRDD}(\langle p \rangle, \langle n \rangle \setminus \langle p \rangle, n)).$$

4. Matrices with the Doubly Diagonal Dominance Property vs. Matrices with the (S_1, S_2) Separation-Induced Doubly Diagonal Dominance Property

We denote the Cartesian product of two nonempty sets X and Y by $X \times Y$.

Theorem 4.1 *Let $n \in \mathbb{N} \setminus \{1,2\}$, and let (S_1,S_2) be a separation of $\langle n \rangle$. Then the elements of the set*

$$\{\mathbf{DD}(n),\mathbf{SDD}(n),\mathbf{IRDD}(n)\}\times\{\mathbf{DD}(S_1,S_2,n),\mathbf{SDD}(S_1,S_2,n),\mathbf{IRDD}(S_1,S_2,n)\}$$

are pairs of incomparable classes.

Proof. It follows from Remark 3.4 that it suffices to consider the case: $S_1=\langle p\rangle$, where $p\in\langle n-1\rangle$. Also, from Remark 3.1 (item (3), and (iii) of item (4)), we see that it suffices to show the existence of $A,B\in\mathbb{C}^{n\times n}$ such that

$$A\in\left[\mathbf{SDD}(n)\cap\mathbf{IR}(n)\right]\setminus\mathbf{DD}(\langle p\rangle,\{p+1,\cdots,n\},n) \tag{4.1}$$

and

$$B\in\left[\mathbf{SDD}(\langle p\rangle,\{p+1,\cdots,n\},n)\cap\mathbf{IR}(n)\right]\setminus\mathbf{DD}(n). \tag{4.2}$$

We consider the following two cases:

Case 1: $p\in\{1,n-1\}$. Assume without loss of generality that $p=1$. Define $A=(a_{ij})$ as follows:

$$a_{11}=2.1, a_{1j}=(n-1)^{-1} \text{ for all } j\in\langle n\rangle\setminus\{1\},$$

$$a_{21}=a_{22}=1 \text{ and } a_{2j}=(n-2)^{-1} \text{ for all } 3\le j\le n,$$

$$a_{ii}=3 \text{ for all } 3\le i\le n,$$

$$a_{ij}=(n-1)^{-1} \text{ for all } (i,j)\in\{(l,m):l\in\{3,\cdots,n\},m\in\langle n\rangle\setminus\{l\}\}. \tag{4.3}$$

Define $B=(b_{ij})$ by

$$b_{11}=1 \text{ and } b_{1j}=2(n-1)^{-1} \text{ for all } j\in\{2,\cdots,n\},$$

$$b_{ii}=2 \text{ and } b_{i1}=0.4 \text{ for all } i\in\{2,\cdots,n\},$$

$$b_{ij}=(n-2)^{-1} \text{ for all } (i,j)\in\{(l,m):l,m\in\{2,\cdots,n\},m\ne l\}. \tag{4.4}$$

Then A and B defined by (4.3) and (4.4) satisfy (4.1) and (4.2), respectively, in this case.

Case 2: $p\in\{2,\cdots,n-2\}$. Define $A=(a_{ij})$ as follows:

$$a_{11}=a_{12}=1 \text{ and } a_{1j}=2(n-2)^{-1} \text{ for all } j\in\{3,\cdots,n\},$$

$$a_{ii}=9.1 \text{ for all } i\in\{2,\cdots,n\},$$

$$a_{ij}=3(n-1)^{-1} \text{ for all } (i,j)\in\{(l,m):l\in\{2,\cdots,n\},m\in\langle n\rangle\setminus\{l\}\}. \tag{4.5}$$

Define $B=(b_{ij})$ by

$$b_{ii}=2 \text{ for all } i\in\langle n\rangle,$$

$$b_{ij}=1/(p-1) \text{ if } (i,j)\in\{(l,m):l,m\in\langle p\rangle,m\ne l\},$$

$$b_{ij}=2/(n-p) \text{ if } i\in\langle p\rangle, j\in\{p+1,\cdots,n\},$$

$$b_{ij}=0.4/p \text{ if } i\in\{p+1,\cdots,n\}, j\in\langle p\rangle$$

and

$$b_{ij}=(n-p-1)^{-1} \text{ if } (i,j)\in\{(l,m):l,m\in\{p+1,\cdots,n\},m\ne l\}. \tag{4.6}$$

Then A and B defined by (4.5) and (4.6) satisfy (4.1) and (4.2), respectively, in this case.

The following corollary is a direct consequence of items (5) and (6) of Lemma 3.3, and Theorem 4.1. The exclusion of $n=2$ in the corollary is by virtue of item (5) of Remark 3.1.

Corollary 4.1 *Let* $n \in \mathbb{N}\setminus\{1,2\}$. *Then the inclusions*
$\mathbf{DD}(n) \subset \bigcup_{i=1}^{n} \mathbf{DD}(\{i\},\langle n\rangle\setminus\{i\},n)$ and $\mathbf{SDD}(n) \subset \bigcup_{i=1}^{n} \mathbf{SDD}(\{i\},\langle n\rangle\setminus\{i\},n)$
are proper.

Remark 4.1 1) It follows from (v) of item (4) of Remark 3.1 that in order to establish sufficient conditions for matrices in $\mathbf{DD}(n)$ ($\mathbf{SDD}(n)$) to be in $\mathbf{DD}(S_1,S_2,n)$ ($\mathbf{SDD}(S_1,S_2,n)$), it suffices to provide such conditions for matrices in the smaller classes $\mathbf{DD}(n)\setminus\mathbf{D}(n)$ ($\mathbf{SDD}(n)\setminus\mathbf{SD}(n)$). Also, from Lemma 3.1 and item (4) of Lemma 3.3, we see that in order to present sufficient conditions for matrices in $\mathbf{IRDD}(n)$ to be in $\mathbf{IRDD}(S_1,S_2,n)$, it suffices to provide such conditions for matrices in the smaller class $\mathbf{IRDD}(n)\setminus\mathbf{D}(n)$. This provides the basis for the set-ups of Theorems 4.2-4.4.

2) Let $n \in \mathbb{N}\setminus\{1\}$, and let $A \in \mathbf{DD}(n)\setminus\mathbf{D}(n)$, $B \in \mathbf{SDD}(n)\setminus\mathbf{SD}(n)$ and $C \in \mathbf{IRDD}(n)\setminus\mathbf{D}(n)$. Then there exist $l,m,s \in \langle n\rangle$ such that $\mathcal{L}(A)=\{l\}$ (see (2.1)), $\mathcal{J}(B)=\langle n\rangle\setminus\{m\}$ and $\mathcal{J}(C)=\langle n\rangle\setminus\{s\}=\langle n\rangle\setminus\mathcal{L}(C)$. Suppose that (S_1,S_2) is a separation of $\langle n\rangle$. Then

$$\mathcal{L}(A)\subset S_1 \text{ or } \mathcal{L}(A)\subset S_2 \text{ (but not both)}, \tag{4.7}$$

$$S_1\subset\mathcal{J}(B) \text{ or } S_2\subset\mathcal{J}(B) \text{ (but not both)}. \tag{4.8}$$

and

$$S_1\subset\mathcal{J}(C) \text{ or } S_2\subset\mathcal{J}(C) \text{ (but not both)}. \tag{4.9}$$

If $\mathcal{L}(A)\not\subset S_1$ and $\mathcal{L}(A)\not\subset S_2$, then from $\mathcal{L}(A)=\{l\}$ we obtain $l\notin S_1\cup S_2$, but this contradicts that $S_1\bigcup S_2=\langle n\rangle$. The "not both" phrase in (4.7) follows from $\mathcal{L}(A)=\{l\}$ and $S_1\bigcap S_2=\varnothing$. If $S_1\not\subset\mathcal{J}(B)$ and $S_2\not\subset\mathcal{J}(B)$, then from $\mathcal{J}(B)=\langle n\rangle\setminus\{m\}$ we get $m\in S_1\bigcap S_2$, but this contradicts that S_1 and S_2 are disjoint. The "not both" phrase in (4.8) follows from $J(B)=\langle n\rangle\setminus\{m\}$ and $S_1\bigcup S_2=\langle n\rangle$. The facts in (4.9) are proved similarly to the ones in (4.8). We will use (4.7), (4.8) and (4.9) in Theorems 4.2, 4.3 and 4.4, respectively.

Theorem 4.2 *Let* $A=(a_{ij})\in\mathbf{DD}(n)\setminus\mathbf{D}(n)$, $n\ge 2$, *and let* $l\in\langle n\rangle$ *be the integer such that* $\mathcal{L}(A)=\{l\}$. *Suppose that* (S_1,S_2) *is a separation of* $\langle n\rangle$ *such that* $\mathcal{L}(A)\subset S_1$. *In addition, assume that A satisfies one of the following two conditions*:

Condition (1): $r_l^{S_1}(A)=0$.

Condition (2): $|a_{jj}|=r_j(A)$ for all $j\in S_2$.

Then $A\in\mathbf{DD}(S_1,S_2,n)$.

Proof. Let $j\in S_2$. Then from $\mathcal{L}(A)=\{l\}\subset S_1$, we get

$$i\in S_1\setminus\{l\}\Rightarrow\left(|a_{ii}|-r_i^{S_1}(A)\right)\left(|a_{jj}|-r_j^{S_2}(A)\right)\ge r_i^{S_2}(A)\,r_j^{S_1}(A) \tag{4.10}$$

and

$$\begin{aligned}&\left(|a_{ll}|-r_l^{S_1}(A)\right)\left(|a_{jj}|-r_j^{S_2}(A)\right)\\&\ge r_l(A)r_j(A)-|a_{ll}|r_j^{S_2}(A)-\left[|a_{jj}|-r_j^{S_2}(A)\right]r_l^{S_1}(A)\\&\ge r_l(A)r_j(A)-r_l(A)r_j^{S_2}(A)-\left[|a_{jj}|-r_j^{S_2}(A)\right]r_l^{S_1}(A)\\&=r_l(A)r_j^{S_1}(A)-\left[|a_{jj}|-r_j^{S_2}(A)\right]r_l^{S_1}(A)\\&=r_l^{S_2}(A)r_j^{S_1}(A)+\left[r_j(A)-|a_{jj}|\right]r_l^{S_1}(A),\end{aligned}\qquad(4.11)$$

where in (4.11), the first inequality follows from $A\in\mathbf{DD}(n)$ and the second inequality follows from $|a_{ll}|<r_l(A)$. From (4.11), we see that if A satisfies either condition (1) or condition (2) then $\left(|a_{ll}|-r_l^{S_1}(A)\right)\left(|a_{jj}|-r_j^{S_2}(A)\right)\ge r_l^{S_2}(A)r_j^{S_1}(A)$. (Note that if A satisfies condition (2) then, from $A\in\mathbf{DD}(n)$ and $\mathcal{L}(A)=\{l\}\subset S_1$, we get $a_{jj}=r_j(A)=0$ for all $j\in S_2$.) Then from (4.10) and the fact that $j\in S_2$ was chosen arbitrarily, the result follows.

