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Applications of Arithmetic Geometric Mean Inequality

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Abstract

The well-known arithmetic-geometric mean inequality for singular values, due to Bhatia and Kittaneh, is one of the most important singular value inequalities for compact operators. The purpose of this study is to give new singular value inequalities for compact operators and prove that these inequalities are equivalent to arithmetic-geometric mean inequality, the way by which several future studies could be done.

Keywords

Compact Operator, Inequality, Positive Operator, Singular Value

1. Fundamental Principles

Let $B(H)$ indicate the set of all bounded linear operators on a complex separable Hilbert space *H*, and let $K(H)$ indicate the two-sided ideal of compact operators in $B(H)$. If $T \in K(H)$, the singular values of *T*, denoted by $s_1(T), s_2(T), \cdots$ are the eigenvalues of the positive operator $|T| = (T^*T)^{1/2}$ ordered as $s_1(T) \geq s_2(T) \geq \cdots$ and repeated according to multiplicity. It is well known that $s_j(T) = s_j(T^*) = s_j(|T|)$ for $j = 1, 2, \cdots$. It follows by Weyl's monotonicity principle (see, e.g., [1], p. 63 or [2], p. 26) that if $S, T \in K(H)$ are positive and $S \leq T$, then $s_j(S) \leq s_j(T)$ for $j = 1, 2, \cdots$. Moreover, for $S, T \in K(H), s_j(S) \leq s_j(T)$ if and only if $s_j(S \oplus S) \leq s_j(T \oplus T)$ for $j = 1, 2, \cdots$. Here, we use the direct sum notation $S \oplus T$ for the block-diagonal operator $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$ defined on $H \oplus H$. The singular values of $S \oplus T$ and $\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}$ are the same, and they consist of those of *S* together with those of *T*.

Bhatia and Kittaneh have proved in [3] that if $A, B \in K(H)$ such that A is self-adjoint, $B \geq 0$, and $\pm A \leq B$, then

$$s_j(A) \leq s_j(B \oplus B) \tag{1.1}$$

for $j = 1, 2, \cdots$.

Audeh and Kittaneh in [4] prove inequality which is equivalent to inequality (1.1):

If $A, B, C \in K(H)$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$s_j(B) \leq s_j(A \oplus C) \tag{1.2}$$

for $j = 1, 2, \cdots$.

The well-known arithmetic-geometric mean inequality for singular values, due to Bhatia and Kittaneh [5], says that if $A, B \in K(H)$, then

$$2s_j(AB^*) \leq s_j(A^*A + B^*B) \tag{1.3}$$

for $j = 1, 2, \cdots$. On the other hand, Zhan has proved in [6] that if $A, B \in K(H)$ are positive, then

$$s_j(A - B) \leq s_j(A \oplus B) \tag{1.4}$$

for $j = 1, 2, \cdots$. Moreover, Tao has proved in [7] that if $A, B, C \in K(H)$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$2s_j(B) \leq s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{1.5}$$

for $j = 1, 2, \cdots$.

Audeh and Kittaneh have proved in [4] that:

If $A, B \in K(H)$ such that A is self-adjoint, $B \geq 0$, and $\pm A \leq B$, then

$$2s_j(A) \leq s_j((B + A) \oplus (B - A)) \tag{1.6}$$

for $j = 1, 2, \cdots$.

It has been pointed out in [4] that the four inequalities (1.3)-(1.6) are equivalent.

Moreover, Tao in [7] uses inequality (1.3) to prove that if A and B are positive operators in $K(H)$, $r \geq 0$. Then

$$2s_j\left(A^{1/2}(A + B)^r B^{1/2}\right) \leq s_j(A + B)^{r+1} \tag{1.7}$$

for $j = 1, 2, \cdots$.

2. Introduction

In this study, we will present several new inequalities, and prove that they are equivalent to arithmetic-geometric mean inequality.

The following are the proved inequalities in this study:

Let A, C and D be operators in $K(H)$ where $A \geq 0$, C and D arbitrary operators. Then

$$2s_j(CAD^*) \leq s_j \begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix} \tag{2.1}$$

for $j=1,2,\cdots$.

