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Minimum Covering Randić Energy of a Graph

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Abstract

Randić energy was first defined in the paper [1]. Using minimum covering set, we have introduced the minimum covering Randić energy $RE_C(G)$ of a graph G in this paper. This paper contains computation of minimum covering Randić energies for some standard graphs like star graph, complete graph, thorn graph of complete graph, crown graph, complete bipartite graph, cocktail graph and friendship graphs. At the end of this paper, upper and lower bounds for minimum covering Randić energy are also presented.

Keywords

Minimum Covering Set, Minimum Covering Randić Matrix, Minimum Covering Randić Eigenvalues, Minimum Covering Randić Energy

1. Introduction

Study on energy of graphs goes back to the year 1978, when I. Gutman [2] defined this while working with energies of conjugated hydrocarbon containing carbon atoms. All graphs considered in this paper are assumed to be simple without loops and multiple edges. Let $A=(a_{ij})$ be the adjacency matrix of the graph G with its eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ assumed in decreasing order. Since A is real symmetric, the eigenvalues of G are real numbers whose sum equal to zero. The sum of the absolute eigenvalues values of G is called the energy $E(G)$ of *G. i.e.*,

$$E(G)=\sum_{i=1}^{n}|\lambda_i|.$$

Theories on the mathematical concepts of graph energy can be seen in the reviews[3], papers [4] [5] [6] and the references cited there in. For various upper and lower bounds for energy of a graph can be found in papers[7] [8] and it was observed that graph energy has chemical applications in the molecular orbital theory of conjugated molecules [9] [10].

1.1. Randić Energy

It was in the year 1975, Milan Randić invented a molecular structure descriptor called Randić index which is defined as [11]

$$R(G) = \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{d_i d_j}}$$

Motivated by this S.B. Bozkurt *et al.* [1] defined Randić matrix and Randić energy as follows. Let G be graph of order n with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Randić matrix of G is a $n \times n$ symmetric matrix defined by $R(G) := (r_{ij})$, where

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

The characteristic equation of $R(G)$ is defined by $f_n(G, \rho) = \det(\rho I - R(G)) = 0$. The roots of this equation is called Randić eigenvalues of G. Since $R(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in decreasing order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Randić energy of G is defined as

$$RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

Further studies on Randić energy can be seen in the papers [12] [13] [14] and the references cited there in.

1.2. Minimum Covering Energy

In the year 2012 C Adiga *et al.* [15] introduced minimum covering energy of a graph, which depends on its particular minimum cover. A subset C of vertex set V is called a covering set of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a minimum covering set. If C is a minimum covering set of a graph G then the minimum covering matrix of G is the $n \times n$ matrix defined by $A_C(G) := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise} \end{cases}$$

The minimum covering eigenvalues of the graph G are roots of the characteristic equation $f_n(G, \lambda) = \det(\lambda I - A_C(G)) = 0$, obtained from the matrix $A_C(G)$. Since $A_C(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in the order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum covering energy of G is defined as $E_C(G) = \sum_{i=1}^{n} |\lambda|_i$.

1.3. Minimum Covering Randić Energy

Results on Randić energy and minimum covering energy of graph G motivates us to define minimum covering Randić energy. Consider a graph G with vertex set

$V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. If C is a minimum covering set of a graph G then the minimum covering Randić matrix of G is the $n \times n$ matrix defined by $R_C(G) := (r_{ij})$, where

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i v_j \in E(G) \\ 1 & \text{if } i = j \text{ and } \upsilon_i \in C \\ 0 & \text{otherwise} \end{cases}.$$

The characteristic polynomial of $R_C(G)$ is defined by $f_n(G, \rho) = \det(\rho I - R_C(G))$. The minimum covering Randić eigenvalues of the graph G are the eigenvalues of $R_C(G)$. Since $R_C(G)$ is real and symmetric matrix so its eigenvalues are real numbers. We label the eigenvalues in order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The minimum covering Randić energy of G is defined as $RE_C(G) = \sum_{i=1}^{n} |\rho_i|$.

Example 1: i) $C_1 = \{v_1, v_2, v_5\}$ ii) $C_2 = \{v_2, v_4, v_5\}$ are the possible minimum covering sets for the **Figure 1** as shown below.

i) $$R_{C_1}(G) = \begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{15}} & 0 \\ \frac{1}{3} & 1 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{15}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 1 & \frac{1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}$$

Minimum covering Randić eigenvalues are

$$\rho_1 \approx -0.3143, \rho_2 \approx -0.1937, \rho_3 \approx -0.0646, \rho_4 \approx 0.8604, \rho_5 \approx 0.9108, \rho_6 \approx 1.8014$$

Minimum covering Randić energy, $RE_{C_1}(G) \approx 4.1453$.