Theorem 4.3 *Let* $B=(b_{ij})\in\mathbf{SDD}(n)\setminus\mathbf{SD}(n)$, $n\ge 2$, *and let* $m\in\langle n\rangle$ *be the integer such that* $\mathcal{J}(B)=\langle n\rangle\setminus\{m\}$. *Suppose that* (S_1,S_2) *is a separation of* $\langle n\rangle$ *such that* $S_2\subset\mathcal{J}(B)$. *In addition, assume that* $r_m^{S_1}(B)=0$. *Then* $B\in\mathbf{SDD}(S_1,S_2,n)$.

Proof. Let $j\in S_2$. Then from $S_2\subset\mathcal{J}(B)=\langle n\rangle\setminus\{m\}$, we deduce that

$$i\in S_1\setminus\{m\}\Rightarrow\left(|b_{ii}|-r_i^{S_1}(B)\right)\left(|b_{jj}|-r_j^{S_2}(B)\right)>r_i^{S_2}(B)r_j^{S_1}(B)$$

and, from $B\in\mathbf{SDD}(n)$ and $r_m^{S_1}(B)=0$, we get

$$\begin{aligned}&\left(|b_{mm}|-r_m^{S_1}(B)\right)\left(|b_{jj}|-r_j^{S_2}(B)\right)>r_m(B)r_j(B)-|b_{mm}|r_j^{S_2}(B)\\&\ge r_m(B)\left(r_j(B)-r_j^{S_2}(B)\right)=r_m^{S_2}(B)r_j^{S_1}(B).\end{aligned}$$

This proves $B\in\mathbf{SDD}(S_1,S_2,n)$.

Theorem 4.4 *Let* $A=(a_{ij})\in\mathbf{IRDD}(n)\setminus\mathbf{D}(n)$, $n\ge 2$. *Let* $l\in\langle n\rangle$ *be the integer such that* $\mathcal{J}(A)=\langle n\rangle\setminus\{l\}$. *Suppose that* (S_1,S_2) *is a separation of* $\langle n\rangle$ *such that* $S_2\subset\mathcal{J}(A)$. *In addition, assume that A satisfies the following two conditions*:

Condition (1): $r_l^{S_1}(A)=0$.

Condition (2): If $\operatorname{card}S_1=1$ then there exists $m\in S_2$ such that

$$\max\left\{|a_{ll}||a_{mm}|-r_l(A)r_m(A),r_m^{S_2}(A)\right\}>0.\qquad(4.12)$$

Then $A\in\mathbf{IRDD}(S_1,S_2,n)$.

Proof. It follows from $A\in\mathbf{IRDD}(n)\setminus\mathbf{D}(n)$ that $A\in\mathbf{DD}(n)\setminus\mathbf{D}(n)$, $A\in\mathbf{IR}(n)$ and $\mathcal{J}(A)\ne\varnothing$. From $S_2\subset\mathcal{J}(A)=\langle n\rangle\setminus\{l\}$ and $\mathcal{L}(A)=\{l\}$ (see item (2) of Remark 4.1), we deduce that $\mathcal{L}(A)=\{l\}\subset S_1$. Then from $A\in\mathbf{DD}(n)\setminus\mathbf{D}(n)$, condition (1) and Theorem 4.2, we infer that $A\in\mathbf{DD}(S_1,S_2,n)$. So, it remains to show that there exist $s_1\in S_1$ and $s_2\in S_2$ such that (2.4) is satisfied. It follows from $\mathcal{L}(A)=\{l\}\subset S_1$ and $\mathcal{J}(A)=\langle n\rangle\setminus\{l\}$ that

$$i\in S_1\setminus\{l\},\,j\in S_2\Rightarrow\left(|a_{ii}|-r_i^{S_1}(A)\right)\left(|a_{jj}|-r_j^{S_2}(A)\right)>r_i^{S_2}(A)r_j^{S_1}(A).\qquad(4.13)$$

If $\operatorname{card} S_1 > 1$, then, with the choice of any $s_1 \in S_1 \setminus \{l\}$ and any $s_2 \in S_2$, we see from (4.13) that (2.4) is satisfied. Thus it remains to consider the case $\operatorname{card} S_1 = 1$. In this case, we deduce from condition (2) that there exists $m \in S_2$ such that (4.12) is satisfied. Hence from $r_l(A) = r_l^{S_2}(A)$ (see condition (1)), $r_m(A) = r_m^{S_1}(A) + r_m^{S_2}(A)$ and $r_l(A) > |a_{ll}|$, we get

$$|a_{ll}|\left(|a_{mm}| - r_m^{S_2}(A)\right) - r_l^{S_2}(A) r_m^{S_1}(A) = |a_{ll}||a_{mm}| - r_l(A) r_m(A) + r_m^{S_2}(A)\left[r_l(A) - |a_{ll}|\right] > 0.$$

Then (2.4) is satisfied with $s_1 = l$ and $s_2 = m$.

Theorem 4.5 provides sufficient conditions for matrices in the classes $\mathbf{DD}(S_1, S_2, n)$ and $\mathbf{IRDD}(S_1, S_2, n)$ to be in $\mathbf{DD}(n)$ and $\mathbf{IRDD}(n)$, respectively. We prove item (2) of the theorem. Item (1) is proven similarly.

Theorem 4.5 *Let* (S_1, S_2) *be a separation of* $\langle n \rangle$, $n \geq 2$, *and let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ *be such that the following two conditions are satisfied*:

Condition (1): $|a_{ii}||a_{jj}| \geq r_i(A) r_j(A)$ for all $i, j \in S_1$ with $i \neq j$, and $|a_{kk}||a_{ll}| \geq r_k(A) r_l(A)$ for all $k, l \in S_2$ with $k \neq l$.

Condition (2): $|a_{ii}| \geq r_i(A) + r_i^{S_1}(A)$ for all $i \in S_1$ and $|a_{jj}| \geq r_j^{S_1}(A)$ for all $j \in S_2$.

Then

1) If $A \in \mathbf{DD}(S_1, S_2, n)$ then $A \in \mathbf{DD}(n)$.

2) If $A \in \mathbf{IRDD}(S_1, S_2, n)$ then $A \in \mathbf{IRDD}(n)$.

Proof. Assume that $A \in \mathbf{IRDD}(S_1, S_2, n)$. Then

$$|a_{ii}||a_{jj}| \geq r_i^{S_2}(A) r_j^{S_1}(A) + |a_{ii}| r_j^{S_2}(A) + |a_{jj}| r_i^{S_1}(A) - r_i^{S_1}(A) r_j^{S_2}(A)$$

for all $i \in S_1$ and $j \in S_2$, and there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that (2.4) is satisfied. Thus from condition (2), we deduce that $|a_{ii}||a_{jj}| \geq r_i(A) r_j(A)$ for all $i \in S_1$ and $j \in S_2$, and $|a_{s_1 s_1}||a_{s_2 s_2}| > r_{s_1}(A) r_{s_2}(A)$. Hence from condition (1) and $A \in \mathbf{IR}(n)$, we see that $A \in \mathbf{IRDD}(n)$.

Remark 4.2 1) In item (2) of Theorem 4.5, the condition $J(A) \neq \varnothing$ was not used to drive the result.

2) If (S_1, S_2) is a separation of $\langle n \rangle$, $n \geq 2$, and $A \in \mathbf{SDD}(S_1, S_2, n)$, sufficient conditions for *A* to be in $\mathbf{SDD}(n)$ could be set by replacing the inequalities in condition (1) of Theorem 4.5 by strict inequalities and keeping condition (2) of the theorem as it is.

5. The Class $\mathbf{GD}(n)$ vs. the Classes $\mathbf{DD}(S_1, S_2, n)$ and $\mathbf{DD}(n)$

The first main result of this section is Theorem 5.1. In item (2) of the theorem, 5 is the smallest integer we were able to find, which satisfies the result. We denote the set of all separations of $\langle n \rangle$ by $\mathbb{S}(\langle n \rangle)$.

Theorem 5.1 *Let* $n \in \mathbb{N} \setminus \{1\}$. *Then*

1) If (S_1, S_2) is a separation of $\langle n \rangle$, then $\mathbf{DD}(S_1, S_2, n) \not\subset \mathbf{GD}(n)$.

2) If $n \geq 5$, then $\mathbf{GD}(n) \not\subset \bigcup_{(S_1, S_2) \in \mathbb{S}(\langle n \rangle)} \mathbf{DD}(S_1, S_2, n)$, and for every separation (S_1, S_2) of $\langle n \rangle$, the pair $(\mathbf{GD}(n), \mathbf{DD}(S_1, S_2, n))$ is a pair of incompar-

able classes.

Proof. 1) Let (S_1, S_2) be a separation of $\langle n\rangle$. Choose $s_1 \in S_1$ and $s_2 \in S_2$, and define $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ by $a_{s_1 s_2} = 1$ and $a_{ij} = 0$ for all $(i, j) \in (\langle n\rangle \times \langle n\rangle) \setminus \{(s_1, s_2)\}$. Then $A \in \mathbf{DD}(S_1, S_2, n) \setminus \mathbf{GD}(n)$.

2) Let $n \in \mathbb{N} \setminus \langle 4\rangle$. We construct $A_n \in \mathbf{GD}(n)$ such that $A_n \notin \mathbf{DD}(S_1, S_2, n)$ for any separation (S_1, S_2) of $\langle n\rangle$. The idea is to perturb the matrix defined in item (6) of Remark 3.2. Consider the following two cases:

Case 1: $n = 5$. Define A_5 by

$$A_5 = (a_{ij}) = \begin{pmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0.01 & 0 \\ 2 & 2 & 4 & 0.01 & 0 \\ 0.01 & 2 & 2 & 6 & 2 \\ 0.01 & 0 & 4 & 4 & 8 \end{pmatrix}. \tag{5.1}$$

Then, with Υ_5 as defined by (3.6), we have $A_5\Upsilon_5 \in \mathbf{SD}(5)$. Table 1 illustrates that $A_5 \notin \mathbf{DD}(S_1, S_2, 5)$ for any separation (S_1, S_2) of $\langle 5\rangle$.

Case 2: $n > 5$. Define A_n by

$$A_n = (a_{ij}) = A_5 \oplus I_{n-5}, \tag{5.2}$$

where A_5 is defined by (5.1). Thus, with $\Upsilon_n = \Upsilon_5 \oplus I_{n-5}$ and Υ_5 given by (3.6), we deduce from (5.1) that $A_n\Upsilon_n \in \mathbf{SD}(n)$. Hence $A_n \in \mathbf{GD}(n)$. Let (S_1, S_2) be a separation of $\langle n\rangle$. Assume first that either $\langle 5\rangle \subset S_1$ or $\langle 5\rangle \subset S_2$.

Table 1. Case 1.