Let C,D and X be arbitrary operators in $K(H)$. Then we have

$$2s_j\left(DXC^*\right)\le s_j\begin{bmatrix} D\left|X^*\right|D^* & DXC^* \\ CX^*D^* & C\left|X\right|C^* \end{bmatrix} \tag{2.2}$$

for $j=1,2,\cdots$.

Let A_1,A_2,A_3,A_4 be operators in $K(H)$. Then

$$2s_j\begin{bmatrix} A_1A_2^* & A_1A_4^* \\ A_3A_2^* & A_3A_4^* \end{bmatrix}\le s_j\left(\left|A_1\right|^2+\left|A_2\right|^2+\left|A_3\right|^2+\left|A_4\right|^2\right) \tag{2.3}$$

for $j=1,2,\cdots$.

If A,B,C,D and X are operators in $K(H)$. Then

$$\begin{aligned} &2s_j\left(A\left|X\right|C^*\pm BXC^*\pm AX^*D^*+B\left|X^*\right|D^*\right)\\ &\le s_j\begin{bmatrix} A\left|X\right|A^*\pm BXA^*\pm AX^*B^*+B\left|X^*\right|B^* & A\left|X\right|C^*\pm BXC^*\pm AX^*D^*+B\left|X^*\right|D^* \\ C\left|X\right|A^*\pm DXA^*\pm CX^*B^*+D\left|X^*\right|B^* & C\left|X\right|C^*\pm DXC^*\pm CX^*D^*+D\left|X^*\right|D^* \end{bmatrix} \end{aligned} \tag{2.4}$$

for $j=1,2,\cdots$.

Let $A_1,A_2,\cdots,A_n$ be positive operators in $K(H), r\ge 0.$ Then

$$\begin{aligned} &2s_j\left[A_1^{1/2}\left(A_1+A_n\right)^r A_n^{1/2}\oplus A_2^{1/2}\left(A_2+A_{n-1}\right)^r A_{n-1}^{1/2}\oplus\cdots\oplus A_n^{1/2}\left(A_n+A_1\right)^r A_1^{1/2}\right]\\ &\le s_j\left[\left(A_1+A_n\right)^{r+1}\oplus\left(A_2+A_{n-1}\right)^{r+1}\oplus\cdots\oplus\left(A_n+A_1\right)^{r+1}\right] \end{aligned} \tag{2.5}$$

for $j=1,2,\cdots$.

3. Main Results

Our first singular value inequality needs the following lemma.

Lemma 1: *Let A be a positive operator in $K(H)$, X be an arbitrary operator in $K(H)$. Then we have*

$$XAX^*\ge 0 \tag{3.1}$$

Now we will prove the first Theorem which is equivalent to arithmetic-geometric mean inequality.

Theorem 3.1 *Let A,C and D be operators in $K(H)$ where $A\ge 0$, C and D arbitrary operators. Then*

$$2s_j\left(CAD^*\right)\le s_j\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix}$$

for $j=1,2,\cdots$.

Proof. Let $X=\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\ge 0$ (because $A\ge 0$ by assumption), and let $M=\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}$. Then we have

$$MXM^*=\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} C^* & D^* \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix}\ge 0$$

From (1.5) we have

$$2s_j\left(CAD^*\right)\le s_j\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix}$$

for $j=1,2,\cdots$.

Now we will prove that Theorem (3.1) is equivalent to arithmetic-geometric mean inequality.

Theorem 3.2 *The following statements are equivalent*:

1) Let $X,Y\in K(H)$, then

$$2s_j\left(XY^*\right)\le s_j\left(X^*X+Y^*Y\right)$$

for $j=1,2,\cdots$.

2) Let A,C and D be operators in $K(H)$ where $A\ge 0$, C and D arbitrary operators. Then

$$2s_j\left(CAD^*\right)\le s_j\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix}$$

for $j=1,2,\cdots$.

Proof. 1) $\to$ 2) Let $X=CA^{1/2},Y=DA^{1/2}$.