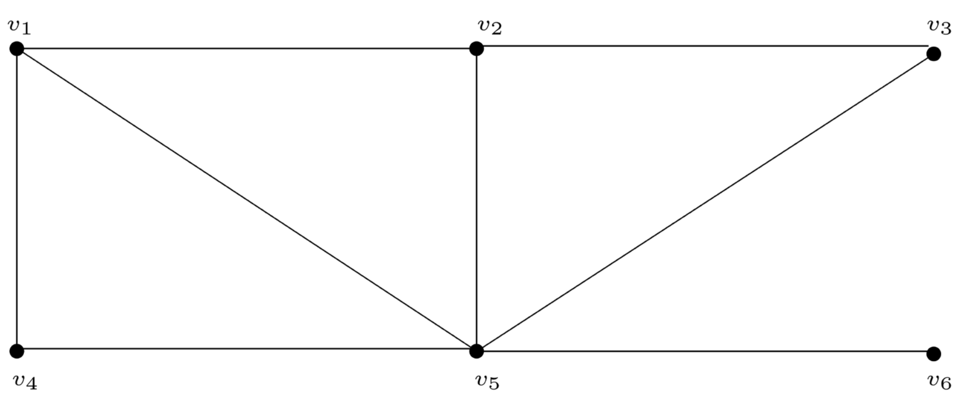

Figure 1. Minimum covering Randić energy depends on the covering set.

ii) $$R_{C_2}(G)=\begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{15}} & 0 \\ \frac{1}{3} & 1 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{15}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & 1 & \frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 1 & \frac{1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}$$

Minimum covering Randić eigenvalues are

$\rho_1 \approx -0.3515, \rho_2 \approx -0.1788, \rho_3 \approx -0.0611, \rho_4 \approx 0.7699, \rho_5 \approx 1.0767, \rho_6 \approx 1.7488$.

Minimum covering Randić energy, $RE_{C_2}(G) \approx 4.1828$.

Therefore minimum covering Randić energy depends on the covering set.

2. Main Results and Discussion

2.1. Minimum Covering Randić Energy of Some Standard Graphs

Theorem 2.1 *For* $n \geq 2$, *the minimum covering Randić energy,* $R_C(G)$ *of complete graph* K_n *is* $\frac{(n-2)^2+\sqrt{4n^2-8n+5}}{n-1}$.

Proof. Let K_n be a complete graph with vertex set $V=\{v_1,v_2,\cdots,v_n\}$. The minimum covering set for K_n is $C=\{v_1,v_2,v_3,\cdots,v_{n-1}\}$. Then

$$R_C(K_n)=\begin{pmatrix} 1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & 1 & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 1 & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 1 & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \end{pmatrix}_{n\times n}$$

Characteristic polynomial is $\frac{(-1)^n\left[(n-1)\rho-(n-2)\right]^{n-2}\left[(n-1)\rho^2-(2n-3)\rho-1\right]}{(n-1)^{n-1}}$

Characteristic equation is $(-1)^n\left[(n-1)\rho-(n-2)\right]^{n-2}\left[(n-1)\rho^2-(2n-3)\rho-1\right]=0$

Minimum covering Randić Spec

$$\left(K_n\right)=\begin{pmatrix} \frac{n-2}{n-1} & \frac{(2n-3)+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{(2n-3)-\sqrt{4n^2-8n+5}}{2(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}$$

Minimum covering Randić energy,

$$RE_C\left(K_n\right)=\left|\frac{n-2}{n-1}\right|(n-2)+\left|\frac{(2n-3)+\sqrt{4n^2-8n+5}}{2(n-1)}\right|(1)+\left|\frac{(2n-3)-\sqrt{4n^2-8n+5}}{2(n-1)}\right|(1)$$

$$=\frac{(n-2)^2}{n-1}+\frac{(2n-3)+\sqrt{4n^2-8n+5}}{2(n-1)}+\frac{\sqrt{4n^2-8n+5}-(2n-3)}{2(n-1)}$$

$$=\frac{(n-2)^2}{n-1}+\frac{\sqrt{4n^2-8n+5}}{(n-1)}=\frac{(n-2)^2+\sqrt{4n^2-8n+5}}{n-1}.$$

Definition 2.1 *Thorn graph of* K_n *is denoted by* $\left(K_n\right)^{+1}$ *and it is obtained by attaching one edge to each vertex of* K_n *.*

Theorem 2.2 *For* $n\geq 2$, *the minimum covering Randić energy,* $R_C(G)$ *of thorn graph* $\left(K_n\right)^{+1}$ *is* $\frac{\left(n^2-2\right)+\sqrt{4n^2+1}}{n}$.