Separation of $\langle 5\rangle$	Testing Pair	$(\lvert a_{ii}\rvert - r_i^{S_1}(A))(\lvert a_{jj}\rvert - r_j^{S_2}(A))$	$r_i^{S_2}(A) r_j^{S_1}(A)$
$(\{1\},\{2,3,4,5\})$	$(1,4)$	$(1)(6-6)=0$	$(0.5)(0.01)=0.005$
$(\{2\},\{1,3,4,5\})$	$(2,5)$	$(2)(8-8.01)=-0.02$	$(2.01)(0)=0$
$(\{3\},\{1,2,4,5\})$	$(3,2)$	$(4)(2-1.01)=3.96$	$(4.01)(1)=4.01$
$(\{4\},\{1,2,3,5\})$	$(4,2)$	$(6)(2-2)=0$	$(6.01)(0.01)=0.0601$
$(\{5\},\{1,2,3,4\})$	$(5,2)$	$(8)(2-2.01)=-0.08$	$(8.01)(0)=0$
$(\{1,2\},\{3,4,5\})$	$(2,3)$	$(2-1)(4-0.01)=3.99$	$(1.01)(4)=4.04$
$(\{1,3\},\{2,4,5\})$	$(3,2)$	$(4-2)(2-0.01)=3.98$	$(2.01)(2)=4.02$
$(\{1,4\},\{2,3,5\})$	$(4,3)$	$(6-0.01)(4-2)=11.98$	$(6)(2.01)=12.06$
$(\{1,5\},\{2,3,4\})$	$(5,2)$	$(8-0.01)(2-1.01)=7.9101$	$(8)(1)=8$
$(\{2,3\},\{1,4,5\})$	$(2,4)$	$(2-1)(6-2.01)=3.99$	$(1.01)(4)=4.04$
$(\{2,4\},\{1,3,5\})$	$(2,3)$	$(2-0.01)(4-2)=3.98$	$(2)(2.01)=4.02$
$(\{2,5\},\{1,3,4\})$	$(2,3)$	$(2)(4-2.01)=3.98$	$(2.01)(2)=4.02$
$(\{3,4\},\{1,2,5\})$	$(3,5)$	$(4-0.01)(8-0.01)=31.8801$	$(4)(8)=32$
$(\{3,5\},\{1,2,4\})$	$(3,4)$	$(4)(6-2.01)=15.96$	$(4.01)(4)=16.04$
$(\{4,5\},\{1,2,3\})$	$(4,3)$	$(6-2)(4-4)=0$	$(4.01)(0.01)=0.0401$

Assume without loss of generality that $\langle 5\rangle \subset S_1$. Choose $j \in S_2$. Then from $5 \in S_1$, (5.1) and (5.2), we get

$$\left(\left|a_{55}\right| - r_5^{S_1}\left(A_n\right)\right)\left(\left|a_{jj}\right| - r_j^{S_2}\left(A_n\right)\right) = (-0.01)(1) < 0 = r_5^{S_2}\left(A_n\right) r_j^{S_1}\left(A_n\right).$$

So, it remains to consider the case:

$$S_1 \cap \langle 5\rangle \neq \varnothing \text{ and } S_2 \cap \langle 5\rangle \neq \varnothing. \tag{5.3}$$

Since (S_1, S_2) is a separation of $\langle n\rangle$ and $n > 5$, we infer from (5.3) that $(S_1 \cap \langle 5\rangle, S_2 \cap \langle 5\rangle)$ is a separation of $\langle 5\rangle$. Thus from (5.2), we see that

$$r_m^{S_1}\left(A_n\right) = r_m^{S_1 \cap \langle 5\rangle}\left(A_n\right) = r_m^{S_1 \cap \langle 5\rangle}\left(A_5\right) \text{ and } r_m^{S_2}\left(A_n\right) = r_m^{S_2 \cap \langle 5\rangle}\left(A_n\right) = r_m^{S_2 \cap \langle 5\rangle}\left(A_5\right)$$

for all $m \in \langle 5\rangle$. Hence from case 1, we deduce there exist $i \in S_1 \cap \langle 5\rangle$ and $j \in S_2 \cap \langle 5\rangle$ such that

$$\left(\left|a_{ii}\right| - r_i^{S_1}\left(A_n\right)\right)\left(\left|a_{jj}\right| - r_j^{S_2}\left(A_n\right)\right) < r_i^{S_2}\left(A_n\right) r_j^{S_1}\left(A_n\right).$$

As (S_1, S_2) was chosen arbitrarily, we infer that $A_n \notin \bigcup_{(S_1,S_2)\in \mathbb{S}(\langle n\rangle)} \mathbf{DD}(S_1, S_2, n)$. For every separation (S_1, S_2) of $\langle n\rangle$, the incomparability of the pair $(\mathbf{GD}(n), \mathbf{DD}(S_1, S_2, n))$ follows from the first part and item (1).

The following theorem provides sufficient conditions for matrices in $\mathbf{DD}(S_1, S_2, n)$ to be in $\mathbf{GD}(n)$.

Theorem 5.2 *Let* $n \in \mathbb{N} \setminus \{1\}$, *and let* (S_1, S_2) *be a separation of* $\langle n\rangle$. *Suppose that* $A = (a_{ij}) \in \mathbf{DD}(S_1, S_2, n)$. *In addition, assume that A satisfies the following two conditions*:

Condition (1): There exists $l \in S_1$ such that $\left|a_{ll}\right| - r_l^{S_1}(A) > 0$.

Condition (2): The sets $T_1 := \left\{i \in S_1 : r_i^{S_2}(A) > 0\right\}$ and $T_2 := \left\{j \in S_2 : r_j^{S_1}(A) > 0\right\}$ are nonempty.

Then

1) We have

$$\min\left\{\left|a_{ii}\right| - r_i^{S_1}(A) : i \in S_1\right\} \geq 0 \text{ and } \min\left\{\left|a_{jj}\right| - r_j^{S_2}(A) : j \in S_2\right\} \geq 0, \tag{5.4}$$

and

$$\min\left\{\left|a_{ii}\right| - r_i^{S_1}(A), \frac{r_i^{S_2}(A)}{\left|a_{ii}\right| - r_i^{S_1}(A)} : i \in T_1\right\} > 0 \text{ and } \min\left\{\left|a_{jj}\right| - r_j^{S_2}(A) : j \in T_2\right\} > 0. \tag{5.5}$$

2) We have

$$0 < \Gamma_1 = \max\left\{\frac{r_i^{S_2}(A)}{\left|a_{ii}\right| - r_i^{S_1}(A)} : i \in T_1\right\} \leq \min\left\{\frac{\left|a_{jj}\right| - r_j^{S_2}(A)}{r_j^{S_1}(A)} : j \in T_2\right\} = \Gamma_2, \tag{5.6}$$

and for every γ satisfying

$$\Gamma_1 \leq \gamma \leq \Gamma_2, \tag{5.7}$$

the diagonal matrix $\Upsilon = \Upsilon(\gamma) = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)$ defined by

$$\gamma_i = \begin{cases} \gamma & \text{if } i \in S_1, \\ 1 & \text{if } i \in S_2, \end{cases} \tag{5.8}$$

is nonsingular and satisfies $A\Upsilon \in \mathbf{D}(n)$. In particular, $A \in \mathbf{GD}(n)$.

3) If there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that (2.4) is satisfied, then either $\left|(A\Upsilon)_{s_1s_1}\right| > r_{s_1}(A\Upsilon)$ or $\left|(A\Upsilon)_{s_2s_2}\right| > r_{s_2}(A\Upsilon)$, where Υ is the nonsingular diagonal matrix defined by (5.7) and (5.8).

Proof. 1) It follows from $A \in \mathbf{DD}(S_1, S_2, n)$ and condition (1) that the second inequality of (5.4) holds. Then from condition (2) and $A \in \mathbf{DD}(S_1, S_2, n)$, we deduce that there exists $s \in S_2$ such that $|a_{ss}| - r_s^{S_2}(A) > 0$, and that the first inequality of (5.4) holds as well. Now, let $i \in T_1$ and $j \in T_2$. It then follows from $A \in \mathbf{DD}(S_1, S_2, n)$ and the definitions of T_1 and T_2 that $\left(|a_{ii}| - r_i^{S_1}(A)\right)\left(|a_{jj}| - r_j^{S_2}(A)\right) > 0$. Thus from (5.4), we get $|a_{ii}| - r_i^{S_1}(A) > 0$ and $|a_{jj}| - r_j^{S_2}(A) > 0$, and from the definition of T_1, we obtain $\left[|a_{ii}| - r_i^{S_1}(A)\right]^{-1} r_i^{S_2}(A) > 0$. This completes the proof of (5.5).

2) It follows from $A \in \mathbf{DD}(S_1, S_2, n)$, condition (2) and (5.5) that (5.6) holds. Let γ be a real satisfying (5.7), and define the diagonal matrix $\Upsilon = \Upsilon(\gamma)$ by (5.8). Hence from $0 < \Gamma_1$ (in (5.6)) and $\gamma \ge \Gamma_1$ (in (5.7)), we infer that Υ is a diagonal matrix with positive diagonal entries. Also, from (5.8) and $\gamma > 0$, it is clear that

$$\left|(A\Upsilon)_{ii}\right| = \gamma |a_{ii}| \text{ and } \left|(A\Upsilon)_{jj}\right| = |a_{jj}| \tag{5.9}$$

for all $i \in S_1$ and $j \in S_2$, and

$$r_m(A\Upsilon) = \gamma\, r_m^{S_1}(A) + r_m^{S_2}(A) \tag{5.10}$$

for all $m \in \langle n \rangle$. From the first strict inequality of (5.5), the definition of Γ_1 (in (5.6)) and $\Gamma_1 \le \gamma$ (in (5.7)), we get $\gamma |a_{ii}| \ge \gamma r_i^{S_1}(A) + r_i^{S_2}(A)$ for all $i \in T_1$. Then from the first equality of (5.9), and (5.10), we see that

$$r_i(A\Upsilon) \le \left|(A\Upsilon)_{ii}\right| \tag{5.11}$$

for all $i \in T_1$. Since $\gamma > 0$, we deduce from the definition of Γ_2 (in (5.6)) and $\gamma \le \Gamma_2$ (in (5.7)) that $\gamma r_j^{S_1}(A) \le |a_{jj}| - r_j^{S_2}(A)$ for all $j \in T_2$. Thus from the second equality of (5.9), and (5.10), we infer that

$$r_j(A\Upsilon) \le \left|(A\Upsilon)_{jj}\right| \tag{5.12}$$

for all $j \in T_2$. Since $\gamma > 0$ and $r_i^{S_2}(A) = 0$ for all $i \in S_1 \setminus T_1$, we see from the first inequality of (5.4), (5.9) and (5.10) that

$$r_i(A\Upsilon) \le \left|(A\Upsilon)_{ii}\right| \tag{5.13}$$

for all $i \in S_1 \setminus T_1$. Since $r_j^{S_1}(A) = 0$ for all $j \in S_2 \setminus T_2$, we deduce from the second inequality of (5.4), (5.9) and (5.10) that $r_j(A\Upsilon) \le \left|(A\Upsilon)_{jj}\right|$ for all $j \in S_2 \setminus T_2$. Hence from (5.11)-(5.13), we infer that $A\Upsilon \in \mathbf{D}(n)$. Then from Υ being a nonsingular diagonal matrix, we see that $A \in \mathbf{GD}(n)$.