Now apply arithmetic-geometric mean inequality to get

$$2s_j\left(CA^{1/2}A^{1/2}D^*\right)\le s_j\left(A^{1/2}C^*CA^{1/2}+A^{1/2}Y^*YA^{1/2}\right)$$

for $j=1,2,\cdots$. But

$$\begin{aligned} s_j\left(A^{1/2}C^*CA^{1/2}+A^{1/2}Y^*YA^{1/2}\right) &= s_j\begin{bmatrix} A^{1/2}C^* & A^{1/2}D^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} CA^{1/2} & 0 \\ DA^{1/2} & 0 \end{bmatrix} \\ &= s_j\begin{bmatrix} CA^{1/2} & 0 \\ DA^{1/2} & 0 \end{bmatrix}\begin{bmatrix} A^{1/2}C^* & A^{1/2}D^* \\ 0 & 0 \end{bmatrix} \\ &= s_j\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix}. \end{aligned}$$

The above steps implies that

$$2s_j\left(CAD^*\right)\le s_j\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix} \quad \text{for} \quad j=1,2,\cdots.$$

2) $\to$ 1) The matrix $\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix}$ can be factorized as $\begin{bmatrix} CA^{1/2} & 0 \\ DA^{1/2} & 0 \end{bmatrix}\begin{bmatrix} A^{1/2}C^* & A^{1/2}D^* \\ 0 & 0 \end{bmatrix}$, but it is well known that $s_j(T)=s_j\left(T^*\right)=s_j\left(|T|\right)$ for $j=1,2,\cdots$. So

$$\begin{aligned} s_j\begin{bmatrix} CAC^* & CAD^* \\ DAC^* & DAD^* \end{bmatrix} &= s_j\begin{bmatrix} CA^{1/2} & 0 \\ DA^{1/2} & 0 \end{bmatrix}\begin{bmatrix} A^{1/2}C^* & A^{1/2}D^* \\ 0 & 0 \end{bmatrix} \\ &= s_j\begin{bmatrix} A^{1/2}C^* & A^{1/2}D^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} CA^{1/2} & 0 \\ DA^{1/2} & 0 \end{bmatrix} \\ &= s_j\left(A^{1/2}C^*CA^{1/2}+A^{1/2}D^*DA^{1/2}\right) \end{aligned}$$

for $j = 1, 2, \cdots$, from (2) we have

$$2s_j\left(CAD^*\right) \le s_j\left(A^{1/2}C^*CA^{1/2} + A^{1/2}D^*DA^{1/2}\right) \quad (3.2)$$

for $j = 1, 2, \cdots$. Now let $A = I$ in Inequality (3.2) we get

$$2s_j\left(CD^*\right) \le s_j\left(C^*C + D^*D\right) \quad (3.3)$$

for $j = 1, 2, \cdots$, which is the arithmetic-geometric mean inequality.

The following lemma which was proved by Bhatia [1] is essential to prove the next theorem.

Lemma 2 *Let* X *be arbitrary operator in* $B(H)$. *Then*

$$\begin{bmatrix} |X| & \pm X^* \\ \pm X & |X^*| \end{bmatrix} \ge 0 \quad (3.4)$$

Now we will prove the following theorem which is more general than Theorem (3.1) and equivalent to arithmetic-geometric mean inequality.

Theorem 3.3 *Let* C, D *and* X *be arbitrary operators in* $K(H)$. *Then we have*

$$2s_j\left(DXC^*\right) \le s_j\begin{bmatrix} D|X^*|D^* & DXC^* \\ CX^*D^* & C|X|C^* \end{bmatrix}$$

for $j = 1, 2, \cdots$.

Proof. Applying Lemma (2) gives $A = \begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \ge 0$ for an arbitrary operator X. Let $N = \begin{bmatrix} 0 & D \\ C & 0 \end{bmatrix}$, by using Inequality (3.1) we have $NAN^* = \begin{bmatrix} D|X^*|D^* & DXC^* \\ CX^*D^* & C|X|C^* \end{bmatrix} \ge 0$. Hence using Inequality (1.5) gives $2s_j\left(DXC^*\right) \le s_j\begin{bmatrix} D|X^*|D^* & DXC^* \\ CX^*D^* & C|X|C^* \end{bmatrix}$.

Remark 1 *Theorem* (3.3) *is generalization of Theorem* (3.1) *because here X is arbitrary operator but there A should be positive operator.*

Remark 2 *Inequality* (2.2) *is equivalent to arithmetic-geometric mean inequality. We can prove this equivalent by similar steps used to prove Theorem* (3.2).

The following theorem is a generalization of Theorem (3.1) and Theorem (3.3).

