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Least-Squares Solutions of Generalized Sylvester Equation with X_i Satisfies Different Linear Constraint

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Abstract

In this paper, an iterative method is constructed to find the least-squares solutions of generalized Sylvester equation $A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 = C$, where $[X_1, X_2, X_3, X_4]$ is real matrices group, and X_i satisfies different linear constraint. By this iterative method, for any initial matrix group $\left[X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}\right]$ within a special constrained matrix set, a least squares solution group $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right]$ with $\tilde{X}_i$ satisfying different linear constraint can be obtained within finite iteration steps in the absence of round off errors, and the unique least norm least-squares solution can be obtained by choosing a special kind of initial matrix group. In addition, a minimization property of this iterative method is characterized. Finally, numerical experiments are reported to show the efficiency of the proposed method.

Keywords

Least-Squares Problem, Centro-Symmetric Matrix, Bisymmetric Matrix, Iterative Method

1. Introduction

A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be a Centro-symmetric matrix if $a_{ij} = a_{n+1-j,n+1-i}$ for all $1 \le i, j \le n$. A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be a Bisymmetric matrix if $a_{ij} = a_{ji} = a_{n+1-j,n+1-i}$ for all $1 \le i, j \le n$. Let

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$R^{m\times n}, SR^{n\times n}, CSR^{n\times n}$ and $BSR^{n\times n}$ denote the set of $m\times n$ real matrices, $n\times n$ real symmetric matrices, $n\times n$ real Centro-symmetric matrices and $n\times n$ real Bisymmetric matrices, respectively. $S_n=(e_n,e_{n-1},\cdots,e_1)$ where e_i denotes *i*th column of $n\times n$ unit matrix. For a matrix $A\in R^{m\times n}$, we denote its transpose, traced by $A^{\mathrm{T}}, tr(A)$ respectively. In space $R^{n\times n}$, we define inner product as: $\langle A,B\rangle = tr(B^{\mathrm{T}}A)$ for all $A,B\in R^{m\times n}$, then the norm of a matrix *A* generated by this inner product is, obviously, Frobenius norm and denoted by $\|A\|$.

Denote

$$K=\left\{[X_1,X_2,X_3,X_4]\,\middle|\, X_1\in R^{n_1\times n_1}, X_2\in SR^{n_2\times n_2}, X_3\in CSR^{n_3\times n_3}, X_4\in BSR^{n_4\times n_4}\right\}.$$

Obviously, *K*, *i.e.* $R^{n_1\times n_1}\times SR^{n_2\times n_2}\times CSR^{n_3\times n_3}\times BSR^{n_4\times n_4}$, is a linear subspace of real number field.

In this paper, we mainly consider the following two problems:

Problem I. Given matrices $A_i\in R^{p\times n_i}$, $B_i\in R^{n_i\times q}$ $(i=1,2,3,4)$, find matrix group $[\tilde{X}_1,\tilde{X}_2,\tilde{X}_3,\tilde{X}_4]\in K$ such that

$$\begin{aligned}&\left\|A_1\tilde{X}_1B_1+A_2\tilde{X}_2B_2+A_3\tilde{X}_3B_3+A_4\tilde{X}_4B_4-C\right\|\\&=\min_{[X_1,X_2,X_3,X_4]\in K}\left\|A_1X_1B_1+A_2X_2B_2+A_3X_3B_3+A_4X_4B_4-C\right\|.\end{aligned}$$

Problem II. Denote by S_E the solution set of Problem I. Find matrix group $[\hat{X}_1,\hat{X}_2,\hat{X}_3,\hat{X}_4]\in S_E$, such that

$$\|\hat{X}_1\|+\|\hat{X}_2\|+\|\hat{X}_3\|+\|\hat{X}_4\|=\min.$$

In fact, Problem II is to find the least norm solution of Problem I.

There are many valuable efforts on formulating solutions of various linear matrix equations with or without linear constraint. For example, Baksalary and Kala [1], Chu [2] [3], Peng [4], Liao, Bai and Lei [5] and Xu, Wei and Zheng [6] considered the nonsymmetric solution of the matrix equation

$$AXB+CYD=E, \tag{1}$$

by using Moore-Penrose generalized inverse and the generalized singular value decomposition of matrices, while Chang and Wang [7] considered the symmetric conditions on the solution of the matrix equations

$$AXA^{\mathrm{T}}+BYB^{\mathrm{T}}=E,\quad AX+YA=C. \tag{2}$$

Zietak [8] [9] discussed the l_p-solution and Chebyshev-solution of the matrix equation

$$AX+YB=C. \tag{3}$$

Peng [10] researched the general linear matrix equation

$$A_1X_1B_1+A_2X_2B_2+\cdots+A_lX_lB_l=C, \tag{4}$$

with the bisymmetric conditions on the solutions. Vec operator and Kronecker product are employed in this paper, so the size of the matrix is enlarged greatly and the computation is very expensive in the process of solving solutions. Iterative algorithms have been received much attention to solve linear matrix equations in recent years. For example, by extending the well-known Jacobi and Gauss-seidel iterations for $Ax=b$, Ding, Liu and Ding in [11] derived iterative solutions of matrix equations $AXB=F$ and generalized Sylvester matrix equations $AXB+CYD=F$. By absorbing the thought of the conjugate gradient method, Peng [12] presented an iterative algorithm to solve Equation (1). Peng [13], Peng, Hu and Zhang [14] put forward an iterative method for bisymmetric solution of Equation (4). These matrix-form CG methods are based on short recurrences, which keep work and storage requirement constant at each iteration. However, these iteration methods are only defined by the Galerkin condition, but lack of a minimization property, which means that the algorithm may exhibit a rather irregular convergence, and often results in a very slow convergence. Lei and Liao [15] presented that a minimal residual algorithm could remedy this problem, and this algorithm satisfies a minimization property, which ensures that this method possesses a smoothly convergence.

However, to our best knowledge, the unknown matrix with different linear constraint of linear matrix equations, such as Equations ((1)-(4)), has not been considered yet. No loss of generality, we research the following case

$$A_1X_1B_1+A_2X_2B_2+A_3X_3B_3+A_4X_4B_4=C, \tag{5}$$

which has four unknown matrices and each is required to satisfy different linear constraint. We should point out that the matrices A_i, B_i, C are experimentally occurring in practices, so they may not satisfy solvability conditions. Hence, we should study the least squares solutions, *i.e.* Problem I. Noting that it is obvious difficulties to solve this problem by conventional methods, such as matrix decomposition and ver operator, hence iterative method is considered. Absorbing the thought of the minimal residual algorithm presented by Lei and Liao [15], and combing the trait of problem, we conduct an iterative method for solving Problem I. This method can both maintain the short recurrence and satisfy a minimization property, *i.e.* the approximation solution minimizes the residual norm of Equation (5) over a special affine subspace, which ensures that this method converges smoothly.

The paper is organized as follows. In Section 2, we first conduct an iterative method for solving Problem I, and then describe the basic properties of this method; we also solve Problem II by using this iterative method. In Section 3, we show that the method possesses a minimization property. In Section 4, we present numerical experiments to show the efficiency of the proposed method, and use some conclusions in Section 5 to end our paper.

2. The Iterative Method for Solving Problem I and II

In this section, we firstly introduce some lemmas which are required for solving Problem I, we then conduct an iterative method to obtain the solution of Problem I. We show that, for any initial matrix group $\left[X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}\right] \in K$, the matrix group sequences $\left[X_1^{(k)}, X_2^{(k)}, X_3^{(k)}, X_4^{(k)}\right]$ generated by the iterative method converge to a solution of Problem I within finite iteration steps in the absence of roundoff errors. We also show that the unique least norm solution of Problem I can be obtained by choosing a special kind of initial matrix group.

Lemma 1. [16] [17]. *A matrix* $X \in CSR^{n\times n}$ *if and only if* $X = S_n X S_n$.

A matrix $X \in BSR^{n\times n}$ if and only if $X = X^{\mathrm{T}} = S_n X S_n$.