3) Assume that there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that (2.4) is satisfied. We consider the following two cases:

Case 1: $s_1 \in T_1$ and $s_2 \in T_2$. In this case, we have $r_{s_1}^{S_2}(A)\, r_{s_2}^{S_1}(A) > 0$. Thus from (2.4), (5.6) and (5.7), we deduce that

$$\text{either}\quad 0<\frac{r_{s_1}^{S_2}(A)}{|a_{s_1s_1}|-r_{s_1}^{S_1}(A)}<\gamma\quad\text{or}\quad 0<\gamma<\frac{|a_{s_2s_2}|-r_{s_2}^{S_2}(A)}{r_{s_2}^{S_1}(A)}.$$

Hence from $r_{s_1}^{S_2}(A)r_{s_2}^{S_1}(A)>0$, (5.9) and (5.10), the result follows.

Case 2: Either $s_1\in S_1\setminus T_1$ or $s_2\in S_2\setminus T_2$. In this case, we have

$$\text{either } r_{s_1}^{S_2}(A)=0 \text{ or } r_{s_2}^{S_1}(A)=0. \tag{5.14}$$

Then from (2.4) and (5.4), we infer that $|a_{s_1s_1}|-r_{s_1}^{S_1}(A)>0$ and $|a_{s_2s_2}|-r_{s_2}^{S_2}(A)>0$. Thus from $\gamma>0$, (5.10) and (5.14), we see that either $\gamma|a_{s_1s_1}|>r_{s_1}(A\Upsilon)$ or $|a_{s_2s_2}|>r_{s_2}(A\Upsilon)$. Hence from (5.9), the result follows.

Corollary 5.1 *Let* $n\in\mathbb{N}\setminus\{1\}$. *Then*

1) For each separation (S_1,S_2) of $\langle n\rangle$, we have $\mathbf{IR}(n)\cap\mathbf{DD}(S_1,S_2,n)\cap\{X\in\mathbb{C}^{n\times n}:\mathcal{J}(X(S_1))\neq\varnothing\}\subset\mathbf{GD}(n)$.

2) $\mathbf{IR}(n)\cap\left[\bigcup_{k=1}^{n}\mathbf{DD}(\{k\},\langle n\rangle\setminus\{k\},n)\right]\subset\mathbf{GD}(n)$.

Proof. 1) Let (S_1,S_2) be a separation of $\langle n\rangle$, and let $A\in\mathbf{IR}(n)\cap\mathbf{DD}(S_1,S_2,n)$ be such that $\mathcal{J}(A(S_1))\neq\varnothing$. We show that A satisfies conditions (1) and (2) of Theorem 5.2. It follows from Remark 1.1 and $\mathcal{J}(A(S_1))\neq\varnothing$ that condition (1) of Theorem 5.2 is satisfied. Also, from A being irreducible, we deduce that condition (2) of Theorem 5.2 is satisfied. Then from $A\in\mathbf{DD}(S_1,S_2,n)$ and Theorem 5.2, we infer that $A\in\mathbf{GD}(n)$.

2) Let $A=(a_{ij})\in\mathbf{IR}(n)\cap\left[\bigcup_{k=1}^{n}\mathbf{DD}(\{k\},\langle n\rangle\setminus\{k\},n)\right]$. Thus there exists $m\in\langle n\rangle$ such that

$$A\in\mathbf{DD}(\{m\},\langle n\rangle\setminus\{m\},n). \tag{5.15}$$

So, from $A\in\mathbf{IR}(n)$ and item (1), the result follows if we can show that $\mathcal{J}(A(\{m\}))\neq\varnothing$. It follows from $A\in\mathbf{IR}(n)$ that $r_m(A)>0$ and there exists $k\in\langle n\rangle\setminus\{m\}$ such that $|a_{km}|>0$. Hence from (5.15), we see that $|a_{mm}|>0$, that is, $\mathcal{J}(A(\{m\}))\neq\varnothing$.

Remark 5.1 The irreducibility condition in Corollary 5.1 cannot be dropped. Let $A=(a_{ij})=\begin{pmatrix}0&0&1\\0&1&1\\0&0&1\end{pmatrix}$. Then A is reducible, $\mathcal{J}(A(\{1,2\}))\neq\varnothing$, $\mathcal{J}(A(\{3\}))\neq\varnothing$ (as $a_{33}=1$) and $A\in\mathbf{DD}(\{1,2\},\{3\},3)$, but $A\notin\mathbf{GD}(3)$.

If $\operatorname{card}S_1=1$ in Theorem 5.2, we could relax condition (2) in the theorem.

Theorem 5.3 *Let* $A=(a_{ij})\in\mathbb{C}^{n\times n}$, $n\geq 2$, *and let* $k\in\langle n\rangle$. *Assume that* $A\in\mathbf{DD}(\{k\},\langle n\rangle\setminus\{k\},n)$, $a_{kk}\neq 0$ *and* $r_k(A)>0$. *Then* $A\in\mathbf{GD}(n)$.

Proof. Without loss of generality, assume that $k=1$. Define the diagonal matrix $\Upsilon=\operatorname{diag}(\gamma_1,\cdots,\gamma_n)$ by

$$\gamma_i=\begin{cases}r_1(A) & \text{if } i=1,\\ |a_{11}| & \text{if } i\neq 1.\end{cases}$$

Then Υ is nonsingular, $r_1(\Upsilon^{-1}A\Upsilon)=|a_{11}|$ and

$$r_j(\Upsilon^{-1}A\Upsilon)=r_j(A)-|a_{j1}|+\frac{|a_{j1}|r_1(A)}{|a_{11}|}$$

for all $j \in \langle n \rangle \setminus \{1\}$. Thus from $A \in \mathbf{DD}(\{1\}, \langle n \rangle \setminus \{1\}, n)$, we deduce that $\Upsilon^{-1} A \Upsilon \in \mathbf{D}(n)$. Hence from Lemma 3.2, we see that $A \in \mathbf{GD}(n)$.

Remark 5.2 Let $n \in \mathbb{N} \setminus \{1,2\}$, and let (S_1, S_2) be a separation of $\langle n \rangle$. It follows from Theorem 4.1 (using the permutation similarity transformation technique) that there exists $B \in [\mathbf{SDD}(S_1, S_2, n) \cap \mathbf{IR}(n)] \setminus \mathbf{DD}(n)$. It is clear that B satisfies conditions (1) and (2) of Theorem 5.2 as well. So, from $B \in \mathbf{DD}(S_1, S_2, n)$ and Theorem 5.2, we deduce that $B \in \mathbf{GD}(n)$. This observation together with (3.8) lead to the following corollary.

Corollary 5.2 *Let* $n \in \mathbb{N} \setminus \{1,2\}$. *Then* $(\mathbf{GD}(n), \mathbf{DD}(n))$ *is a pair of incomparable classes.*

The following corollary establishes the first inclusion of (3.3).

Corollary 5.3 *Let* $n \in \mathbb{N} \setminus \{1\}$, *and let* (S_1, S_2) *be a separation of* $\langle n \rangle$. *Then* $\mathbf{IRDD}(S_1, S_2, n) \subset \mathbf{IRGD}(n)$.

Proof. Let $A = (a_{ij}) \in \mathbf{IRDD}(S_1, S_2, n)$. Then from (iv) of item (4) of Remark 3.1, we deduce that there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that (2.4) and (3.1) are satisfied. Thus condition (1) of Theorem 5.2 is satisfied. Also, A satisfies condition (2) of Theorem 5.2 by virtue of being irreducible. Hence from $A \in \mathbf{DD}(S_1, S_2, n)$ and item (2) of Theorem 5.2, we infer that $A\Upsilon \in \mathbf{D}(n)$, where $\Upsilon = \Upsilon(\gamma)$ is the nonsingular diagonal matrix defined by (5.7) and (5.8). Also, since s_1 and s_2 satisfy (2.4), we see from item (3) of Theorem 5.2 that $J(A\Upsilon) \cap \{s_1, s_2\} \neq \varnothing$. Finally, from $A \in \mathbf{IR}(n)$ and Υ being nonsingular diagonal matrix, we deduce that $A\Upsilon \in \mathbf{IR}(n)$. This completes the proof that $A\Upsilon \in \mathbf{IRD}(n)$, that is, $A \in \mathbf{IRGD}(n)$.

It follows from (v) of item (4) of Remark 3.1 that in order to establish sufficient conditions for matrices in $\mathbf{GD}(n)$, $n \geq 2$, to be in $\mathbf{DD}(S_1, S_2, n)$, it suffices to provide such conditions for matrices in $\mathbf{GD}(n) \setminus \mathbf{D}(n)$. In the following theorem, if a set is empty its maximum is understood to be 0.

Theorem 5.4 *Let* $A = (a_{ij}) \in \mathbf{GD}(n) \setminus \mathbf{D}(n)$, $n \geq 2$, *and let* (S_1, S_2) *be a separation of* $\langle n \rangle$. *Let* Υ *be a diagonal matrix in* $\mathbb{C}^{n \times n}$ *with positive diagonal entries such that* $\Upsilon^{-1} A \Upsilon \in \mathbf{D}(n)$. *Assume that* A *and* Υ *satisfy the following conditions*:

Condition (1): $\mathcal{L}(A) \subset S_1$.

Condition (2): $|a_{ii}| > r_i^{S_1}(A)$ for all $i \in \mathcal{L}(A)$.

Condition (3): For all $j \in S_2$, we have $r_j^{S_1}(\Upsilon^{-1} A \Upsilon) \geq r_j^{S_1}(A)$ and $r_j^{S_2}(\Upsilon^{-1} A \Upsilon) \geq r_j^{S_2}(A)$.

Then

1) For every $i \in \mathcal{L}(A)$, we have $r_i^{S_2}(A) > 0$.

2) If $j \in S_2$ and $r_j^{S_1}(A) > 0$, then $r_j^{S_1}(\Upsilon^{-1} A \Upsilon) > 0$ and $|a_{jj}| - r_j^{S_2}(\Upsilon^{-1} A \Upsilon) > 0$.

3) If A and Υ satisfy the additional condition:

Condition (4):

$$\max \left\{ \frac{r_j^{S_1}(\Upsilon^{-1} A \Upsilon)}{|a_{jj}| - r_j^{S_2}(\Upsilon^{-1} A \Upsilon)} : j \in S_2, r_j^{S_1}(A) > 0 \right\} \leq \min \left\{ \frac{|a_{ii}| - r_i^{S_1}(A)}{r_i^{S_2}(A)} : i \in \mathcal{L}(A) \right\},$$

then $A \in \mathbf{DD}(S_1, S_2, n)$.

Proof. We first observe that the existence of the diagonal matrix Υ with positive diagonal entries, which satisfies $\Upsilon^{-1}A\Upsilon \in \mathbf{D}(n)$, is ensured by virtue of $A \in \mathbf{GD}(n)$ and Lemma 3.2.