Theorem 3.4 *Let* A, B, C, D *and* X *be arbitrary operators in* $K(H)$. *Then we have*

$$\begin{aligned} &2s_j\left(A|X|C^* \pm BXC^* \pm AX^*D^* + B|X^*|D^*\right) \\ &\le s_j\begin{bmatrix} A|X|A^* \pm BXA^* \pm AX^*B^* + B|X^*|B^* & A|X|C^* \pm BXC^* \pm AX^*D^* + B|X^*|D^* \\ C|X|A^* \pm DXA^* \pm CX^*B^* + D|X^*|B^* & C|X|C^* \pm DXC^* \pm CX^*D^* + D|X^*|D^* \end{bmatrix} \end{aligned}$$

for $j = 1, 2, \cdots$.

Proof. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, Z = \begin{bmatrix} |X| & \pm X^* \\ \pm X & |X^*| \end{bmatrix}$. Then

$MZM^* = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} |X| & \pm X^* \\ \pm X & |X^*| \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \geq 0.$ Hence

$$\begin{bmatrix} A|X|A^* \pm BXA^* \pm AX^*B^* + B|X^*|B^* & A|X|C^* \pm BXC^* \pm AX^*D^* + B|X^*|D^* \\ C|X|A^* \pm DXA^* \pm CX^*B^* + D|X^*|B^* & C|X|C^* \pm DXC^* \pm CX^*D^* + D|X^*|D^* \end{bmatrix} \geq 0,$$

use Inequality (1.5) to get the required result.

Remark 3 *Replace B, D by* 0 *in Inequality* (2.4) *will gives Inequality* (2.1).

Remark 4 *Replace A, C by* 0 *in Inequality* (2.4) *will also gives Inequality* (2.1).

Now we will use Inequality (1.3) to prove the following theorem, then we will show that they are equivalent.

Theorem 3.5 *Let* A_1, A_2, A_3, A_4 *be operators in* $K(H)$. *Then*

$$2s_j \begin{bmatrix} A_1A_2^* & A_1A_4^* \\ A_3A_2^* & A_3A_4^* \end{bmatrix} \leq s_j \left(|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 \right)$$

for $j = 1, 2, \cdots$.

Proof. Let $A = \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix}, B = \begin{bmatrix} A_2 & 0 \\ A_4 & 0 \end{bmatrix}$. Then $AB^* = \begin{bmatrix} A_1A_2^* & A_1A_4^* \\ A_3A_2^* & A_3A_4^* \end{bmatrix}$, and $A^*A + B^*B = |A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2$. Now use Inequality (1.3) we get

$$2s_j \begin{bmatrix} A_1A_2^* & A_1A_4^* \\ A_3A_2^* & A_3A_4^* \end{bmatrix} \leq s_j \left(|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 \right)$$

for $j = 1, 2, \cdots$.

Now we will prove that Inequality (2.3) is equivalent to Inequality (1.3).

Theorem 3.6 *The following statements are equivalent*:

1) Let $A, B \in K(H)$. Then

$$2s_j \left(AB^* \right) \leq s_j \left(A^*A + B^*B \right)$$

for $j = 1, 2, \cdots$.

2) Let A_1, A_2, A_3, A_4 be operators in $K(H)$. Then

$$2s_j \begin{bmatrix} A_1A_2^* & A_1A_4^* \\ A_3A_2^* & A_3A_4^* \end{bmatrix} \leq s_j \left(|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2 \right)$$

for $j = 1, 2, \cdots$.

Proof. 1) $\to$ 2) It is the proof of Theorem (3.5).

2) $\to$ 1) By replacing $A_2 = A_4 = B$ and $A_1 = A_3 = A$ in Inequality (2.3), we get $2s_j \begin{bmatrix} AB^* & AB^* \\ AB^* & AB^* \end{bmatrix} \leq s_j \left(|A|^2 + |B|^2 + |A|^2 + |B|^2 \right)$. From this we reach to $4s_j \left(AB^* \right) \leq 2s_j \left(A^*A + B^*B \right)$ which implies that $2s_j \left(AB^* \right) \leq s_j \left(A^*A + B^*B \right)$ for $j = 1, 2, \cdots$.

In the rest of this paper, we will prove new inequality which is equivalent to Inequality (1.7).