Lemma 2. *Suppose that a matrix* $X \in R^{n\times n}$, *then* $X + S_n X S_n \in CSR^{n\times n}$.

Suppose that a matrix $X \in SR^{n\times n}$, then $X + S_n X S_n \in BSR^{n\times n}$.

Proof: Its proof is easy to obtain from Lemma 1. □

Lemma 3. *Suppose that* $A \in R^{n\times n}$, $X \in SR^{n\times n}$, $Y \in CSR^{n\times n}$, $Z \in BSR^{n\times n}$ *then*

$$\langle A, X\rangle = \left\langle \frac{1}{2}\left[A + A^{\mathrm{T}}\right], X\right\rangle$$

$$\langle A, Y\rangle = \left\langle \frac{1}{2}\left[A + S_n A S_n\right], Y\right\rangle$$

$$\langle A, Z\rangle = \left\langle \frac{1}{4}\left[A + A^{\mathrm{T}} + S_n\left(A + A^{\mathrm{T}}\right)S_n\right], Z\right\rangle.$$

Proof: It is easy to verify from direct computation. □

Lemma 4. (*Projection Theorem*) [18]. *Let X be a finite dimensional inner product space, M be a subspace of X, and* $M^{\perp}$ *be the orthogonal complement subspace of M. For a given* $x \in X$, *there always exists an* $m_0 \in M$ *such that*

$$\|x - m_0\| \le \|x - m\|, \quad \forall m \in M,$$

where $\|.\|$ is the norm associated with the inner product defined in *X*. Moreover, $m_0 \in M$ is the unique minimization vector in *M* if and only if

$$\left(x - m_0\right) \perp M \quad i.e, \quad \left(x - m_0\right) \in M^{\perp}.$$

Lemma 5. *Suppose* $\tilde{R}$ *is the residual of matrix group* $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right] \in K$ *corresponding to Equation* (5), *i.e.* $\tilde{R} = C - A_1\tilde{X}_1B_1 - A_2\tilde{X}_2B_2 - A_3\tilde{X}_3B_3 - A_4\tilde{X}_4B_4$, *if the following conditions are satisfied simultaneously,*

$$\begin{aligned}
&A_1^{\mathrm{T}}\tilde{R}B_1^{\mathrm{T}} = 0,\\
&A_2^{\mathrm{T}}\tilde{R}B_2^{\mathrm{T}} + B_2\tilde{R}^{\mathrm{T}}A_2 = 0,\\
&A_3^{\mathrm{T}}\tilde{R}B_3^{\mathrm{T}} + S_{n_3}A_3^{\mathrm{T}}RB_3^{\mathrm{T}}S_{n_3} = 0,\\
&A_4^{\mathrm{T}}\tilde{R}B_4^{\mathrm{T}} + B_4\tilde{R}^{\mathrm{T}}A_4 + S_{n_4}\left(A_4^{\mathrm{T}}\tilde{R}B_4^{\mathrm{T}} + B_4\tilde{R}^{\mathrm{T}}A_4\right)S_{n_4} = 0,
\end{aligned} \tag{6}$$

then the matrix group $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right]$ is a solution of Problem I.

Proof: Let

$$L = \left\{ Z \mid Z = A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4, \left[X_1, X_2, X_3, X_4\right] \in K \right\},$$

obviously, Z is a linear subspace of $R^{p\times q}$. For matrix group $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right] \in K$, denote

$$\tilde{C} = A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4,$$

then $\tilde{C} \in L$. Applying to Lemma 4, we know that $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right]$ is a solution of Problem I if and only if

$$\left(\tilde{C} - C\right) \perp L,$$

i.e. for all $\left[X_1, X_2, X_3, X_4\right] \in K$,

$$\left\langle \tilde{R}, A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 \right\rangle = 0.$$

By Lemma 3, it is easy to verify that if the equations of (6) are satisfied simultaneously, the expression above holds, which means $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right]$ is a solution of Problem I. □

Lemma 6. *Suppose that matrix group* $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right]$ *is a solution of Problem I, then arbitrary matrix group* $\left[X_1, X_2, X_3, X_4\right] \in S_E$ *can be express as* $\left[\tilde{X}_1 + \bar{X}_1, \tilde{X}_2 + \bar{X}_2, \tilde{X}_3 + \bar{X}_3, \tilde{X}_4 + \bar{X}_4\right]$ *where matrix group* $\left[\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\right] \in K$ *satisfies*

$$A_1\bar{X}_1B_1 + A_2\bar{X}_2B_2 + A_3\bar{X}_3B_3 + A_4\bar{X}_4B_4 = 0. \tag{7}$$

Proof: Assume that matrix group $\left[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4\right]$ is a solution of Problem I. If $\left[\tilde{X}_1 + \bar{X}_1, \tilde{X}_2 + \bar{X}_2, \tilde{X}_3 + \bar{X}_3, \tilde{X}_4 + \bar{X}_4\right] \in S_E$, then by Lemma 5 and its proof process, we have

$$\begin{aligned}
&\left\|A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 - C\right\|^2 \\
&= \left\|A_1\left(\tilde{X}_1 + \bar{X}_1\right)B_1 + A_2\left(\tilde{X}_2 + \bar{X}_2\right)B_2 + A_3\left(\tilde{X}_3 + \bar{X}_3\right)B_2 + A_4\left(\tilde{X}_4 + \bar{X}_4\right)B_4 - C\right\|^2 \\
&= \left\|A_1\bar{X}_1B_1 + A_2\bar{X}_2B_2 + A_3\bar{X}_3B_3 + A_4\bar{X}_4B_4 - \tilde{R}\right\|^2 \\
&= \left\|A_1\bar{X}_1B_1 + A_2\bar{X}_2B_2 + A_3\bar{X}_3B_3 + A_4\bar{X}_4B_4\right\|^2 + \left\|\tilde{R}\right\|^2
\end{aligned}$$

which implies matrix group $\left[\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\right]$ satisfies (7).

Conversely, if matrix group

$$\left[Y_1, Y_2, Y_3, Y_4\right] = \left[\tilde{X}_1 + \bar{X}_1, \tilde{X}_2 + \bar{X}_2, \tilde{X}_3 + \bar{X}_3, \tilde{X}_4 + \bar{X}_4\right] \in K,$$

where matrix group $\left[\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\right] \in K$ satisfies (7), then

$$\begin{aligned}
&\left\|A_1Y_1B_1 + A_2Y_2B_2 + A_3Y_3B_3 + A_4Y_4B_4 - C\right\| \\
&= \left\|A_1\bar{X}_1B_1 + A_2\bar{X}_2B_2 + A_3\bar{X}_3B_3 + A_4\bar{X}_4B_4 - \tilde{R}\right\| \\
&= \left\|A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 - C\right\|
\end{aligned}$$

which means matrix group $\left[Y_1, Y_2, Y_3, Y_4\right] \in S_E$. □

Next, we develop iterative algorithm for the least-squares solutions with X_i satisfies different linear constraint of matrix equation

$$A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 = C,$$

where $A_i \in R^{p\times n_i}$, $B_i \in R^{n_i\times q}, (i = 1,2,3,4)$ and C are given constant matrices, and $\left[X_1, X_2, X_3, X_4\right] \in K$ is the unknown matrices group to be solved.