Proof. $\left(K_n\right)^{+1}$ is a thorn graph of complete graph K_n with vertex set $V=\{v_1,v_2,\cdots,v_n,v_1',v_2',\cdots,v_n'\}$. The minimum covering set for thorn graph $\left(K_n\right)^{+1}$ is $C=\{v_1,v_2,v_3,\cdots,v_n\}$. Then

$$R_C\left(\left(K_n\right)^{+1}\right)=\left(\begin{array}{c|ccccc|ccccc} & v_1 & v_2 & v_3 & \cdots & v_n & v_1' & v_2' & v_3' & \cdots & v_n' \\ \hline v_1 & 1 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{\sqrt{n}} & 0 & 0 & 0 & 0 \\ v_2 & \frac{1}{n} & 1 & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{\sqrt{n}} & 0 & \cdots & 0 \\ v_3 & \frac{1}{n} & \frac{1}{n} & 1 & \cdots & \frac{1}{n} & 0 & 0 & \frac{1}{\sqrt{n}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & 1 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{n}} \\ \hline v_1' & \frac{1}{\sqrt{n}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ v_2' & 0 & \frac{1}{\sqrt{n}} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ v_3' & 0 & 0 & \frac{1}{\sqrt{n}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n' & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{n}} & 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

Characteristic polynomial is

$$\frac{[\rho-1]^{n-1}[n\rho+1]^{n-2}\left[n\rho^2-(2n-1)\rho-1\right]}{n^n}.$$

Characteristic equation is $[\rho-1]^{n-1}[n\rho+1]^{n-2}\left[n\rho^2-(2n-1)\rho-1\right]=0$
Minimum covering Randić Spec

$$\left((K_n)^{+1}\right)=\begin{pmatrix} 1 & -\frac{1}{n} & \frac{(2n-1)+\sqrt{4n^2+1}}{2n} & \frac{(2n-1)-\sqrt{4n^2+1}}{2n} \\ n-1 & n-2 & 1 & 1 \end{pmatrix}$$

Minimum covering Randić energy is,

$$RE_C\left((K_n)^{+1}\right)=|1|(n-1)+\left|-\frac{1}{n}\right|(n-2)+\left|\frac{(2n-1)+\sqrt{4n^2+1}}{2n}\right|(1)+\left|\frac{(2n-1)-\sqrt{4n^2+1}}{2n}\right|(1)$$

$$=(n-1)+\frac{n-2}{n}+\frac{(2n-1)+\sqrt{4n^2+1}}{2n}+\frac{\sqrt{4n^2+1}-(2n-1)}{2n}$$

$$=(n-1)+\frac{n-2}{n}+\frac{\sqrt{4n^2+1}}{n}=\frac{(n^2-2)+\sqrt{4n^2+1}}{n}.$$

Definition 2.2 *Cocktail party graph is denoted by* $K_{n\times 2}$, *is a graph having the vertex set* $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$ *and the edge set* $E=\{u_iu_j,v_iv_j:i\neq j\}\bigcup\{u_iv_j,v_iu_j:1\le i<j\le n\}$.

Theorem 2.3 *The minimum covering Randić energy,* $R_C(G)$ *of cocktail party graph* $K_{n\times 2}$ *is* $\dfrac{(2n^2-6n+5)+\sqrt{4n^2-8n+5}}{n-1}$.