Theorem 3.7 *Let* $A_1, A_2, \cdots, A_n$ *be positive operators in* $K(H)$, *n is an even integer,* $r \geq 0$. *Then*

$$\begin{aligned}&2s_j\left[A_1^{1/2}(A_1+A_n)^r A_n^{1/2} \oplus A_2^{1/2}(A_2+A_{n-1})^r A_{n-1}^{1/2} \oplus \cdots \oplus A_n^{1/2}(A_n+A_1)^r A_1^{1/2}\right] \\ &\leq s_j\left[(A_1+A_n)^{r+1} \oplus (A_2+A_{n-1})^{r+1} \oplus \cdots \oplus (A_n+A_1)^{r+1}\right]\end{aligned} \tag{3.5}$$

for $j = 1, 2, \cdots$.

Proof. Let $C = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$, $D = \begin{bmatrix} A_n & 0 & \cdots & 0 \\ 0 & A_{n-1} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & A_1 \end{bmatrix}$. Then we have

$$C^{1/2}(C+D)^r D^{1/2} = \begin{bmatrix} A_1^{1/2}(A_1+A_n)^r A_n^{1/2} & 0 & \cdots & 0 \\ 0 & A_2^{1/2}(A_2+A_{n-1})^r A_{n-1}^{1/2} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^{1/2}(A_n+A_1)^r A_1^{1/2} \end{bmatrix},$$

and $(C+D)^{r+1} = \begin{bmatrix} (A_1+A_n)^{r+1} & 0 & \cdots & 0 \\ 0 & (A_2+A_{n-1})^{r+1} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & (A_n+A_1)^{r+1} \end{bmatrix}$. Now apply Inequality (1.7) we get the result.

We will prove that Inequality (1.7) is equivalent to Inequality (3.5).

Theorem 3.8 *The following statements are equivalent*:

1) Let A and B be positive operators in $K(H)$, $r \geq 0$. Then

$$2s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \leq s_j (A+B)^{r+1}$$

for $j = 1, 2, \cdots$.

2) Let $A_1, A_2, \cdots, A_n$ be positive operators in $K(H)$, *n* is even integer, $r \geq 0$. Then

$$\begin{aligned}&2s_j\left[A_1^{1/2}(A_1+A_n)^r A_n^{1/2} \oplus A_2^{1/2}(A_2+A_{n-1})^r A_{n-1}^{1/2} \oplus \cdots \oplus A_n^{1/2}(A_n+A_1)^r A_1^{1/2}\right] \\ &\leq s_j\left[(A_1+A_n)^{r+1} \oplus (A_2+A_{n-1})^{r+1} \oplus \cdots \oplus (A_n+A_1)^{r+1}\right]\end{aligned}$$

for $j = 1, 2, \cdots$.

Proof. 1) $\to$ 2) This implication follows from the proof of Theorem 3.7.

2) $\to$ 1) Let $A_2 = A_3 = \cdots = A_{n-1} = 0$ in Inequality (3.5) to get

$$2s_j\left[A_1^{1/2}(A_1+A_n)^r A_n^{1/2} \oplus A_n^{1/2}(A_n+A_1)^r A_1^{1/2}\right] \leq s_j\left[(A_1+A_n)^{r+1} \oplus (A_n+A_1)^{r+1}\right]$$

for $j = 1, 2, \cdots$. But $s_j(X^*) = s_j(X)$ and $s_j(X \oplus X) \leq s_j(Y \oplus Y)$ for $j = 1, 2, \cdots$.

If and only if $s_j(X) \leq s_j(Y)$, this gives

$$2s_j\left[A_1^{1/2}(A_1+A_n)^r A_n^{1/2}\right] \leq s_j (A_1+A_n)^{r+1}$$

for $j = 1, 2, \cdots$, replace A_1 by A, A_n by B in this inequality we will get

$$2s_j\left(A^{1/2}\left(A+B\right)^r B^{1/2}\right)\le s_j\left(A+B\right)^{r+1}$$

for $j=1,2,\cdots$.

4. Conclusion

Since this study has been completed, we can conclude that several singular value inequalities for compact operators are equivalent to arithmetic-geometric mean inequality, which in turns have many crucial applications in operator theory, and from this point we advise interested authors to join these results with results in other studies to make connection between several branches in operator theory.

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