Algorithm 1. For an arbitrary initial matrix group $\left[X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}\right] \in K$, compute

Step 1. $R_0 = C - \sum_{r=1}^{4} A_r X_r^{(0)} B_r$;

$$P_{0,1} = A_1^{\mathrm{T}} R_0 B_1^{\mathrm{T}};$$

$$P_{0,2} = \frac{1}{2}\left(A_2^{\mathrm{T}} R_0 B_2^{\mathrm{T}} + B_2 R_0^{\mathrm{T}} A_2\right);$$

$$P_{0,3} = \frac{1}{2}\left(A_3^{\mathrm{T}} R_0 B_3^{\mathrm{T}} + S_{n_3} A_3^{\mathrm{T}} R B_3^{\mathrm{T}} S_{n_3}\right);$$

$$P_{0,4} = \frac{1}{4}\left(A_4^{\mathrm{T}} R_0 B_4^{\mathrm{T}} + B_4 R_0^{\mathrm{T}} A_4 + S_{n_4}\left(A_4^{\mathrm{T}} R_0 B_4^{\mathrm{T}} + B_4 R_0^{\mathrm{T}} A_4\right) S_{n_4}\right);$$

$$Q_{0,r} = P_{0,r}, \quad r = 1,2,3,4.$$

Step 2. If $\sum_{r=1}^{4} \left\|P_{k,r}\right\|^2 = 0$, then stop; else, $k := k+1$, and compute

Step 3. $M_k = \sum_{r=1}^{4} A_r Q_{k,r} B_r$;

$$\alpha_k = \frac{\sum_{r=1}^{4} \left\|P_{k,r}\right\|^2}{\left\|M_k\right\|^2}, \quad X_r^{(k+1)} = X_r^{(k)} + \alpha_k Q_{k,r}, \quad r = 1,2,3,4;$$

$$R_{k+1} = R_k - \alpha_k M_k;$$

$$P_{k+1,1} = A_1^{\mathrm{T}} R_{k+1} B_1^{\mathrm{T}} = P_{k,1} - \alpha_k A_1^{\mathrm{T}} M_k B_1^{\mathrm{T}};$$

$$P_{k+1,2} = \frac{1}{2}\left(A_2^{\mathrm{T}} R_{k+1} B_2^{\mathrm{T}} + B_2 R_{k+1}^{\mathrm{T}} A_2\right) = P_{k,2} - \frac{\alpha_k}{2}\left[A_2^{\mathrm{T}} M_k B_2^{\mathrm{T}} + B_2 M_k^{\mathrm{T}} A_2\right];$$

$$P_{k+1,3} = \frac{1}{2}\left(A_3^{\mathrm{T}} R_{k+1} B_3^{\mathrm{T}} + S_{n_3} A_3^{\mathrm{T}} R_{k+1} B_3^{\mathrm{T}} S_{n_3}\right)$$

$$= P_{k,3} - \frac{\alpha_k}{2}\left[A_3^{\mathrm{T}} M_k B_3^{T} + S_{n_3} A_3^{\mathrm{T}} M_k B_3^{\mathrm{T}} S_{n_3}\right];$$

$$P_{k+1,4} = \frac{1}{4}\left(A_4^{\mathrm{T}} R_{k+1} B_4^{\mathrm{T}} + B_4 R_{k+1}^{\mathrm{T}} A_4 + S_{n_4}\left(A_4^{\mathrm{T}} R_{k+1} B_4^{\mathrm{T}} + B_4 R_{k+1}^{\mathrm{T}} A_4\right) S_{n_4}\right)$$

$$= P_{k,4} - \frac{\alpha_k}{4}\left[A_4^{\mathrm{T}} M_k B_4^{\mathrm{T}} + B_4 M_k^{\mathrm{T}} A_4 + S_{n_4}\left(A_4^{\mathrm{T}} M_k B_4^{\mathrm{T}} + B_4 M_k^{\mathrm{T}} A_4\right) S_{n_4}\right];$$

$$\beta_k = \frac{\sum_{r=1}^{4} \left\|P_{k+1,r}\right\|^2}{\sum_{r=1}^{4} \left\|P_{k,r}\right\|^2}, \quad Q_{k+1,r} = P_{k+1,r} + \beta_k Q_{k,r}, \quad r = 1,2,3,4;$$

Step 4. Go to step 2.

Remark 1. 1) *Obviously, matrices sequence* $P_{i,r}, Q_{i,r}, X_r^{(i)}, (r = 1,2,3,4)$ *generated by Algorithm* 1 *satisfies*

$$P_{i,1} \in R^{n_1\times n_1}, \quad P_{i,2} \in SR^{n_2\times n_2}, \quad P_{i,3} \in CSR^{n_3\times n_3}, \quad P_{i,4} \in BSR^{n_4\times n_4}$$

$$Q_{i,1} \in R^{n_1\times n_1}, \quad Q_{i,2} \in SR^{n_2\times n_2}, \quad Q_{i,3} \in CSR^{n_3\times n_3}, \quad Q_{i,4} \in BSR^{n_4\times n_4}$$

$$X_1^{(i)} \in R^{n_1\times n_1}, \quad X_2^{(i)} \in SR^{n_2\times n_2}, \quad X_3^{(i)} \in CSR^{n_3\times n_3}, \quad X_4^{(i)} \in BSR^{n_4\times n_4};$$

2) R_i is the residual of Equation (5), when $i = 0,1,2,\cdots$

3) Algorithm 1 implies that if $P_{k,r} = 0\,(r = 1,2,3,4)$, then the corresponding matrix group $\left[X_1^{(k)}, X_2^{(k)}, X_3^{(k)}, X_4^{(k)}\right]$ is the solution of Problem I.

In the next part, we will show the basic properties of iteration method by induction. First for convenience of discussion in the later context, we introduce the following conclusions from Algorithm 1. For all i, j, t

$$\left\langle R_i - R_j, \sum_{r=1}^{4} A_r P_{t,r} B_r \right\rangle = \sum_{r=1}^{4} \left\langle A_r^{\mathrm{T}} \left(R_i - R_j\right) B_r^{\mathrm{T}}, P_{t,r} \right\rangle$$

$$= \left\langle A_1^{\mathrm{T}} \left(R_i - R_j\right) B_1^{\mathrm{T}}, P_{t,1} \right\rangle + \frac{1}{2} \left\langle A_2^{\mathrm{T}} \left(R_i - R_j\right) B_2^{\mathrm{T}} + B_2 \left(R_i - R_j\right)^{\mathrm{T}} A_2, P_{t,2} \right\rangle$$

$$+ \frac{1}{2} \left\langle A_3^{\mathrm{T}} \left(R_i - R_j\right) B_3^{\mathrm{T}} + S_{n_3} A_3^{\mathrm{T}} \left(R_i - R_j\right) B_3^{\mathrm{T}} S_{n_3}, P_{t,3} \right\rangle$$

$$+ \frac{1}{4} \left\langle A_4^{\mathrm{T}} \left(R_i - R_j\right) B_4^{\mathrm{T}} + B_4 \left(R_i - R_j\right)^{\mathrm{T}} A_4 + S_{n_4} \left[A_4^{\mathrm{T}} \left(R_i - R_j\right) B_4^{\mathrm{T}} + B_4 \left(R_i - R_j\right)^{\mathrm{T}} A_4 \right] S_{n_4}, P_{t,4} \right\rangle$$

$$= \sum_{r=1}^{4} \left\langle P_{i,r}, P_{t,r} \right\rangle + \sum_{r=1}^{4} \left\langle P_{j,r}, P_{t,r} \right\rangle.$$

Lemma 7. *For matrices* $P_{i,r}$, $Q_{i,r}$ $(r = 1, 2, 3, 4)$ *and* M_i *generated by Algorithm* 1, *if there exist a positive integer k such that* $\sum_{r=1}^{4} \left\| P_{i,r} \right\|^2 \neq 0$, $\alpha_i \neq 0$, *and* $\alpha_i \neq \infty$ *for all* $i = 0,1,2,\cdots,k$, *then we have*

1) $\sum_{r=1}^{4} \left\langle P_{i,r}, P_{j,r} \right\rangle = 0, \quad i, j = 0,1,2,\cdots,k, \ i \neq j;$

2) $\left\langle M_i, M_j \right\rangle = 0, \quad i, j = 0,1,2,\cdots,k, \ i \neq j;$

3) $\sum_{r=1}^{4} \left\langle P_{i,r}, Q_{j,r} \right\rangle = 0, \quad 0 \le j < i \le k.$