Proof. Consider cocktail party graph $K_{n\times 2}$ with vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$. The minimum covering set of cocktail party graph $K_{n\times 2}$ is $C=\bigcup_{i=1}^{n-1}\{u_i,v_i\}$. Then

$$R_C(K_{n\times 2})=\left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \cdots & u_n & v_1 & v_2 & v_3 & \cdots & v_n \\ \hline
u_1 & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\
u_2 & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\
u_3 & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 \\ \hline
v_1 & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\
v_2 & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\
v_3 & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \cdots & \frac{1}{2n-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_n & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0
\end{array}\right)$$

Characteristic polynomial is,

$$\frac{\rho(\rho-1)^{n-1}\left[(n-1)\rho-(n-2)\right]^{n-2}\left[(n-1)\rho^2-(2n-3)\rho-1\right]}{(n-1)^{n-1}}.$$

Characteristic equation is,

$$\rho(\rho-1)^{n-1}\left[(n-1)\rho-(n-2)\right]^{n-2}\left[(n-1)\rho^2-(2n-3)\rho-1\right]=0.$$

Minimum covering Randić Spec

$$(K_{n\times 2})=\begin{pmatrix} 1 & \frac{n-2}{n-1} & \frac{(2n-3)+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{(2n-3)-\sqrt{4n^2-8n+5}}{2(n-1)} \\ n-1 & n-2 & 1 & 1 \end{pmatrix}$$

Minimum covering Randić energy,

$$\begin{aligned} RE_C(K_{n\times 2}) &= |1|(n-1)+\left|\frac{n-2}{n-1}\right|(n-2)+ \\ &\left|\frac{(2n-3)+\sqrt{4n^2-8n+5}}{2(n-1)}\right|(1)+\left|\frac{(2n-3)-\sqrt{4n^2-8n+5}}{2(n-1)}\right|(1) \\ &=(n-1)+\frac{(n-2)^2}{n-1}+\frac{(2n-3)+\sqrt{4n^2-8n+5}}{2(n-1)}+\frac{\sqrt{4n^2-8n+5}-(2n-3)}{2(n-1)} \\ &=(n-1)+\frac{(n-2)^2}{n-1}+\frac{\sqrt{4n^2-8n+5}}{(n-1)} \\ &=\frac{(n-1)^2+(n-2)^2+\sqrt{4n^2-8n+5}}{n-1} \\ &=\frac{(2n^2-6n+5)+\sqrt{4n^2-8n+5}}{n-1}. \end{aligned}$$

Theorem 2.4 *For* $n \geq 2$, *minimum covering Randić energy,* $R_C(G)$ *of star graph* $K_{1,n-1}$ *is equal to* $\sqrt{5}$.

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V=\{\upsilon_0,\upsilon_1,\upsilon_2,\cdots,\upsilon_{n-1}\}$. Then its Minimum covering set is $C=\{v_0\}$.

$$R_C(K_{1,n-1})=\begin{pmatrix} 1 & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \cdots & \frac{1}{\sqrt{n-1}} \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

Characteristic equation is $(-1)^n \rho^{n-2}\left[\rho^2-\rho-1\right]=0.$

Minimum covering Randić Spec

$$\left(K_{1,n-1}\right)=\begin{pmatrix} 0 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Minimum covering Randić energy,

$$RE_C\left(K_{1,n-1}\right)=|0|(n-2)+\left|\frac{1+\sqrt{5}}{2}\right|(1)+\left|\frac{1-\sqrt{5}}{2}\right|(1)=0+\frac{1+\sqrt{5}}{2}+\frac{\sqrt{5}-1}{2}=\sqrt{5}.$$

Definition 2.3 *Crown graph* S_n^0 *for an integer* $n\geq 2$ *is the graph with vertex set* $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ *and edge set* $\{u_i v_j : 1\leq i, j\leq n, i\neq j\}$.

Theorem 2.5 *For* $n\geq 2$, *minimum covering Randić energy,* $R_C(G)$ *of the crown graph* S_n^0 *is equal to* $\sqrt{5}+\sqrt{n^2-2n+5}$.

Proof. For the crown graph S_n^0 with vertex set $V=\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$, minimum covering set of crown graph S_n^0 is $C=\{u_1,u_2,\cdots,u_n\}$. Then

$$R_C\left(S_n^0\right)=\left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \cdots & u_n & v_1 & v_2 & v_3 & \cdots & v_n \\ \hline
u_1 & 1 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\
u_2 & 0 & 1 & 0 & \cdots & 0 & \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\
u_3 & 0 & 0 & 1 & \cdots & 0 & \frac{1}{n-1} & \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & 0 & 0 & 0 & \cdots & 1 & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 0 \\ \hline
v_1 & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 & 0 & 0 & \cdots & 0 \\
v_2 & \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 & 0 & 0 & \cdots & 0 \\
v_3 & \frac{1}{n-1} & \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_n & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right)$$