1) Let $i \in \mathcal{L}(A)$. Since $|a_{ii}| < r_i(A)$, we deduce from condition (2) that $r_i^{S_2}(A) > 0$.

2) Let $j \in S_2$, and assume that $r_j^{S_1}(A) > 0$. Then from condition (3), $\Upsilon^{-1}A\Upsilon \in \mathbf{D}(n)$ and $|a_{jj}| = \left|\left(\Upsilon^{-1}A\Upsilon\right)_{jj}\right|$, we infer that $r_j^{S_1}\left(\Upsilon^{-1}A\Upsilon\right) > 0$ and $|a_{jj}| - r_j^{S_2}\left(\Upsilon^{-1}A\Upsilon\right) > 0$.

3) Assume that A and Υ also satisfy condition (4). We first observe that the condition is logically viable by virtue of condition (2) and items (1) and (2). It follows from condition (1) that (2.3) is satisfied for all $i \in S_1 \setminus \mathcal{L}(A)$ and $j \in S_2$. Also, from conditions (1) and (2), we infer that (2.3) is satisfied for all $i \in \mathcal{L}(A)$ and $j \in S_2$ with $r_j^{S_1}(A) = 0$. Thus, it remains to consider the case $i \in \mathcal{L}(A)$ and $j \in S_2$ with $r_j^{S_1}(A) > 0$. It follows from condition (3) and item (2) that

$$\frac{r_j^{S_1}\left(\Upsilon^{-1}A\Upsilon\right)}{|a_{jj}| - r_j^{S_2}\left(\Upsilon^{-1}A\Upsilon\right)} \ge \frac{r_j^{S_1}(A)}{|a_{jj}| - r_j^{S_2}(A)}.$$

Hence from condition (4), we see that (2.3) also holds for all $i \in \mathcal{L}(A)$ and $j \in S_2$ with $r_j^{S_1}(A) > 0$.

Example 5.1 Let $A = (a_{ij}) = \begin{pmatrix} 1.2 & 0.75 & 0.25 & 0.3 & 0.1 \\ 1 & 2 & 0.2 & 0.6 & 0.1 \\ 0.1 & 0.1 & 2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 & 2 & 0.1 \\ 0.5 & 0.5 & 0 & 0 & 3 \end{pmatrix}$, and let $S_1 = \langle 2 \rangle$ and $S_2 = \{3,4,5\}$. Then $\mathcal{L}(A) = \{1\} \subset S_1$ and, with $\Upsilon = \operatorname{diag}(1,1,1/2,1/4,1)$, we see that $\Upsilon^{-1}A\Upsilon \in \mathbf{D}(5)$ and conditions (1)-(4) of Theorem 5.4 are satisfied with

$$\max\left\{\frac{r_j^{S_1}\left(\Upsilon^{-1}A\Upsilon\right)}{|a_{jj}| - r_j^{S_2}\left(\Upsilon^{-1}A\Upsilon\right)} : j \in S_2, r_j^{S_1}(A) > 0\right\} = \frac{2}{3} < \frac{9}{13} = \frac{|a_{11}| - r_1^{S_1}(A)}{r_1^{S_2}(A)}.$$

Then $A \in \mathbf{DD}(S_1, S_2, 5)$.

6. Row-Column Diagonally Dominant Matrices with Index α vs. Matrices with Other Variants of the Diagonal Dominance Property

In this section, we investigate the relations between the class $\{\mathbf{RCD}(\alpha;n), \mathbf{SRCD}(\alpha;n), \mathbf{IRRCD}(\alpha;n)\}$ and the other classes introduced in Definition 2.1. There has not been too much attention in the literature to discuss such relations.

For a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, define the sets $\mathcal{R}(A)$, $\mathcal{C}(A)$, $\mathcal{E}(A)$, $\mathcal{R}_1(A)$ and $\mathcal{C}_1(A)$ by

$$\begin{aligned}&\mathcal{R}(A)=\{i\in\langle n\rangle : r_i(A)>c_i(A)\},\mathcal{C}(A)=\{i\in\langle n\rangle : c_i(A)>r_i(A)\},\\&\mathcal{E}(A)=\{i\in\langle n\rangle : r_i(A)=c_i(A)\},\mathcal{R}_1(A)=\{i\in\mathcal{R}(A) : c_i(A)>0\},\\&\mathcal{C}_1(A)=\{j\in\mathcal{C}(A) : r_j(A)>0\}.\end{aligned}\tag{6.1}$$

It is clear that $\langle n\rangle$ is decomposed into the three mutually disjoint sets $\mathcal{R}(A)$, $\mathcal{C}(A)$ and $\mathcal{E}(A)$. Theorem 6.2 investigates the relation between the classes $\mathbf{SD}(n)$ and $\mathbf{SRCD}(\alpha;n)$. A characterization of the class $\mathbf{SRCD}(\alpha;n)$ is given in Theorem 5 of [27]. We will use the following slightly modified version of the result in Theorem 6.2.

Theorem 6.1 *Let* $A=(a_{ij})\in\mathbb{C}^{n\times n}$, $n\geq 2$. *Define the sets* $\mathcal{R}_1(A)$ *and* $\mathcal{C}_1(A)$ *by* (6.1). *Then the following statements hold*:

1) If A satisfies the condition

$$|a_{kk}|>\min\{r_k(A),c_k(A)\},\forall k\in\langle n\rangle,\tag{6.2}$$

then

$$\begin{aligned}&|a_{kk}|>0\text{ for all }k\in\langle n\rangle,\\&i\in\mathcal{R}_1(A)\Rightarrow\log_{\frac{r_i(A)}{c_i(A)}}\frac{|a_{ii}|}{c_i(A)}>0,\\&j\in\mathcal{C}_1(A)\Rightarrow\log_{\frac{c_j(A)}{r_j(A)}}\frac{c_j(A)}{|a_{jj}|}<1.\end{aligned}\tag{6.3}$$

2) $A\in\mathbf{SRCD}(\alpha;n)$ for some $\alpha\in(0,1)$ if and only if A satisfies condition (6.2) and the condition:

$$\mathcal{R}_1(A)\neq\varnothing,\mathcal{C}_1(A)\neq\varnothing\Rightarrow\min\left\{\log_{\frac{r_i(A)}{c_i(A)}}\frac{|a_{ii}|}{c_i(A)}:i\in\mathcal{R}_1(A)\right\}>\max\left\{\log_{\frac{c_j(A)}{r_j(A)}}\frac{c_j(A)}{|a_{jj}|}:j\in\mathcal{C}_1(A)\right\}.\tag{6.4}$$

Also, if A satisfies conditions (6.2) and (6.4), then the reals α_1 and α_2 defined by

$$\begin{aligned}&\alpha_1=\begin{cases}1 & \text{if }\mathcal{R}_1(A)=\varnothing,\\ \min\left\{1,\min\left\{\log_{\frac{r_i(A)}{c_i(A)}}\frac{|a_{ii}|}{c_i(A)}:i\in\mathcal{R}_1(A)\right\}\right\} & \text{if }\mathcal{R}_1(A)\neq\varnothing,\end{cases}\\&\alpha_2=\begin{cases}0 & \text{if }\mathcal{C}_1(A)=\varnothing,\\ \max\left\{0,\max\left\{\log_{\frac{c_j(A)}{r_j(A)}}\frac{c_j(A)}{|a_{jj}|}:j\in\mathcal{C}_1(A)\right\}\right\} & \text{if }\mathcal{C}_1(A)\neq\varnothing,\end{cases}\end{aligned}\tag{6.5}$$

satisfy $0\leq\alpha_2<\alpha_1\leq 1$ and $A\in\mathbf{SRCD}(\alpha;n)$ for all $\alpha\in(\alpha_2,\alpha_1)$.

Theorem 6.2 *Let* $n\in\mathbb{N}\setminus\{1\}$. *Then*

$$\mathbf{SD}(n)\subset\bigcup_{\alpha_0,0<\alpha_0<1}\ \bigcap_{\alpha,\alpha_0\leq\alpha<1}\mathbf{SRCD}(\alpha;n).$$

Proof. Let $A=(a_{ij})\in\mathbf{SD}(n)$. Then A satisfies condition (6.2). Also, from $A\in\mathbf{SD}(n)$ and the definition of $\mathcal{R}_1(A)$ (in (6.1)), we get

$$i\in\mathcal{R}_1(A)\Rightarrow\log_{\frac{r_i(A)}{c_i(A)}}\frac{|a_{ii}|}{c_i(A)}>1.\tag{6.6}$$

As A satisfies (6.2), we deduce from item (1) of Theorem 6.1 that A satisfies (6.3). Thus from (6.6), we infer that A satisfies (6.4) and the real α_1 defined by the first equality of (6.5) satisfies $\alpha_1 = 1$. Hence from the fact that A satisfies (6.2) and item (2) of Theorem 6.1, we see that $A \in \mathbf{SRCD}(\alpha;n)$ for all $\alpha \in (\alpha_2, 1)$, where $\alpha_2 \in [0,1)$ is the real defined by the second equality of (6.5).

Remark 6.1 There is no set inclusion between the classes $\mathbf{D}(n)$ and $\mathbf{RCD}(\alpha;n)$, or between the classes $\mathbf{IRD}(n)$ and $\mathbf{IRRCD}(\alpha;n)$ as the one established in Theorem 6.2 between the classes $\mathbf{SD}(n)$ and $\mathbf{SRCD}(\alpha;n)$. However, Theorem 6.3 provides sufficient conditions for matrices in the classes $\mathbf{D}(n)$ and $\mathbf{IRD}(n)$ to be in the classes $\mathbf{RCD}(\alpha;n)$ and $\mathbf{IRRCD}(\alpha;n)$, respectively.

We will use in Theorem 6.3 and other parts in the section the following remark.

Remark 6.2 Let $a \in (0,\infty)$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\varphi(x) = a^x$ for all $x \in \mathbb{R}$. Then φ is continuous. Moreover,

1) If $0 < a < 1$, then φ is strictly decreasing and $\varphi([0,1]) = [a,1]$.

2) If $a = 1$, then φ is constant and $\varphi([0,1]) = \{1\}$.

3) If $a > 1$, then φ is strictly increasing and $\varphi([0,1]) = [1,a]$.

Theorem 6.3 *Let* $n \in \mathbb{N} \setminus \{1\}$, *and let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. *Assume that* A *satisfies the following condition*:

Condition (1): There exists a nonempty subset S of $\langle n \rangle$ such that for every $k \in S$,

$$\text{either } r_k(A)c_k(A) = 0 \text{ or } r_k(A) \geq 1 \geq c_k(A) > 0. \tag{6.7}$$

Then

1) For each $k \in S$, we have $r_k(A) \geq (r_k(A))^{\alpha} (c_k(A))^{1-\alpha}$ for all $\alpha \in (0,1)$.