Proof: For $k = 1$, it follows that

$$\sum_{r=1}^{4} \left\langle P_{0,r}, P_{1,r} \right\rangle = \left\langle P_{0,1}, P_{0,1} - \alpha_0 A_1^{\mathrm{T}} M_0 B_1^{\mathrm{T}} \right\rangle + \left\langle P_{0,2}, P_{0,2} - \frac{\alpha_0}{2} \left(A_2^{\mathrm{T}} M_0 B_2^{\mathrm{T}} + B_2 M_0^{\mathrm{T}} A_2 \right) \right\rangle$$

$$+ \left\langle P_{0,3}, P_{0,3} - \frac{\alpha_0}{2} \left(A_3^{\mathrm{T}} M_0 B_3^{\mathrm{T}} + S_{n_3} A_3^{\mathrm{T}} M_0 B_3^{\mathrm{T}} S_{n_3} \right) \right\rangle$$

$$+ \left\langle P_{0,4}, P_{0,4} - \frac{\alpha_0}{4} \left[A_4^{\mathrm{T}} M_0 B_4^{\mathrm{T}} + B_4 M_0^{\mathrm{T}} A_4 + S_{n_4} \left(A_4^{\mathrm{T}} M_0 B_4^{\mathrm{T}} + B_4 M_0^{\mathrm{T}} A_4 \right) S_{n_4} \right] \right\rangle$$

$$= \sum_{r=1}^{4} \left\| P_{0,r} \right\|^2 - \alpha_0 \sum_{r=1}^{4} \left\langle A_r^{\mathrm{T}} P_{0,r} B_r^{\mathrm{T}}, M_0 \right\rangle = \sum_{r=1}^{4} \left\| P_{0,r} \right\|^2 - \alpha_0 \left\langle M_0, M_0 \right\rangle = 0.$$

$$\left\langle M_0, M_1 \right\rangle = \left\langle M_0, \sum_{r=1}^{4} A_r \left(P_{1,r} + \beta_0 Q_{0,r} \right) B_r \right\rangle = \left\langle M_0, \sum_{r=1}^{4} A_r P_{1,r} B_r \right\rangle + \beta_0 \left\| M_0 \right\|^2$$

$$= \frac{1}{\alpha_0} \left\langle R_0 - R_1, \sum_{r=1}^{4} A_r P_{1,r} B_r \right\rangle + \beta_0 \left\| M_0 \right\|^2 = -\frac{1}{\alpha_0} \sum_{r=1}^{4} \left\| P_{1,r} \right\|^2 + \beta_0 \left\| M_0 \right\|^2 = 0.$$

$$\sum_{r=1}^{4} \left\langle P_{1,r}, Q_{0,r} \right\rangle = \sum_{r=1}^{4} \left\langle P_{1,r}, P_{0,r} \right\rangle = 0.$$

Assume that the conclusions

$$\sum_{r=1}^{4} \left\langle P_{j,r}, P_{s,r} \right\rangle = 0; \quad \left\langle M_j, M_s \right\rangle = 0; \quad \sum_{r=1}^{4} \left\langle P_{s,r}, Q_{j,r} \right\rangle = 0$$

hold for all $j \le s-1 \left(0 < s < k\right)$, then

$$\begin{aligned}\sum_{r=1}^{4}\left\langle P_{j,r},P_{s+1,r}\right\rangle &= \left\langle P_{j,1},P_{s,1}-\alpha_s A_1^{\mathrm T}M_sB_1^{\mathrm T}\right\rangle+\left\langle P_{j,2},P_{s,2}-\frac{\alpha_s}{2}\left(A_2^{\mathrm T}M_sB_2^{\mathrm T}+B_2M_s^{\mathrm T}A_2\right)\right\rangle\\ &\quad+\left\langle P_{j,3},P_{s,3}-\frac{\alpha_s}{2}\left(A_3^{\mathrm T}M_sB_3^{\mathrm T}+S_{n_3}A_3^{\mathrm T}M_sB_3^{\mathrm T}S_{n_3}\right)\right\rangle\\ &\quad+\left\langle P_{j,4},P_{s,4}-\frac{\alpha_s}{4}\left[A_4^{\mathrm T}M_sB_4^{\mathrm T}+B_4M_s^{\mathrm T}A_4+S_{n_4}\left(A_4^{\mathrm T}M_sB_4^{\mathrm T}+B_4M_s^{\mathrm T}A_4\right)S_{n_4}\right]\right\rangle\\ &=-\alpha_s\sum_{r=1}^{4}\left\langle P_{j,r},A_r^{\mathrm T}M_sB_r^{\mathrm T}\right\rangle=-\alpha_s\sum_{r=1}^{4}\left\langle A_r\left(Q_{j,r}-\beta_{j-1}Q_{j-1,r}\right)B_r,M_s\right\rangle\\ &=-\alpha_s\left\langle M_j-\beta_{j-1}M_{j-1},M_s\right\rangle=0.\end{aligned}$$

$$\begin{aligned}\left\langle M_j,M_{s+1}\right\rangle &= \left\langle M_j,\sum_{r=1}^{4}A_r\left(P_{s+1,r}+\beta_sQ_{s,r}\right)B_r\right\rangle=\left\langle M_j,\sum_{r=1}^{4}A_rP_{s+1,r}B_r\right\rangle\\ &=\frac{1}{\alpha_j}\left\langle R_j-R_{j+1},\sum_{r=1}^{4}A_rP_{s+1,r}B_r\right\rangle=\frac{1}{\alpha_j}\left[\sum_{r=1}^{4}\left\langle P_{j,r},P_{s+1,r}\right\rangle-\sum_{r=1}^{4}\left\langle P_{j+1,r},P_{s+1,r}\right\rangle\right]\\ &=-\frac{1}{\alpha_j}\sum_{r=1}^{4}\left\langle P_{j+1,r},P_{s+1,r}\right\rangle.\end{aligned}$$

$$\begin{aligned}\sum_{r=1}^{4}\left\langle Q_{j,r},P_{s+1,r}\right\rangle &= \left\langle Q_{j,1},P_{s,1}-\alpha_s A_1^{\mathrm T}M_sB_1^{\mathrm T}\right\rangle+\left\langle Q_{j,2},P_{s,2}-\frac{\alpha_s}{2}\left[A_2^{\mathrm T}M_sB_2^{\mathrm T}+B_2M_s^{\mathrm T}A_2\right]\right\rangle\\ &\quad+\left\langle Q_{j,3},P_{s,3}-\frac{\alpha_s}{2}\left[A_3^{\mathrm T}M_sB_3^{\mathrm T}+S_{n_3}A_3^{\mathrm T}M_sB_3^{\mathrm T}S_{n_3}\right]\right\rangle\\ &\quad+\left\langle Q_{j,4},P_{s,4}-\frac{\alpha_k}{4}\left[A_4^{\mathrm T}M_sB_4^{\mathrm T}+B_4M_s^{\mathrm T}A_4+S_{n_4}\left(A_4^{\mathrm T}M_sB_4^{\mathrm T}+B_4M_s^{\mathrm T}A_4\right)S_{n_4}\right]\right\rangle\\ &=-\alpha_s\sum_{r=1}^{4}\left\langle A_rQ_{j,r}B_r,M_s\right\rangle=-\alpha_s\left\langle M_j,M_s\right\rangle=0.\end{aligned}$$

By the assumption of Equation (3), we have

$$\sum_{r=1}^{4}\left\langle Q_{s,r},P_{s,r}\right\rangle=\sum_{r=1}^{4}\left\langle P_{s,r}+\beta_{s-1}Q_{s-1,r},P_{s,r}\right\rangle=\sum_{r=1}^{4}\left\|P_{s,r}\right\|^2.$$