Characteristic polynomial is

$$\frac{\left(\rho^2-\rho-1\right)\left[(n-1)^2\rho^2-(n-1)^2\rho-1\right]^{n-1}}{(n-1)^{2(n-1)}}.$$

Characteristic equation is

$$\left(\rho^2-\rho-1\right)\left[(n-1)^2\rho^2-(n-1)^2\rho-1\right]^{n-1}=0$$

Minimum covering Randić Spec

$$S_n^0 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{(n-1)+\sqrt{n^2-2n+5}}{2(n-1)} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2(n-1)} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$$

Minimum covering Randić energy,

$$RE_C\left(S_n^0\right) = \left|\frac{1+\sqrt{5}}{2}\right|(1) + \left|\frac{1-\sqrt{5}}{2}\right|(1) + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2(n-1)}\right|(n-1) + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2(n-1)}\right|(n-1)$$

$$= \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} + \frac{\left[(n-1)+\sqrt{n^2-2n+5}\right]}{2(n-1)}(n-1) + \frac{\left[\sqrt{n^2-2n+5}-(n-1)\right]}{2(n-1)}(n-1)$$

$$= \sqrt{5} + \sqrt{n^2-2n+5}.$$

Theorem 2.6 *The minimum covering Randić energy,* $R_C(G)$ *of the complete bipartite graph* $RE_C(K_{m,n})$ *is equal to* $m+\sqrt{5}-1$*.*

Proof. For the complete bipartite graph $K_{m,n}$ $(m \le n)$ with vertex set $V = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$, minimum covering set is $C = \{u_1, u_2, \cdots, u_m\}$. Then

$$R_C(K_{m,n}) = \left(\begin{array}{c|ccccc|ccccc} & u_1 & u_2 & u_3 & \cdots & u_m & v_1 & v_2 & v_3 & \cdots & v_n \\ \hline u_1 & 1 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} \\ u_2 & 0 & 1 & 0 & \cdots & 0 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} \\ u_3 & 0 & 0 & 1 & \cdots & 0 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m & 0 & 0 & 0 & \cdots & 1 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} \\ \hline v_1 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \cdots & 0 \\ v_2 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \cdots & 0 \\ v_3 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \cdots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

Characteristic equation is

$$(-1)^{m+n} \rho^{n-1} (\rho-1)^{m-1} \left[\rho^2 - \rho - 1\right] = 0$$

Minimum covering Randić Spec

$$(K_{m,n}) = \begin{pmatrix} 0 & 1 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ n-1 & m-1 & 1 & 1 \end{pmatrix}$$

Minimum covering Randić energy,

$$RE_C\left(K_{m,n}\right)=|0|(n-1)+|1|(m-1)+\left|\frac{1+\sqrt{5}}{2}\right|(1)+\left|\frac{1-\sqrt{5}}{2}\right|(1)$$

$$=(m-1)+\frac{1+\sqrt{5}}{2}+\frac{\sqrt{5}-1}{2}=m+\sqrt{5}-1.$$

Definition 2.4 *Friendship graph is the graph obtained by taking n copies of the cycle graph* C_3 *with a vertex in common. It is denoted by* F_3^n. *Friendship graph* F_3^n *contains* $2n+1$ *vertices and* $3n$ *edges.*

Theorem 2.7 *The minimum covering Randić energy,* $R_C(G)$ *of friendship graph* $RE_C\left(F_3^n\right)$ *is equal to* $\dfrac{\left(1+\sqrt{17}-2\sqrt{2}\right)+2\sqrt{2}n}{2}$.

Proof. For a friendship graph F_3^n with vertex set $V=\{v_0,v_1,v_2,\cdots,v_n,v_{n+1},v_{n+2},\cdots,v_{2n}\}$, minimum covering set is $C=\{v_0,v_1,v_3,v_5,\cdots,v_{2n-1}\}$. Then

$R_C\left(F_3^n\right)=$

$$\left(\begin{array}{c|cccccccccc}
 & v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & \cdots & v_{2n} \\
\hline
v_0 & 1 & \frac{1}{\sqrt{4n}} & \frac{1}{\sqrt{4n}} & \frac{1}{\sqrt{4n}} & \frac{1}{\sqrt{4n}} & \frac{1}{\sqrt{4n}} & \frac{1}{\sqrt{4n}} & \frac{1}{\sqrt{4n}} & \cdots & \frac{1}{\sqrt{4n}} \\
v_1 & \frac{1}{\sqrt{4n}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
v_2 & \frac{1}{\sqrt{4n}} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
v_3 & \frac{1}{\sqrt{4n}} & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 \\
v_4 & \frac{1}{\sqrt{4n}} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 \\
v_5 & \frac{1}{\sqrt{4n}} & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & \cdots & 0 \\
v_6 & \frac{1}{\sqrt{4n}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
v_7 & \frac{1}{\sqrt{4n}} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \frac{1}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{2n} & \frac{1}{\sqrt{4n}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots & 0
\end{array}\right)_{(2n+1)\times(2n+1)}$$