2) If $S = \langle n \rangle$ and $A \in \mathbf{D}(n)$ then $A \in \bigcap_{\alpha, 0<\alpha<1} \mathbf{RCD}(\alpha;n)$.

3) If $S = \langle n \rangle$ and $A \in \mathbf{IRD}(n)$ then $A \in \bigcap_{\alpha, 0<\alpha<1} \mathbf{IRRCD}(\alpha;n)$.

Proof. 1) Let $k \in S$, and let $\alpha \in (0,1)$. Since

$$r_k(A)c_k(A) = 0 \Rightarrow r_k(A) \geq (r_k(A))^{\alpha} (c_k(A))^{1-\alpha},$$

we deduce from (6.7) that it remains to consider the case:
$r_k(A) \geq 1 \geq c_k(A) > 0$. In this case, we infer from Remark 6.2 that
$r_k(A) \geq (r_k(A))^{\alpha}$ and $1 \geq (c_k(A))^{1-\alpha}$. Then $r_k(A) \geq (r_k(A))^{\alpha} (c_k(A))^{1-\alpha}$ holds.

2) The result follows from item (1).

3) The result follows from item (1).

The following theorem discusses the relation between the classes $\{\mathbf{RCD}(\alpha;n), \mathbf{SRCD}(\alpha;n), \mathbf{IRRCD}(\alpha;n)\}$ and $\{\mathbf{DD}(S_1,S_2,n), \mathbf{SDD}(S_1,S_2,n), \mathbf{IRDD}(S_1,S_2,n)\}$.

Theorem 6.4 *Let* $n \in \mathbb{N} \setminus \{1\}$, *and let* (S_1, S_2) *be a separation of* $\langle n \rangle$. *Then*

1) $\mathbf{SDD}(S_1,S_2,n) \cap \mathbf{IR}(n) \not\subset \bigcup_{\alpha, 0<\alpha<1} \mathbf{RCD}(\alpha;n)$.

2) We have

$$\mathbf{DD}(S_1,S_2,n)\not\subset \bigcup_{\alpha,0<\alpha<1}\mathbf{RCD}(\alpha;n),$$
$$\mathbf{SDD}(S_1,S_2,n)\not\subset \bigcup_{\alpha,0<\alpha<1}\mathbf{SRCD}(\alpha;n),$$
$$\mathbf{IRDD}(S_1,S_2,n)\not\subset \bigcup_{\alpha,0<\alpha<1}\mathbf{IRRCD}(\alpha;n).$$

3) If $n>2$ and $\alpha\in(0,1)$, then $\mathbf{SRCD}(\alpha;n)\cap\mathbf{IR}(n)\not\subset\mathbf{DD}(S_1,S_2,n)$, $\mathbf{RCD}(\alpha;n)\not\subset\mathbf{DD}(S_1,S_2,n)$, $\mathbf{SRCD}(\alpha;n)\not\subset\mathbf{SDD}(S_1,S_2,n)$ and $\mathbf{IRRCD}(\alpha;n)\not\subset\mathbf{IRDD}(S_1,S_2,n)$.

Proof. It follows from Remark 3.4 that it suffices to prove the theorem in the case: $S_1=\langle p\rangle$, where $p\in\langle n-1\rangle$.

1) Define $A=(a_{ij})\in\mathbb{C}^{n\times n}$ by

$$a_{ij}=\begin{cases}2 & \text{if } i=j, i\in\langle p\rangle,\\ 18 & \text{if } i=j, i\in\{p+1,\cdots,n\},\\ 3 & \text{if } (i,j)\in\{(1,n),(n,1)\},\\ 1/(p-1) & \text{if } p>1, (i,j)\in\{(l,m):l,m\in\langle p\rangle, l\neq m\}\cup\{(l,n):l\in\{2,\cdots,p\}\},\\ 1/(n-p-1) & \text{if } n-p>1, (i,j)\in\{(l,m):l\in\langle p\rangle, m\in\{p+1,\cdots,n-1\}\},\\ 1/(n-p-1) & \text{if } n-p>1, (i,j)\in\{(m,1):m\in\{p+1,\cdots,n-1\}\},\\ 1/(p-1) & \text{if } p>1, i\in\{p+1,\cdots,n\}, j\in\{2,\cdots,p\},\\ 1/(n-p-1) & \text{if } n-p>1, (i,j)\in\{(l,m):l,m\in\{p+1,\cdots,n\}, l\neq m\}.\end{cases} \tag{6.8}$$

Then $A\in\mathbf{IR}(n)$. It can be shown by considering the cases $\min\{p,n-p\}=1$ and $\min\{p,n-p\}\geq 2$ that

$$\left(|a_{ii}|-r_i^{\langle p\rangle}(A)\right)\left(|a_{jj}|-r_j^{\langle n\rangle\setminus\langle p\rangle}(A)\right)\geq 17>16\geq r_i^{\langle n\rangle\setminus\langle p\rangle}(A)\,r_j^{\langle p\rangle}(A)$$

for all $i\in\langle p\rangle$ and $j\in\langle n\rangle\setminus\langle p\rangle$. So, $A\in\mathbf{SDD}(\langle p\rangle,\langle n\rangle\setminus\langle p\rangle,n)\cap\mathbf{IR}(n)$. Also, from (6.8), we have

$$|a_{11}|=2<3^{\alpha}\,3^{(1-\alpha)}=3\leq\left(r_1(A)\right)^{\alpha}\left(c_1(A)\right)^{(1-\alpha)}$$

for all $\alpha\in(0,1)$. Then $A\notin\mathbf{RCD}(\alpha;n)$.

2) The result follows from item (1), and Remark 3.1 ((iii) of item (4), and (i) and (ii) of item (6)).

3) Assume that $n>2$ and that $\alpha\in(0,1)$. Let $B=(b_{ij})>0$ be a matrix in $\mathbb{C}^{n\times n}$, which satisfies the following conditions:

$$c_1(B)<r_1(B) \text{ and } c_{p+1}(B)<r_{p+1}(B), \tag{6.9}$$

$$b_{ii}>\left(r_i(B)\right)^{\alpha}\left(c_i(B)\right)^{1-\alpha} \text{ for all } i\in\langle n\rangle. \tag{6.10}$$

and

$$r_1^{\langle p\rangle}(B)<b_{11}<r_1(B) \text{ and } r_{p+1}^{\langle n\rangle\setminus\langle p\rangle}(B)<b_{p+1,p+1}<r_{p+1}(B) \tag{6.11}$$

The construction of B is possible by virtue of $n\geq 3$, and

$$c_1(B)\leq\left(r_1(B)\right)^{\alpha}\left(c_1(B)\right)^{1-\alpha}<r_1(B),\ c_{p+1}(B)\leq\left(r_{p+1}(B)\right)^{\alpha}\left(c_{p+1}(B)\right)^{1-\alpha}<r_{p+1}(B). \tag{6.12}$$

We observe that (6.12) follows from $B>0$, (6.9) and $\alpha\in(0,1)$. As $B>0$, we have $B\in\mathbf{IR}(n)$. From (6.10), it is clear that $B\in\mathbf{SRCD}(\alpha;n)$. Also, it follows from (6.11) that $B \notin \mathbf{DD}(\langle p\rangle, \langle n\rangle \setminus \langle p\rangle, n)$. This proves $\mathbf{SRCD}(\alpha; n) \cap \mathbf{IR}(n) \not\subset \mathbf{DD}(\langle p\rangle, \langle n\rangle \setminus \langle p\rangle, n)$. It then follows from Remark 3.1 ((ii) and (iii) of item (4), and (ii) of item (6)) that the remaining statements hold.

Remark 6.3 It is clear that the irreducible matrix A defined in item (1) of Theorem 6.4 satisfies $A \notin \mathbf{D}(n)$. Then from $\mathbf{SD}(n) \cup \mathbf{IRD}(n) \subset \mathbf{D}(n)$, and (ii) and (iii) of item (4) of Remark 3.1, we see that the inclusions in (v) of item (4) of Remark 3.1 and Lemma 3.1 are all proper.

Let $\alpha = 0.5$. In Examples 6.1 and 6.2, we construct a matrix B which satisfies conditions (6.9)-(6.11). The case $\min\{p, n-p\} = 1$ is considered in Example 6.1, while the case $\min\{p, n-p\} \geq 2$ is considered in Example 6.2.

Example 6.1 Let $n \in \mathbb{N} \setminus \{1,2\}$. Define $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ by

$$b_{ij} = \begin{cases} \sqrt{1.8} & \text{if } i = j = 1, \\ 3.3 & \text{if } i = j = 2, \\ \sqrt{3.5} & \text{if } i = j = 3, \\ 2 & \text{if } n > 3,\ j = i, i \notin \langle 3\rangle, \\ 3.25 & \text{if } i = 1,\ j = 3, \\ 1/4 & \text{if } i = 2,\ j = 1, \\ 1/4(n-2) & \text{if } (i,j) \in \{(1,m) : m \in \langle n\rangle \setminus \{1,3\}\} \cup \{(3,m) : m \in \langle n\rangle \setminus \{3\}\}, \\ 13/4(n-2) & \text{if } (i,j) \in \{(2,m) : m \in \langle n\rangle \setminus \langle 2\rangle\}, \\ 1/4(n-2) & \text{if } n > 3, (i,j) \in \{(l,m) : l \in \langle n\rangle \setminus \langle 3\rangle, m \in \langle n\rangle, m \neq l\}. \end{cases}$$

Then $B \in \mathbf{SRCD}(0.5; n) \cap \mathbf{IR}(n)$, and

$$\sqrt{1.8}(3.3 - 3.25) = |b_{11}|(|b_{22}| - r_2(B) + |b_{21}|) < r_1(B)|b_{21}| = (3.5)(0.25).$$

So, $B \notin \mathbf{DD}(\{1\}, \langle n\rangle \setminus \{1\}, n)$. From (i) of item (4) of Remark 3.1, it is clear that $B \notin \mathbf{DD}(\langle n\rangle \setminus \{1\}, \{1\}, n)$.