Then for $j = s$,

$$\begin{aligned}\sum_{r=1}^{4}\left\langle P_{s,r},P_{s+1,r}\right\rangle &= \left\langle P_{s,1},P_{s,1}-\alpha_s A_1^{\mathrm T}M_sB_1^{\mathrm T}\right\rangle+\left\langle P_{s,2},P_{s,2}-\frac{\alpha_s}{2}\left(A_2^{\mathrm T}M_sB_2^{\mathrm T}+B_2M_s^{\mathrm T}A_2\right)\right\rangle\\ &\quad+\left\langle P_{s,3},P_{s,3}-\frac{\alpha_s}{2}\left(A_3^{\mathrm T}M_sB_3^{\mathrm T}+S_{n_3}A_3^{\mathrm T}M_sB_3^{\mathrm T}S_{n_3}\right)\right\rangle\\ &\quad+\left\langle P_{s,4},P_{s,4}-\frac{\alpha_s}{4}\left[A_4^{\mathrm T}M_sB_4^{\mathrm T}+B_4M_s^{\mathrm T}A_4+S_{n_4}\left(A_4^{\mathrm T}M_sB_4^{\mathrm T}+B_4M_s^{\mathrm T}A_4\right)S_{n_4}\right]\right\rangle\\ &=\sum_{r=1}^{4}\left\|P_{s,1}\right\|^2-\alpha_s\sum_{r=1}^{4}\left\langle A_rP_{s,r}B_r,M_s\right\rangle\\ &=\sum_{r=1}^{4}\left\|P_{s,1}\right\|^2-\alpha_s\left\langle\sum_{r=1}^{4}A_r\left(Q_{s,r}-\beta_{s-1}Q_{s-1,r}\right)B_r,M_s\right\rangle\\ &=\sum_{r=1}^{4}\left\|P_{s,1}\right\|^2-\alpha_s\left\|M_s\right\|^2=0.\end{aligned}$$

$$\begin{aligned}\left\langle M_s,M_{s+1}\right\rangle &= \left\langle M_s,\sum_{r=1}^{4}A_r\left(P_{s+1,r}+\beta_sQ_{s,r}\right)B_r\right\rangle=\beta_s\left\|M_s\right\|^2+\left\langle M_s,\sum_{r=1}^{4}A_rP_{s+1,r}B_r\right\rangle\\ &=\beta_s\left\|M_s\right\|^2+\frac{1}{\alpha_s}\left\langle R_s-R_{s+1},\sum_{r=1}^{4}A_rP_{s+1,r}B_r\right\rangle=\beta_s\left\|M_s\right\|^2-\frac{1}{\alpha_s}\sum_{r=1}^{4}\left\langle P_{s+1,r},P_{s+1,r}\right\rangle=0.\end{aligned}$$

$$\sum_{r=1}^{4}\left\langle Q_{s,r},P_{s+1,r}\right\rangle=\left\langle Q_{s,1},P_{s,1}-\alpha_s A_1^{\mathrm{T}}M_sB_1^{\mathrm{T}}\right\rangle+\left\langle Q_{s,2},P_{s,2}-\frac{\alpha_s}{2}\left(A_2^{\mathrm{T}}M_sB_2^{\mathrm{T}}+B_2M_s^{\mathrm{T}}A_2\right)\right\rangle$$
$$+\left\langle Q_{s,3},P_{s,3}-\frac{\alpha_s}{2}\left(A_3^{\mathrm{T}}M_sB_3^{\mathrm{T}}+S_{n_3}A_3^{\mathrm{T}}M_sB_3^{\mathrm{T}}S_{n_3}\right)\right\rangle$$
$$+\left\langle Q_{s,4},P_{s,4}-\frac{\alpha_s}{4}\left[A_4^{\mathrm{T}}M_sB_4^{\mathrm{T}}+B_4M_s^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}M_sB_4^{\mathrm{T}}+B_4M_s^{\mathrm{T}}A_4\right)S_{n_4}\right]\right\rangle$$
$$=\sum_{r=1}^{4}\left\langle Q_{s,r},P_{s,r}\right\rangle-\alpha_s\sum_{r=1}^{4}\left\langle Q_{s,r},A_r^{\mathrm{T}}M_sB_r^{\mathrm{T}}\right\rangle=\sum_{r=1}^{4}\left\|P_{s,r}\right\|^2-\alpha_s\left\|M_s\right\|^2=0.$$

Then the conclusion $\sum_{r=1}^{4}\left\langle P_{s,r},P_{s+1,r}\right\rangle=0$ and the assumption $\sum_{r=1}^{4}\left\langle P_{j,r},P_{s,r}\right\rangle=0$ show that $\left\langle M_j,M_{s+1}\right\rangle=0$ for all $j\le s-1$. By the principal of induction, we know that Eq.(3) holds for all $0\le j<i\le k$, and Equation (1) and Equation (2) hold for all $i,j=0,1,2,\cdots,k,i\ne j$ due to the fact that $\langle A,B\rangle=\langle B,A\rangle$ holds for all matrices *A* and *B* in $R^{m\times n}$. □

Lemma 7. shows that the matrix sequence

$$\begin{pmatrix}P_{0,1}&&&\\&P_{0,2}&&\\&&P_{0,3}&\\&&&P_{0,4}\end{pmatrix},\begin{pmatrix}P_{1,1}&&&\\&P_{1,2}&&\\&&P_{1,3}&\\&&&P_{1,4}\end{pmatrix},\cdots$$

generated by Algorithm 1 are orthogonal each other in the finite dimension matrix space $R^{\sum_{r=1}^{4}n_r\times\sum_{r=1}^{4}n_r}$. Hence the iterative method will be terminated at most $\sum_{r=1}^{4}n_r^2$ steps in the absence of roundoff errors.

It is worth to note that the conclusions of Lemma 7 may not be true without the assumptions $\alpha_i\ne 0$ and $\alpha_i\ne\infty$. Hence it is necessary to consider the case that $\alpha_i=0$ or $\alpha_i=\infty$.

If $\alpha_i=0$, which implies $\sum_{r=1}^{4}\left\|P_{i,r}\right\|=0$, it follows that $P_{i,r}=0,r=1,2,3,4$.

If $\alpha_i=\infty$, which implies $\left\|M_i\right\|=0$, then we have $\sum_{r=1}^{4}A_rQ_{i,r}B_r=0$, making inner product with R_i by both side, yields

$$\left\langle\sum_{r=1}^{4}A_rQ_{i,r}B_r,R_i\right\rangle=\sum_{r=1}^{4}\left\langle Q_{i,r},A_r^{\mathrm{T}}R_iB_r^{\mathrm{T}}\right\rangle=\sum_{r=1}^{4}\left\langle Q_{i,r},P_{i,r}\right\rangle=\sum_{r=1}^{4}\left\|P_{i,r}\right\|^2=0.$$

So the discussions above show that if there exist a positive integer *i* such that the coefficient $\alpha_i=0$ or $\alpha_i=\infty$, then the corresponding matrix group $\left[X_1^{(i)},X_2^{(i)},X_3^{(i)},X_4^{(i)}\right]\in K$ is just the solution of Problem I.

Together with Lemma 7 and the discussion about the coefficient α_i, we can conclude the following theorem.

Theorem 1. *For an arbitrary initial matrix group* $\left[X_1^{(0)},X_2^{(0)},X_3^{(0)},X_4^{(0)}\right]\in K$, *the matrix group sequence* $\left[X_1^{(k)},X_2^{(k)},X_3^{(k)},X_4^{(k)}\right]\in K$ *generated by Algorithm* 1 *will converge to a solution of Problem I at infinite iteration steps in exact arithmetic.*

By choosing a special kind of initial matrix group, we can obtain the unique least norm of Problem I. To this end, we first define a matrix set as follows

$$S=\left\{\left[X_1,X_2,X_3,X_4\right]\left|\begin{array}{l}X_1=A_1^{\mathrm{T}}HB_1^{\mathrm{T}},\\X_2=\frac{1}{2}\left(A_2^{\mathrm{T}}HB_2^{\mathrm{T}}+B_2H^{\mathrm{T}}A_2\right),\\X_3=\frac{1}{2}\left(A_2^{\mathrm{T}}HB_2^{\mathrm{T}}+S_{n_3}A_2^{\mathrm{T}}HB_2^{\mathrm{T}}S_{n_3}\right),\\X_4=\frac{1}{4}\left(A_4^{\mathrm{T}}HB_4^{\mathrm{T}}+B_4H^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}HB_4^{\mathrm{T}}+B_4H^{\mathrm{T}}A_4\right)S_{n_4}\right)\end{array}\right.\right\}$$

where $H\in R^{p\times q}$. Evidently, *S* is a linear subspace of *K*.