Characteristic equation is

$$\frac{(2\rho-1)\left(2\rho^2-3\rho-1\right)\left(4\rho^2-4\rho-1\right)^{n-1}}{4^n}=0.$$

Minimum covering Randić Spec

$$RE_C\left(F_3^n\right)=\begin{pmatrix} \frac{1}{2} & \frac{3+\sqrt{17}}{4} & \frac{3-\sqrt{17}}{4} & \frac{1+\sqrt{2}}{2} & \frac{1-\sqrt{2}}{2} \\ 1 & 1 & 1 & n-1 & n-1 \end{pmatrix}$$

Minimum covering Randić energy,

$$RE_C\left(F_3^n\right)=\left|\frac{1}{2}\right|(1)+\left|\frac{3+\sqrt{17}}{4}\right|(1)+\left|\frac{3-\sqrt{17}}{4}\right|(1)+\left|\frac{1+\sqrt{2}}{2}\right|(n-1)+\left|\frac{1-\sqrt{2}}{2}\right|(n-1)$$
$$=\frac{1}{2}+\frac{3+\sqrt{17}}{4}+\frac{\sqrt{17}-3}{4}+\frac{1+\sqrt{2}}{2}(n-1)+\frac{\sqrt{2}-1}{2}(n-1)$$
$$=\frac{1}{2}+\frac{\sqrt{17}}{2}+\frac{n-1}{2}\left(2\sqrt{2}\right)$$
$$=\frac{\left(1+\sqrt{17}-2\sqrt{2}\right)+2\sqrt{2}n}{2}.$$

2.2. Properties of Minimum Covering Randić Eigenvalues

Theorem 2.8 Let G be a graph with vertex set $V=\{v_1,v_2,\cdots,v_n\}$, edge set E and $D=\{u_1,u_2,\cdots,u_k\}$ be a minimum covering set. If $\rho_1,\rho_2,\cdots,\rho_n$ are the eigenvalues of minimum covering Randić matrix $R_C(G)$ then (i) $\sum_{i=1}^{n}\rho_i=|C|$ (ii) $\sum_{i=1}^{n}\rho_i^2=|C|+2\sum_{i<j}\frac{1}{d_id_j}$.

Proof. **i)** We know that the sum of the eigenvalues of $R_C(G)$ is the trace of $R_C(G)$

$$\therefore \sum_{i=1}^{n}\rho_i=\sum_{i=1}^{n}r_{ii}=|C|.$$

ii) Similarly the sum of squares of the eigenvalues of $R_C(G)$ is trace of $\left[R_C(G)\right]^2$

$$\therefore \sum_{i=1}^{n}\rho_i^2=\sum_{i=1}^{n}\sum_{j=1}^{n}r_{ij}r_{ji}=\sum_{i=1}^{n}(r_{ii})^2+\sum_{i\neq j}r_{ij}r_{ji}=\sum_{i=1}^{n}(r_{ii})^2+2\sum_{i<j}(r_{ij})^2=\sum_{i=1}^{n}(r_{ii})^2+2\sum_{i<j}\left(\frac{1}{\sqrt{d_id_j}}\right)^2$$
$$=|C|+2\sum_{i<j}\frac{1}{d_id_j}=|C|+2M,\ \text{where } M=\sum_{i<j}\frac{1}{d_id_j}.$$

2.3. Bounds for Minimum Covering Randić Energy

Mclelland's [8] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for $RE_C(G)$ are given in the following theorem.

Theorem 2.9 *Let G be a simple graph with n vertices and m edges . If C is the minimum covering set and* $P=\left|\det R_C(G)\right|$ *then*

$$\sqrt{|C|+2\sum_{i<j}\frac{1}{d_id_j}+n(n-1)P^{\frac{2}{n}}}\le RE_C(G)\le\sqrt{n\left(|C|+2\sum_{i<j}\frac{1}{d_id_j}\right)}.$$

Proof.