Example 6.2 Let $n \in \mathbb{N} \setminus \{1,2,3\}$, and let $p \in \langle n\rangle$ be such that $\min\{p, n-p\} \geq 2$. Define $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ by

$$b_{ij} = \begin{cases} \sqrt{1.8} & \text{if } i = j = 1, \\ 3.3 & \text{if } i = j = p+1, \\ \sqrt{3.5} & \text{if } i = j = p+2, \\ 2 & \text{if } i = j, i \in \langle n\rangle \setminus \{1, p+1, p+2\}, \\ 1/8(p-1) & \text{if } i \in \{1, p+1\},\ j \in \langle p\rangle \setminus \{1\}, \\ 1/8(n-p-1) & \text{if } i = 1, j \in \langle n\rangle \setminus (\langle p\rangle \cup \{p+2\}), \\ 3.25 & \text{if } i = 1,\ j = p+2, \\ 0.125 & \text{if } i = p+1,\ j = 1, \\ 3.125 & \text{if } i = p+1,\ j = p+2, \\ 1/8(n-p-2) & \text{if } n-p > 2, i = p+1,\ j \in \{p+3, \cdots, n\}, \\ 1/4(n-1) & \text{if } i = p+2,\ j \in \langle n\rangle \setminus \{p+2\}, \\ 1/8(n-1) & \text{if } (i,j) \in \{(l,m) : l \in \langle n\rangle \setminus \{1, p+1, p+2\}, m \in \langle n\rangle, m \neq l\}. \end{cases}$$

Then $B \in \mathbf{SRCD}(0.5; n) \cap \mathbf{IR}(n)$, and

$$\begin{aligned} (\sqrt{1.8} - 0.125)(3.3 - 3.25) &= \left(|b_{11}| - r_1^{\langle p\rangle}(B)\right)\left(|b_{p+1,p+1}| - r_{p+1}^{\langle n\rangle\setminus\langle p\rangle}(B)\right) \\ &< r_1^{\langle n\rangle\setminus\langle p\rangle}(B)\, r_{p+1}^{\langle p\rangle}(B) = (3.375)(0.25). \end{aligned}$$

So, $B \notin \mathbf{DD}(\langle p \rangle, \langle n \rangle \setminus \langle p \rangle, n)$.

Theorem 6.5 investigates the relations between the class $\{\mathbf{RCD}(\alpha;n), \mathbf{SRCD}(\alpha;n), \mathbf{IRRCD}(\alpha;n)\}$ and each of the classes $\{\mathbf{DD}(n), \mathbf{SDD}(n), \mathbf{IRDD}(n)\}$ and $\{\mathbf{GD}(n), \mathbf{SGD}(n), \mathbf{IRGD}(n)\}$.

Theorem 6.5 *Let* $n \in \mathbb{N} \setminus \{1\}$. *Then*

1) $\mathbf{SGD}(n) \cap \mathbf{SDD}(n) \cap \mathbf{IR}(n) \not\subset \bigcup_{\alpha, 0<\alpha<1} \mathbf{RCD}(\alpha;n)$.

2) $\mathbf{GD}(n) \cap \mathbf{DD}(n) \not\subset \bigcup_{\alpha, 0<\alpha<1} \mathbf{RCD}(\alpha;n)$, $\mathbf{SGD}(n) \cap \mathbf{SDD}(n) \not\subset \bigcup_{\alpha, 0<\alpha<1} \mathbf{SRCD}(\alpha;n)$ and $\mathbf{IRGD}(n) \cap \mathbf{IRDD}(n) \not\subset \bigcup_{\alpha, 0<\alpha<1} \mathbf{IRRCD}(\alpha;n)$.

3) If $\alpha \in (0,1)$ and $n > 2$, then $\mathbf{SRCD}(\alpha;n) \cap \mathbf{IR}(n) \not\subset \mathbf{DD}(n)$, $\mathbf{RCD}(\alpha;n) \not\subset \mathbf{DD}(n)$, $\mathbf{SRCD}(\alpha;n) \not\subset \mathbf{SDD}(n)$ and $\mathbf{IRRCD}(\alpha;n) \not\subset \mathbf{IRDD}(n)$.

4) If $\alpha \in (0,1)$, then $\mathbf{RCD}(\alpha;n) \not\subset \mathbf{GD}(n)$.

Proof. 1) Define the matrix A by

$$A = \begin{pmatrix} 1 & 2 \\ 1.1 & 7 \end{pmatrix}, \tag{6.13}$$

and define A_n by

$$A_n = \begin{cases} A & \text{if } n = 2, \\ \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} & \text{if } n > 2, \end{cases} \tag{6.14}$$

where A is given by (6.13) and $B = (b_{ij}) \in \mathbb{C}^{2\times(n-2)}$ and $C = (c_{ij}) \in \mathbb{C}^{(n-2)\times(n-2)}$ are given by

$$b_{ij} = 1/(n-1) \text{ for } i = 1,2 \text{ and all } j \in \langle n-2 \rangle \tag{6.15}$$

and

$$c_{ij} = \begin{cases} 7 & \text{if } j = i, \\ 1/(n-1) & \text{if } j \neq i. \end{cases} \tag{6.16}$$

It follows from (6.13) - (6.16) that $A_n \in \mathbf{SDD}(n) \cap \mathbf{IR}(n)$, and with Υ_n defined by

$$\Upsilon_n = \begin{cases} \operatorname{diag}(4,1) & \text{if } n = 2, \\ \operatorname{diag}(4,1) \oplus I_{n-2} & \text{if } n > 2, \end{cases}$$

we have $A_n \Upsilon_n \in \mathbf{SD}(n)$. Also, since $1 < (2)^{\alpha} (1.1)^{1-\alpha}$ for all $\alpha \in (0,1)$, we deduce from (6.13)-(6.16) that $A_n \notin \bigcup_{\alpha, 0<\alpha<1} \mathbf{RCD}(\alpha;n)$.

2) The statements follow from item (1), and items (1)-(3) and (6) of Remark 3.1.

3) Let $\alpha \in (0,1)$, and let $n > 2$. Let $B = (b_{ij}) > 0$ be a matrix in $\mathbf{SRCD}(\alpha;n)$, which satisfies the following condition: $\max\{c_i(B), b_{ii}\} < r_i(B)$, $i = 1,2$. (Such matrix exists; see item (3) of Theorem 6.4.) Then $B \in (\mathbf{SRCD}(\alpha;n) \cap \mathbf{IR}(n)) \setminus \mathbf{DD}(n)$. It then follows from items (2), (3) and (6) of Remark 3.1 that the remaining statements also hold.

4) Let $\alpha \in (0,1)$. Define $C = (c_{ij}) \in \mathbb{C}^{n\times n}$ by $c_{12} = 1$ and $c_{ij} = 0$ for all $(i,j) \in \langle n \rangle \times \langle n \rangle \setminus \{(1,2)\}$. Then $C \in \mathbf{RCD}(\alpha;n) \setminus \mathbf{GD}(n)$.

Remark 6.4 Let $n \in \mathbb{N} \setminus \{1\}$. It is clear that the matrix $A \in \mathbf{SGD}(n) \cap \mathbf{SDD}(n) \cap \mathbf{IR}(n)$ defined in item (1) of Theorem 6.5 satisfies $A \notin \mathbf{D}(n)$. Then from $\mathbf{SGD}(n) \subset \mathbf{GD}(n)$, $\mathbf{SD}(n) \subset \mathbf{D}(n)$ and $\mathbf{IRD}(n) \subset \mathbf{D}(n)$, we see that the respective inclusions $\mathbf{D}(n) \subset \mathbf{GD}(n)$, $\mathbf{SD}(n) \subset \mathbf{SGD}(n)$ and $\mathbf{IRD}(n) \subset \mathbf{IRGD}(n)$ in item (1) of Remark 3.1 are all proper.

Remark 6.5 We are not able to determine whether $\mathbf{SRCD}(\alpha;n) \subset \mathbf{SGD}(n)$ or $\mathbf{SRCD}(\alpha;n) \not\subset \mathbf{SGD}(n)$. However, we show in Theorem 6.6 that a subclass of $\mathbf{SRCD}(\alpha;n)$ containing $\mathbf{SRCD}(\alpha;n) \cap \mathbf{IR}(n)$ is indeed a subclass of $\mathbf{SGD}(n)$. The theorem also establishes $\mathbf{IRRCD}(\alpha;n) \subset \mathbf{IRGD}(n)$.

Definition 6.1 Let $n \in \mathbb{N} \setminus \{1\}$, and let $\alpha \in (0,1)$. Define the class $\mathbf{C}(n)$ by

$$\mathbf{C}(n) = \left\{ A \in \mathbb{C}^{n \times n} : r_i(A) c_i(A) > 0 \text{ for all } i \in \langle n \rangle \right\}. \tag{6.17}$$

For every $i \in \langle n \rangle$, define the function $\epsilon_i^{(\alpha)} : \mathbf{C}(n) \to (0,\infty)$ by

$$\epsilon_i^{(\alpha)}(A) = \left(r_i(A)\right)^{\alpha} \left(c_i(A)\right)^{1-\alpha} \tag{6.18}$$

for all $A \in \mathbf{C}(n)$. Also, define the class $\mathbf{C}(\alpha;n)$ by

$$\mathbf{C}(\alpha;n) = \left\{ A \in \mathbf{C}(n) : \operatorname{diag}\left(\frac{1}{\epsilon_1^{(\alpha)}(A)}, \cdots, \frac{1}{\epsilon_n^{(\alpha)}(A)} \right) \left[|A| - |D(A)| \right] \text{ has a positive eigenvector} \right\}. \tag{6.19}$$

To simplify notation, we denote for every $A \in \mathbf{C}(n)$, the matrix $\operatorname{diag}\left(\frac{1}{\epsilon_1^{(\alpha)}(A)}, \cdots, \frac{1}{\epsilon_n^{(\alpha)}(A)} \right) \left[|A| - |D(A)| \right]$ by $A(\alpha)$.

Theorem 6.6 *Let* $n \in \mathbb{N} \setminus \{1\}$, *and let* $\alpha \in (0,1)$. *Then*

1) $\mathbf{IR}(n) \subset \mathbf{C}(\alpha;n)$.

2) If $A = (a_{ij}) \in \mathbf{C}(\alpha;n)$ and $(\gamma_1, \cdots, \gamma_n)^t$ is a positive eigenvector of $A(\alpha)$, then the diagonal matrix $\Upsilon = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)$ satisfies

$$r_i\left(\Upsilon^{-1} A \Upsilon\right) \le \left(r_i(A)\right)^{\alpha} \left(c_i(A)\right)^{1-\alpha} \tag{6.20}$$

for all $i \in \langle n \rangle$.

3) $\mathbf{RCD}(\alpha;n) \cap \mathbf{C}(\alpha;n) \subset \mathbf{GD}(n)$, $\mathbf{SRCD}(\alpha;n) \cap \mathbf{C}(\alpha;n) \subset \mathbf{SGD}(n)$ and $\mathbf{IRRCD}(\alpha;n) \subset \mathbf{IRGD}(n)$, and the three inclusions are proper.

4) $\mathbf{IRRCD}(\alpha;n) \subset \mathbf{GL}(n)$.

Proof. 1) Let $B \in \mathbf{IR}(n)$. Then $r_i(B) c_i(B) > 0$ for all $i \in \langle n \rangle$. Thus from (6.17), we deduce that $B \in \mathbf{C}(n)$. Also, from $B \in \mathbf{IR}(n)$ and (6.18), it is clear that $B(\alpha)$ (see Definition 6.1) is both nonnegative and irreducible. Hence from Theorem 8.4.4 of [26], we infer that $B(\alpha)$ has a positive eigenvector. Then from $B \in \mathbf{C}(n)$ and (6.19), we see that $B \in \mathbf{C}(\alpha;n)$.