Theorem 2. *If we choose the initial matrix group* $\left[X_1^{(0)},X_2^{(0)},X_3^{(0)},X_4^{(0)}\right]\in S$, *especially, let*

$X_i^{(0)}=0\left(i=1,2,3,4\right)$, *we can obtain the least norm solution of Problem I.*

Proof: By the Algorithm 1 and Theorem 1, if we choosing initial matrix group $\left[X_1^{(0)},X_2^{(0)},X_3^{(0)},X_4^{(0)}\right]\in S$, we can obtain the solution $\left[\hat{X}_1,\hat{X}_2,\hat{X}_3,\hat{X}_4\right]$ of Problem I with finite iteration steps and there exist a matrix $\tilde{H}\in R^{p\times q}$ such that the solution $\left[\hat{X}_1,\hat{X}_2,\hat{X}_3,\hat{X}_4\right]$ can be represented that

$$\hat{X}_1=A_1^{\mathrm{T}}\tilde{H}B_1^{\mathrm{T}},$$
$$\hat{X}_2=\frac{1}{2}\left(A_2^{\mathrm{T}}\tilde{H}B_2^{\mathrm{T}}+B_2\tilde{H}^{\mathrm{T}}A_2\right),$$
$$\hat{X}_3=\frac{1}{2}\left(A_2^{\mathrm{T}}\tilde{H}B_2^{\mathrm{T}}+S_{n_3}A_2^{\mathrm{T}}\tilde{H}B_2^{\mathrm{T}}S_{n_3}\right),$$
$$\hat{X}_4=\frac{1}{4}\left(A_4^{\mathrm{T}}\tilde{H}B_4^{\mathrm{T}}+B_4\tilde{H}^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}\tilde{H}B_4^{\mathrm{T}}+B_4\tilde{H}^{\mathrm{T}}A_4\right)S_{n_4}\right).$$

By Lemma 6 we know that arbitrary solution of Problem I can be express as

$$\left[\hat{X}_1+\bar{X}_1,\hat{X}_2+\bar{X}_2,\hat{X}_3+\bar{X}_3,\hat{X}_4+\bar{X}_4\right],$$

where matrix group $\left[\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4\right]\in K$ satisfies (7).

Then

$$\begin{aligned}\sum_{r=1}^{4}\left\langle\hat{X}_r,\bar{X}_r\right\rangle&=\left\langle A_1^{\mathrm{T}}\tilde{H}B_1^{\mathrm{T}},\bar{X}_1\right\rangle+\frac{1}{2}\left\langle A_2^{\mathrm{T}}\tilde{H}B_2^{\mathrm{T}}+B_2\tilde{H}^{\mathrm{T}}A_2,\bar{X}_2\right\rangle\\&\quad+\frac{1}{2}\left\langle A_2^{\mathrm{T}}\tilde{H}B_2^{\mathrm{T}}+S_{n_3}A_2^{\mathrm{T}}\tilde{H}B_2^{\mathrm{T}}S_{n_3},\bar{X}_3\right\rangle\\&\quad+\frac{1}{4}\left\langle A_4^{\mathrm{T}}\tilde{H}B_4^{\mathrm{T}}+B_4\tilde{H}^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}\tilde{H}B_4^{\mathrm{T}}+B_4\tilde{H}^{\mathrm{T}}A_4\right)S_{n_4},\bar{X}_4\right\rangle\\&=\left\langle\tilde{H},\sum_{r=1}^{4}A_r\bar{X}_rB_r\right\rangle=0.\end{aligned}$$

So we have

$$\sum_{r=1}^{4}\left\|\hat{X}_r+\bar{X}_r\right\|^2=\sum_{r=1}^{4}\left\|\hat{X}_r\right\|^2+\sum_{r=1}^{4}\left\|\bar{X}_r\right\|^2,$$

which implies that matrix group $\left[\hat{X}_1,\hat{X}_2,\hat{X}_3,\hat{X}_4\right]$ is the least norm solution of Problem I. □

Remark 2. *Since the solution of Problem I is no empty, so the* S_E *is a closed convex linear subspace, hence it is certain that the least norm solution group* $\left[\hat{X}_1,\hat{X}_2,\hat{X}_3,\hat{X}_4\right]$ *of Problem I is unique, and* $\left[\hat{X}_1,\hat{X}_2,\hat{X}_3,\hat{X}_4\right]\in S$. *If matrix group* $\left[X_1,X_2,X_3,X_4\right]\in S$ *is a solution of Problem I, then it just be the unique least norm solution of Problem I, i.e.* $X_i=\hat{X}_i\left(i=1,2,3,4\right)$.

3. The Minimization Property of Iterative Method

In this section, the minimization property of Algorithm 1 is characterized, which ensures the Algorithm 1 converges smoothly.

Theorem 3. *For an arbitrary initial matrix group* $\left[X_1^{(0)},X_2^{(0)},X_3^{(0)},X_4^{(0)}\right]\in K$, *the matrix group* $\left[X_1^{(k)},X_2^{(k)},X_3^{(k)},X_4^{(k)}\right]$ *generated by Algorithm 1 at the kth iteration step satisfies the following minimization problem*

$$\left\|A_1X_1^{(k)}B_1+\cdots+A_4X_4^{(k)}B_4-C\right\|^2=\min_{[X_1,X_2,X_3,X_4]\in F}\left\|A_1X_1B_1+\cdots+A_4X_4B_4-C\right\|^2$$

where F denote a affine subspace which has the following form

$$F=\left[X_1^{(0)},\cdots,X_4^{(0)}\right]+span\left\{\left[Q_{0,1},\cdots,Q_{0,4}\right],\left[Q_{1,1},\cdots,Q_{1,4}\right],\cdots,\left[Q_{k-1,1},\cdots,Q_{k-1,4}\right]\right\}.$$

Proof: For arbitrary matrix group $\left[X_1,X_2,X_3,X_4\right]\in F$, there exist a set of real number $\left\{t_i\right\}_0^{k-1}$ such that

$$\left[X_1,\cdots,X_4\right]=\left[X_1^{(0)},\cdots,X_4^{(0)}\right]+\sum_{i=0}^{k-1}t_i\left[Q_{i,1},\cdots,Q_{i,4}\right].$$

Denote

$$g\left(t_0,\cdots,t_{k-1}\right)=\left\|A_1\left(X_1^{(0)}+\sum_{i=0}^{k-1}t_iQ_{i,1}\right)B_1+\cdots+A_4\left(X_4^{(0)}+\sum_{i=0}^{k-1}t_iQ_{i,4}\right)B_4-C\right\|^2$$

by the conclusion Equation (2) in Lemma 7, we have

$$\begin{aligned}g\left(t_0,\cdots,t_{k-1}\right)&=\left\|A_1X_1^{(0)}B_1+\ldots+A_4X_4^{(0)}B_4-C+\sum_{i=0}^{k-1}t_i\left(A_1Q_{i,1}B_1+\cdots+A_4Q_{i,4}B_4\right)\right\|^2\\&=\left\|R_0\right\|^2+\sum_{i=0}^{k-1}t_i^2\left\|A_1Q_{i,1}B_1+\cdots+A_4Q_{i,4}B_4\right\|^2-2\sum_{i=0}^{k-1}t_i\left\langle A_1Q_{i,1}B_1+\cdots+A_4Q_{i,4}B_4,R_0\right\rangle\end{aligned}$$

where R_0 is the corresponding residual of initial matrix group $\left[X_1^{(0)},X_2^{(0)},X_3^{(0)},X_4^{(0)}\right]$. Algorithm 1 show that the matrix R_0 can be express as

$$R_0=R_i+\alpha_{i-1}\sum_{r=1}^{4}A_rQ_{i-1},rB_r+\alpha_{i-2}\sum_{r=1}^{4}A_rQ_{i-2},rB_r+\cdots+\alpha_0\sum_{r=1}^{4}A_rQ_0,rB_r.$$