Canchy Schwarz inequality is $\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$

If $a_i=1,b_i=|\rho_i|$ then $\left(\sum_{i=1}^{n}|\rho_i|\right)^2\le\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\rho_i^2\right)$

$$\left[RE_C(G)\right]^2 \le n\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right) \quad \text{[From theorem 2.8]}$$

$$\Rightarrow RE_C(G) \le \sqrt{n\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right)}.$$

Since arithmetic mean is greater than or equal to geometric mean we have

$$\frac{1}{n(n-1)}\sum_{i\ne j}|\rho_i||\rho_j| \ge \left[\prod_{i\ne j}|\rho_i||\rho_j|\right]^{\frac{1}{n(n-1)}} = \left[\prod_{i=1}^{n}|\rho_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$= \left[\prod_{i=1}^{n}|\rho_i|\right]^{\frac{2}{n}} = \left|\prod_{i=1}^{n}\rho_i\right|^{\frac{2}{n}} = \left|\det R_C(G)\right|^{\frac{2}{n}} = P^{\frac{2}{n}}$$

$$\therefore \sum_{i\ne j}|\rho_i||\rho_j| \ge n(n-1)P^{\frac{2}{n}}. \tag{2.1}$$

Now consider, $\left[RE_C(G)\right]^2 = \left(\sum_{i=1}^{n}|\rho_i|\right)^2 = \sum_{i=1}^{n}|\rho_i|^2 + \sum_{i\ne j}|\rho_i||\rho_j|$

$$\therefore \left[RE_C(G)\right]^2 \ge |C| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)P^{\frac{2}{n}} \quad \text{[From (2.1)]}$$

$$\text{i.e.,} RE_C(G) \ge \sqrt{|C| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)P^{\frac{2}{n}}}.$$

Theorem 2.10 *If* $\rho_1(G)$ *is the largest minimum covering Randić eigenvalue of* $R_C(G)$, *then* $\rho_1(G) \ge \dfrac{|C| + R(G)}{n}$.

Proof. For any nonzero vector *X*, we have by [16], $\rho_1(A) = \max_{X\ne 0}\left\{\dfrac{X'AX}{X'X}\right\}$

$$\therefore \rho_1(G) \ge \frac{J'AJ}{J'J} = \frac{|C| + 2\sum_{i<j}\frac{1}{\sqrt{d_i d_j}}}{n} = \frac{|C| + R(G)}{n}$$ where J is a unit column matrix.

Just like Koolen and Moulton's [17] upper bound for energy of a graph, an upper bound for $RE_C(G)$ is given in the following theorem.

Theorem 2.11 *If G is a graph with n vertices and m edges and* $\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right) \ge n$ *then*

$$RE_C(G) \le \frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} + \sqrt{(n-1)\left[|C| + 2\sum_{i<j}\frac{1}{d_i d_j} - \left(\frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)^2\right]}.$$

Proof.

Cauchy-Schwartzin equality is $\left[\sum_{i=2}^{n} a_i b_i\right]^2 \le \left(\sum_{i=2}^{n} a_i^2\right)\left(\sum_{i=2}^{n} b_i^2\right)$.

Put $a_i = 1, b_i = |\rho_i|$ then $\left(\sum_{i=2}^{n} |\rho_i|\right)^2 = \sum_{i=2}^{n} 1 \sum_{i=2}^{n} |\rho_i|^2$

$$\Rightarrow \left[RE_C(G) - \rho_1\right]^2 \le (n-1)\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2\right)$$

$$\Rightarrow RE_C(G) \le \rho_1 + \sqrt{(n-1)\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2\right)}.$$

Let

$$f(x) = x + \sqrt{(n-1)\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2\right)}$$

For decreasing function $f'(x) \le 0 \Rightarrow 1 - \dfrac{x(n-1)}{\sqrt{(n-1)\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2\right)}} \le 0$

$$\Rightarrow x \ge \sqrt{\frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}}$$

Since $\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right) \ge n$, we have $\sqrt{\dfrac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}} \le \dfrac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} \le \rho_1$

$$\therefore f(\rho_1) \le f\left(\frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)$$

$$i.e., RE_C(G) \le f(\rho_1) \le f\left(\frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)$$

$$i.e., RE_C(G) \le f\left(\frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)$$

$$i.e., RE_C(G) \le \frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} + \sqrt{(n-1)\left[|C| + 2\sum_{i<j}\frac{1}{d_i d_j} - \left(\frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}\right)^2\right]}.$$

Milovanović [18] bounds for minimum covering Randić energy of a graph are given in the following theorem.