2) Let $A = (a_{ij}) \in \mathbf{C}(\alpha;n)$, and let $(\gamma_1, \cdots, \gamma_n)^t$ be a positive eigenvector of $A(\alpha)$. Thus from $A(\alpha) \ge 0$ (see Definition 6.1) and Corollary 8.1.30 of [26], we deduce that the eigenvalue of $A(\alpha)$ corresponding to the positive eigenvector $(\gamma_1, \cdots, \gamma_n)^t$ of $A(\alpha)$ is $\rho(A(\alpha))$. Hence from the definition of $A(\alpha)$, we get

$$\frac{1}{\gamma_i} \sum_{j=1, j \ne i}^{n} |a_{ij}| \gamma_j = \rho(A(\alpha)) \epsilon_i^{(\alpha)}(A) \tag{6.21}$$

for all $i \in \langle n \rangle$. It follows from $\mathbf{SRCD}(n;\alpha) \subset \mathbf{GL}(n)$ (in (3.4)) and (6.18) that for each matrix $C = (c_{ij}) \in \mathbb{C}^{n\times n}$, we have

$$\sigma(C) \subset \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - c_{ii}| \le \epsilon_i^{(\alpha)}(C) \right\}. \tag{6.22}$$

Let $A^{(1)}(\alpha) = \operatorname{diag}\left(\epsilon_1^{(\alpha)}(A), \cdots, \epsilon_n^{(\alpha)}(A)\right)\left[A(\alpha) - \rho(A(\alpha)) I_n\right]$. Then from (6.18) and the definition of $A(\alpha)$, we obtain $\epsilon_i^{(\alpha)}\left(A^{(1)}(\alpha)\right) = \epsilon_i^{(\alpha)}(A)$ for all $i \in \langle n \rangle$. Thus from $A^{(1)}(\alpha)$ being singular (as $\rho(A(\alpha))$ is an eigenvalue of $A(\alpha)$) and (6.22), we infer that there exists $k \in \langle n \rangle$ such that $\left|\left(A^{(1)}(\alpha)\right)_{kk}\right| \le \epsilon_k^{(\alpha)}(A)$. Hence from the definitions of $A(\alpha)$ and $A^{(1)}(\alpha)$, we see that $\rho(A(\alpha)) \le 1$. Then from (6.18) and (6.21), we deduce that with $\Upsilon = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)$, inequality (6.20) holds for all $i \in \langle n \rangle$.

3) We prove $\mathbf{SRCD}(\alpha;n) \cap \mathbf{C}(\alpha;n) \subset \mathbf{SGD}(n)$. The other two set inclusions are proven similarly. Let $A = (a_{ij}) \in \mathbf{SRCD}(\alpha;n) \cap \mathbf{C}(\alpha;n)$. Thus from the definition of $\mathbf{C}(\alpha;n)$, there exists a positive eigenvector $(\gamma_1, \cdots, \gamma_n)^t$ of $A(\alpha)$. Hence from item (2), we infer that, with $\Upsilon = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)$, inequality (6.20) is satisfied for all $i \in \langle n \rangle$. Then from $A \in \mathbf{SRCD}(\alpha;n)$ and $\left(\Upsilon^{-1} A \Upsilon\right)_{ii} = a_{ii}$, $i = 1, \cdots, n$, we see that $\Upsilon^{-1} A \Upsilon \in \mathbf{SD}(n)$. Thus from Lemma 3.2, we deduce that $A \in \mathbf{SGD}(n)$.

It follows from Remark 3.1 (item (1), and (ii) of item (6)) that in order to show that the three set inclusions are proper, it suffices to show that there exists $A_n \in \mathbf{SGD}(n) \cap \mathbf{IR}(n)$ such that $A_n \notin \mathbf{RCD}(\alpha;n)$. Define $A_n = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } j = i = 1, \\ 3 & \text{if } j = i, i \ne 1, \\ 1.5/(n-1) & \text{if } j \ne i. \end{cases} \tag{6.23}$$

Hence $A_n \notin \mathbf{RCD}(\alpha;n)$. Also, with $\Upsilon = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)$, $\gamma_1 = 1.6$ and $\gamma_n = 1$ for all $n \ge 2$, we deduce from (6.23) that $A_n \Upsilon_n \in \mathbf{SD}(n)$. So, from $A \in \mathbf{IR}(n)$, the result follows.

4) The result follows from $\mathbf{IRRCD}(\alpha;n) \subset \mathbf{IRGD}(n)$ in item (3), and item (1) of Lemma 3.3.

We now provide sufficient conditions for matrices in the classes $\mathbf{DD}(n)$, $\mathbf{SDD}(n)$ and $\mathbf{IRDD}(n)$ to be in the classes $\mathbf{RCD}(\alpha;n)$, $\mathbf{SRCD}(\alpha;n)$ and $\mathbf{IRRCD}(\alpha;n)$, respectively. It follows from Theorems 6.2 and 6.3 that in order to establish such conditions, it suffices to consider matrices in the smaller classes $\mathbf{DD}(n) \setminus \mathbf{D}(n)$, $\mathbf{SDD}(n) \setminus \mathbf{SD}(n)$ and $\mathbf{IRDD}(n) \setminus \mathbf{IRD}(n)$. The integer n in Theorems 6.7-6.9 is assumed to satisfy $n \ge 2$.

Theorem 6.7 *Let* $A = (a_{ij}) \in \mathbf{DD}(n) \setminus \mathbf{D}(n)$, *and let l be the integer in* $\langle n \rangle$, *which satisfies* $\mathcal{L}(A) = \{l\}$. *Assume that A satisfies the following two conditions*:

Condition (1): Every $k \in \langle n \rangle \setminus \{l\}$ satisfies (6.7).

Condition (2): Either $c_l(A) = 0$, or $r_l(A) \ge 1$ and $0 < c_l(A) < \min\{|a_{ll}|, 1\}$.

Then there exists $\alpha_0 \in (0,1)$ such that $A \in \bigcap_{\alpha, 0<\alpha\le\alpha_0} \mathbf{RCD}(\alpha;n)$.

Proof. Since $|a_{kk}| \ge r_k(A)$ for all $k \in \langle n \rangle \setminus \{l\}$, we deduce from condition (1) and item (1) of Theorem 6.2 that

$$|a_{kk}| \ge (r_k(A))^{\alpha}(c_k(A))^{1-\alpha} \tag{6.24}$$

for all $k \in \langle n \rangle \setminus \{l\}$ and $\alpha \in (0,1)$. It is clear that

$$c_l(A) = 0 \Rightarrow |a_{ll}| \ge (r_l(A))^{\alpha}(c_l(A))^{1-\alpha}, \forall \alpha \in (0,1). \tag{6.25}$$

Then from condition (2), it remains to consider the case in which l satisfies the conditions $r_l(A) \ge 1$ and $0 < c_l(A) < \min\{|a_{ll}|, 1\}$. In this case, we infer from Remark 6.2 that there exist $\alpha_1 \in (0,1)$ and $\alpha_0 \in (0, \alpha_1)$ such that

$$0 < (c_l(A))^{1-\alpha_1} < |a_{ll}| \text{ and } (r_l(A))^{\alpha} < \frac{|a_{ll}|}{(c_l(A))^{1-\alpha_1}} \tag{6.26}$$

for all $\alpha \in [0, \alpha_0]$. Since $c_l(A) \in (0,1)$, we see from Remark 6.2 that the function $f(\alpha) = (c_l(A))^{1-\alpha}$, $0 \le \alpha \le 1$, is an increasing function. Thus from (6.26) and $0 < \alpha_0 < \alpha_1 < 1$, we see that $|a_{ll}| > (r_l(A))^{\alpha}(c_l(A))^{1-\alpha}$ for all $\alpha \in [0, \alpha_0]$. Hence from (6.24) and (6.25), the result follows.

The following two theorems are proven similarly as Theorem 6.7.

Theorem 6.8 *Let* $A = (a_{ij}) \in \mathbf{SDD}(n) \setminus \mathbf{SD}(n)$, *and let m be the integer in* $\langle n \rangle$, *which satisfies* $\mathcal{J}(A) = \langle n \rangle \setminus \{m\}$. *Assume that A satisfies the following two condition*:

Condition (1): Every $k \in \langle n \rangle \setminus \{m\}$ satisfies (6.7).

Condition (2): Either $c_m(A) = 0$, or $r_m(A) \ge 1$ and $0 < c_m(A) < \min\{|a_{mm}|, 1\}$.

Then there exists $\alpha_0 \in (0,1)$ such that $A \in \bigcap_{\alpha, 0<\alpha\le\alpha_0} \mathbf{SRCD}(\alpha; n)$.

It follows from item (4) of Lemma 3.3 that $\mathbf{IRDD}(n) \setminus \mathbf{IRD}(n) = \mathbf{IRDD}(n) \setminus \mathbf{D}(n)$. We use this fact in the following theorem. We omit the proof.

Theorem 6.9 *Let* $n \in \mathbb{N} \setminus \{1\}$, *and let* $A = (a_{ij}) \in \mathbf{IRDD}(n) \setminus \mathbf{D}(n)$. *Suppose that l is the integer in* $\langle n \rangle$, *which satisfies* $\mathcal{L}(A) = \{l\} = \langle n \rangle \setminus \mathcal{J}(A)$. *Assume that A satisfies the following two conditions*:

Condition (1): For every $k \in \langle n \rangle \setminus \{l\}$, we have $r_k(A) \ge 1 \ge c_k(A) > 0$.

Condition (2): $r_l(A) \ge 1$ and $0 < c_l(A) < \min\{|a_{ll}|, 1\}$.

Then there exists $\alpha_0 \in (0,1)$ such that

$$A \in \mathbf{IR}(n) \cap \left(\bigcap_{\alpha, 0<\alpha\le\alpha_0} \mathbf{SRCD}(\alpha; n) \right).$$

In particular, $A \in \bigcap_{\alpha, 0<\alpha\le\alpha_0} \mathbf{IRRCD}(\alpha; n)$.

Remark 6.6 If $c_m = 0$ in Theorem 6.8, the fact: $|a_{mm}| > (r_m(A))^{\alpha}(c_m(A))^{1-\alpha}$ for all $\alpha \in (0,1)$ follows from $|a_{mm}| > 0$ (as $A \in \mathbf{SDD}(n)$).

Remark 6.7 1) Theorems 5.4 and 6.6 could be used to establish sufficient conditions for matrices in $\mathbf{RCD}(\alpha; n)$ to be in $\mathbf{DD}(S_1, S_2, n)$.

2) Theorem 4.5, item (2) of Remark 4.2, and Theorems 6.7-6.9 could be used to present sufficient conditions for matrices in $\mathbf{DD}(S_1, S_2, n)$, $\mathbf{SDD}(S_1, S_2, n)$ and $\mathbf{IRDD}(S_1, S_2, n)$ to be in $\mathbf{RCD}(\alpha; n)$, $\mathbf{SRCD}(\alpha; n)$ and $\mathbf{IRRCD}(\alpha; n)$, respectively.

3) Theorems 4.5, 5.4 and 6.6 could be used to provide sufficient conditions for matrices in $\mathbf{RCD}(\alpha;n)$ to be in $\mathbf{DD}(n)$.

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