Because $g\left(t_0,\cdots,t_{k-1}\right)$ is a continuous and differentiable function with respect to the *k* variable $t_i,\left(i=0,1,\cdots,k-1\right)$, we easily know that

$$g\left(t_0,t_1,\cdots,t_{k-1}\right)=\min$$

if and only if

$$\frac{\partial g\left(t_0,t_1,\cdots,t_{k-1}\right)}{\partial t_i}=0.$$

It follows from the conclusion in Lemma 7 that

$$t_i=\frac{\sum_{r=1}^{4}\left\langle A_rQ_{i,r}B_r,R_0\right\rangle}{\left\|\sum_{r=1}^{4}A_rQ_{i,r}B_r\right\|^2}=\frac{\sum_{r=1}^{4}\left\langle A_rQ_{i,r}B_r,R_i\right\rangle}{\left\|\sum_{r=1}^{4}A_rQ_{i,r}B_r\right\|^2}=\frac{\sum_{r=1}^{4}\left\langle Q_{i,r},P_{i,r}\right\rangle}{\left\|\sum_{r=1}^{4}A_rQ_{i,r}B_r\right\|^2}=\frac{\sum_{r=1}^{4}\left\|P_{i,r}\right\|^2}{\left\|\sum_{r=1}^{4}A_rQ_{i,r}B_r\right\|^2}=\alpha_i.$$

By the fact that

$$\min_{t_i}\left(t_0,t_1,\cdots,t_{k-1}\right)=\min_{\left[X_1,X_2,X_3,X_4\right]\in F}\left\|A_1X_1B_1+\cdots+A_4X_4B_4-C\right\|^2.$$

We complete the proof. □

Theorem 3 shows that the approximation solution $\left[X_1^{(k)},X_2^{(k)},X_3^{(k)},X_4^{(k)}\right]$ minimizes the residual norm in the affine subspace *F* for all initial matrix group within *K*. Furthermore, by the fact $\left[X_1^{(k-1)},X_2^{(k-1)},X_3^{(k-1)},X_4^{(k-1)}\right]\in F$, then we have

$$\left\|A_1X_1^{(k)}B_1+\cdots+A_4X_4^{(k)}B_4-C\right\|\le\left\|A_1X_1^{(k-1)}B_1+\cdots+A_4X_4^{(k-1)}B_4-C\right\|$$

which shows that the sequence

$$\left\|A_1X_1^{(0)}B_1+\cdots+A_4X_4^{(0)}B_4-C\right\|,\left\|A_1X_1^{(1)}B_1+\cdots+A_4X_4^{(1)}B_4-C\right\|,\cdots$$

is monotonically decreasing. The descent property of the residual norm of Equation (5) ensures that the Algorithm 1 possesses fast and smoothly convergence.

4. Numerical Examples

In this section, we present numerical examples to illustrate the efficiency of the proposed iteration method. All the tests are performed using Matlab 7.0 which has a machine precision of around 10^{-16}. Because of the error of calculation, the iteration will not stop within finite steps. Hence, we regard the approximation solution group $\left[X_1^{(k)}, X_2^{(k)}, X_3^{(k)}, X_4^{(k)}\right]$ as the solution of Problem I if the corresponding $P_{k,r}\left(r=1,2,3,4\right)$ satisfies $\sum_{r=1}^{4}\left\|P_{k,r}\right\|^2 \le 10e-010$.

Example 1. Given matrices $A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4$ and C as follows:

$$A_1=\begin{pmatrix} 1 & 1 & 5 & 2 & -3 & 0 \\ 1 & 0 & 0 & 6 & 0 & 0 \\ 3 & 12 & 7 & 6 & -12 & 3 \\ -2 & -2 & 2 & -2 & -6 & 4 \\ 2 & 2 & -2 & 2 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 11 & 23 & 4 & -14 & 9 \end{pmatrix}, B_1=\begin{pmatrix} -9 & 4 & 4 & -9 & 6 \\ 8 & 12 & 3 & 8 & 8 \\ 5 & 7 & 0 & 5 & -12 \\ -5 & 15 & 0 & -3 & 20 \\ 7 & 56 & 1 & 7 & 14 \\ -12 & 6 & 0 & -12 & -7 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 3 & 4 & 11 & -5 & 9 & 14 & 30 & 5 \\ 3 & -2 & 6 & 5 & 4 & 12 & -9 & 8 \\ 7 & 1 & 6 & 17 & -9 & 2.3 & 5.9 & -7.8 \\ 12 & 2.9 & 8 & -6 & 3.8 & 44 & 6.9 & -5.6 \\ -7.9 & 3.2 & 4.3 & 9.1 & -9.2 & 8 & -6.4 & 9 \\ 0 & -1.5 & 0 & 5 & 0 & 0 & 1 & 0 \\ 12 & 2.9 & 8 & -6 & 3.8 & 44 & 6.9 & -5.6 \end{pmatrix},$$

$$B_2=\begin{pmatrix} 4 & 4 & -5 & 4 & 4.2 \\ -1 & 12 & -2 & -1 & 21 \\ 3 & 7 & 3 & 3 & 5 \\ -2 & 9 & 2 & -2 & -9 \\ 5 & -8 & -10 & 5 & 12 \\ -9.1 & -9.4 & -12.5 & -9.1 & 7.6 \\ 12 & 7 & 22 & 12 & 11 \\ -9 & 13 & 15.9 & -9 & -6.3 \end{pmatrix}, B_3=\begin{pmatrix} 5 & 0 & 0 & 5 & 2.4 \\ 2 & 3 & 0 & 2 & 0 \\ 23 & 6 & 0 & 23 & 0 \\ 3 & 3 & 0 & 3 & 1 \\ 12 & -15 & 0 & 12 & 0 \\ -10 & 9 & 0 & -10 & 0 \\ -14 & -11.4 & 0 & -14 & 8 \end{pmatrix},$$

$$A_3=\begin{pmatrix} 0 & -2 & 3 & 5 & 0 & -1 & -1 \\ -1 & -3 & 2 & 4 & -5 & 1 & -21 \\ 0 & -2 & 3 & 5 & 0 & -1 & -14 \\ -1 & -3 & 2 & 4 & -5 & 1 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -3 & 2 & 1 & -5 & 1 & -2 \\ 21 & -4.8 & 10.9 & 44.2 & 13.5 & 2.87 & 3.65 \end{pmatrix},$$

$$A_4=\begin{pmatrix} 1 & 2 & 5 & 15 & 31 & -6 & -7.6 & 22 \\ -1 & -3 & -4 & -11 & 14 & 4 & 21.1 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 3 & 12 & 30 & 10 & 22.4 & -35.8 \\ 0 & -2 & 23 & 2 & -6 & 0 & 14.3 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 15 & 31 & -6 & -7.6 & 22.0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} -2 & -2 & -4 & 0 & -2 \\ -1 & -1 & -4 & 0 & -1 \\ -3 & -1 & -6 & 0 & -1 \\ 8 & 8 & -22 & 0 & 8 \\ 4 & 4 & 6 & 0 & 4 \\ 22 & 22 & 12 & 0 & 22 \\ 21 & 21 & 21.8 & 0 & 21 \\ 10.3 & 0 & 36.5 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} -72 & -49 & 26 & 220 & 134 \\ 342 & -67 & -50 & 251 & 39 \\ 134 & 24.9 & 130 & -128 & 60 \\ 190 & -28.6 & 225 & 41 & 120 \\ 87 & 136 & 44 & 52 & 23 \\ -45 & 35 & -24.9 & 35.7 & -63 \\ 41 & -96 & 35.9 & 64.8 & -66 \end{pmatrix}.$$