Theorem 2.12 *Let G be a graph with n vertices and m edges. Let* $|\rho_1| \ge |\rho_2| \ge \cdots \ge |\rho_n|$ *be a non-increasing order of minimum covering Randić eigenvalues of* $R_C(G)$ *and C is minimum covering set then*

$$RE_C(G) \ge \sqrt{n\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right) - \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2} \quad \text{where} \quad \alpha(n) = n\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)$$

and $[x]$ *denotes the integral part of a real number.*

Proof. For real numbers $a, a_1, a_2, \cdots, a_n, A$ and $b, b_1, b_2, \cdots, b_n, B$ with $a \le a_i \le A$ and $b \le b_i \le B \ \forall i = 1, 2, \cdots n$ the following inequality is proved in [19].

$$\left|n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \le \alpha(n)(A-a)(B-b) \quad \text{where} \quad \alpha(n) = n\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) \quad \text{and}$$

equality holds if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.

If $a_i = |\rho_i|$, $b_i = |\rho_i|$, $a = b = |\rho_n|$ and $A = B = |\rho_1|$, then

$$\left|n\sum_{i=1}^{n}|\rho_i|^2 - \left(\sum_{i=1}^{n}|\rho_i|\right)^2\right| \le \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2$$

But $\sum_{i=1}^{n}|\rho_i|^2 = |C| + 2\sum_{i<j}\frac{1}{d_i d_j}$ and $RE_C(G) \le \sqrt{n\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right)}$ then the above inequality becomes

$$n\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right) - \left(RE_C(G)\right)^2 \le \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2$$

$$i,e.,\ RE_C(G) \ge \sqrt{n\left(|C| + 2\sum_{i<j}\frac{1}{d_i d_j}\right) - \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2}.$$

Theorem 2.13 *Let G be a graph with n vertices and m edges. Let* $|\rho_1| \ge |\rho_2| \ge \cdots \ge |\rho_n| > 0$ *be a non-increasing order of minimum covering eigenvalues of* $R_C(G)$ *then* $RE_C(G) \ge \dfrac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j} + n|\rho_1||\rho_n|}{\left(|\rho_1| + |\rho_n|\right)}$.

Proof. Let $a_i \ne 0, b_i, r$ and R be real numbers satisfying $ra_i \le b_i \le Ra_i$, then the following inequality is proved in [20].

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \le (r+R)\sum_{i=1}^{n} a_i b_i$$

Put $b_i = |\rho_i|$, $a_i = 1$, $r = |\rho_n|$ and $R = |\rho_1|$ then

$$\sum_{i=1}^{n}|\rho_i|^2 + |\rho_1||\rho_n|\sum_{i=1}^{n}1 \le \left(|\rho_1| + |\rho_n|\right)\sum_{i=1}^{n}|\rho_i|$$

$$i.e., |C| + 2\sum_{i<j}\frac{1}{d_i d_j} + |\rho_1||\rho_n| n \le \left(|\rho_1| + |\rho_n|\right) RE_C(G)$$

$$\therefore RE_C(G) \geq \frac{|C| + 2\sum_{i<j}\frac{1}{d_i d_j} + n|\rho_1||\rho_n|}{(|\rho_1| + |\rho_n|)}.$$

The question of when does the graph energy becomes a rational number was answered by Bapat and S. pati in their paper [21]. Similar result for minimum covering Randić energy is obtained in the following theorem.

Theorem 2.14 *Let G be a graph with a minimum covering set C. If the minimum covering Randić energy* $RE_C(G)$ *is a rational number, then* $RE_C(G) \equiv |C| \pmod 2$.

Proof. Proof is similar to theorem 3.7 of [15].

3. Conclusion

It was proved in this paper that the minimum covering Randić energy of a graph G depends on the covering set that we take for consideration. Upper and lower bounds for minimum covering Randić energy are established. A generalized expression for minimum covering Randić energies for star graph, complete graph, thorn graph of complete graph, crown graph, complete bipartite graph, cocktail party graph and friendship graphs are also computed.

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Authors Contributions

Both the authors worked together for the preparation of the manuscript and both of us take the full responsibility for the content of the paper. However second author typed the paper and both of us read and approved the final manuscript.

Conflict of Interests

The authors hereby declares that there are no issues regarding the publication of this paper.

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