Choose the initial matrices $X_i^{(0)} = 0\left(i = 1,2,3,4\right)$ where 0 denotes zero matrix in appropriate dimension. Using Algorithm 1 and iterating 74 steps, we have the unique least norm solution $\left[X_1^{(74)}, X_2^{(74)}, X_3^{(74)}, X_4^{(74)}\right] \in K$ as follows:

$$X_1^{(74)} = \begin{pmatrix} 0.3488 & 0.3512 & -0.6731 & -0.8191 & 0.4135 & -0.3588 \\ 1.4504 & 0.8483 & -2.2031 & -4.2978 & 1.2281 & -0.6214 \\ 0.9954 & 0.5572 & -1.6690 & -1.5996 & 0.7736 & -0.4921 \\ 0.6201 & 0.8376 & -1.4185 & -1.7915 & 0.6907 & -0.9980 \\ -1.5711 & -0.5939 & 2.2658 & 4.3840 & -0.7899 & 0.3719 \\ 0.3831 & 0.1950 & -0.4999 & -0.8051 & 0.5968 & -0.0280 \end{pmatrix},$$

$$X_2^{(74)} = \begin{pmatrix} 0.0837 & 0.7782 & 0.1083 & -0.3906 & 0.3401 & 0.3817 & -0.1904 & 0.0314 \\ 0.7782 & 0.7113 & 0.7072 & 0.1963 & -0.4179 & -0.3313 & 0.4221 & -0.5675 \\ 0.1083 & 0.7072 & 0.1762 & -0.3788 & 0.2916 & 0.1869 & 0.2184 & -0.2188 \\ -0.3906 & 0.1963 & -0.3788 & -0.0473 & -0.0796 & 0.3926 & -0.2816 & 0.3870 \\ 0.3401 & -0.4179 & 0.2916 & -0.0796 & 0.0976 & -0.4232 & -0.0532 & -0.2994 \\ 0.3817 & -0.3313 & 0.1869 & 0.3926 & -0.4232 & 0.3104 & -0.0502 & -0.2933 \\ -0.1904 & 0.4221 & 0.2184 & -0.2816 & -0.0532 & -0.0502 & -0.1573 & 0.3146 \\ 0.0314 & -0.5675 & -0.2188 & 0.3870 & -0.2994 & -0.2933 & 0.3146 & -0.8281 \end{pmatrix},$$

$$X_3^{(74)} = \begin{pmatrix} -0.2550 & 0.0991 & 0.0210 & 0.0156 & -0.0884 & 0.0125 & -0.0180 \\ -0.2521 & -0.0117 & -0.0403 & 0.0352 & 0.2365 & -0.0112 & 0.2756 \\ 0.8419 & -0.0292 & -0.0156 & 0.0419 & -0.2079 & 0.0161 & 0.1924 \\ 0.3561 & 0.0560 & 0.0341 & 0.1414 & 0.0341 & 0.0560 & 0.3561 \\ 0.1924 & 0.0161 & -0.2079 & 0.0419 & -0.0156 & -0.0292 & 0.8419 \\ 0.2756 & -0.0112 & 0.2365 & 0.0352 & -0.0403 & -0.0117 & -0.2521 \\ -0.0180 & 0.0125 & -0.0884 & 0.0156 & 0.0210 & 0.0991 & -0.2550 \end{pmatrix},$$

$$X_4^{(74)} = \begin{pmatrix} -0.2988 & 0.0807 & -0.0264 & 0.1130 & -0.1272 & 0.1788 & -0.0045 & 0.0002 \\ 0.0807 & 0.1392 & 0.1494 & -0.1014 & 0.0934 & -0.0472 & -0.0271 & -0.0045 \\ -0.0264 & 0.1494 & 0.0308 & -0.1096 & -0.0239 & 0.0030 & -0.0472 & 0.1788 \\ 0.1130 & -0.1014 & -0.1096 & -0.1113 & -0.2907 & -0.0239 & 0.0934 & -0.1272 \\ -0.1272 & 0.0934 & -0.0239 & -0.2907 & -0.1113 & -0.1096 & -0.1014 & 0.1130 \\ 0.1788 & -0.0472 & 0.0030 & -0.0239 & -0.1096 & 0.0308 & 0.1494 & -0.0264 \\ -0.0045 & -0.0271 & -0.0472 & 0.0934 & -0.1014 & 0.1494 & 0.1392 & 0.0807 \\ 0.0002 & -0.0045 & 0.1788 & -0.1272 & 0.1130 & -0.0264 & 0.0807 & -0.2988 \end{pmatrix},$$

with

$$\sum_{r=1}^{4}\left\|P_{74,r}\right\|^2 = 3.4778e-011, \text{ and } \sum_{r=1}^{4}\left\|X_i^{(74)}\right\| = 14.0628.$$

And

$$\min_{[X_1,X_2,X_3,X_4]\in K}\left\|A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 - C\right\| = 57.0635.$$

If we let the initial matrix $X_1^{(0)} = -2I_6, X_2^{(0)} = 5I_8, X_3^{(0)} = I_7, X_4^{(0)} = 3I_8$, noting that $[-2I_6, 5I_8, I_7, 3I_8]$ within K but not within S, then we have

$$\tilde{X}_1^{(82)} = \begin{pmatrix} -1.6182 & 0.5769 & -0.7270 & -0.5419 & 0.7736 & -0.5307 \\ 1.6801 & -1.2937 & -2.4655 & -3.7133 & 0.3589 & -0.6108 \\ 1.6186 & 0.9272 & -3.6464 & -1.7355 & 1.8356 & -0.1625 \\ 0.6687 & 0.8635 & -1.6254 & -2.9306 & 0.7439 & -1.0791 \\ -2.2221 & 0.1884 & 2.4556 & 4.3019 & -1.1991 & -0.5606 \\ 0.8887 & -0.0064 & -0.6313 & -1.1530 & 0.2455 & -1.5494 \end{pmatrix}$$

$$\tilde{X}_2^{(82)} = \begin{pmatrix} 4.3975 & 0.4156 & 0.0790 & -0.1835 & -0.1415 & 0.1494 & -0.2318 & 1.1825 \\ 0.4156 & 4.8742 & 0.2354 & 0.7305 & -0.1579 & -1.9460 & -0.5984 & -0.4272 \\ 0.0790 & 0.2354 & 4.9505 & -0.4095 & 0.6208 & 0.1762 & -0.4792 & -0.6346 \\ -0.1835 & 0.7305 & -0.4095 & 3.4749 & 1.1343 & 1.0277 & 0.7031 & -0.5118 \\ -0.1415 & -0.1579 & 0.6208 & 1.1343 & 4.2586 & -0.8962 & -0.6739 & 0.9514 \\ 0.1494 & -1.9460 & 0.1762 & 1.0277 & -0.8962 & 2.2692 & 1.5568 & -0.3421 \\ -0.2318 & -0.5984 & -0.4792 & 0.7031 & -0.6739 & 1.5568 & 1.6349 & 0.1487 \\ 1.1825 & -0.4272 & -0.6346 & -0.5118 & 0.9514 & -0.3421 & 0.1487 & 2.2651 \end{pmatrix},$$

$$\tilde{X}_3^{(82)} = \begin{pmatrix} 0.0400 & 0.4583 & -0.3783 & 0.2769 & 0.3650 & 0.2883 & -0.1037 \\ -0.2169 & 0.9919 & 0.0212 & 0.0438 & 0.2006 & 0.0163 & 0.1040 \\ 0.4283 & 0.1148 & 0.7425 & 0.1080 & -0.0350 & 0.0991 & 0.2474 \\ 0.3206 & 0.2510 & 0.0548 & 1.3436 & 0.0548 & 0.2510 & 0.3206 \\ 0.2474 & 0.0991 & -0.0350 & 0.1080 & 0.7425 & 0.1148 & 0.4283 \\ 0.1040 & 0.0163 & 0.2006 & 0.0438 & 0.0212 & 0.9919 & -0.2169 \\ -0.1037 & 0.2883 & 0.3650 & 0.2769 & -0.3783 & 0.4583 & 0.0400 \end{pmatrix},$$

$$\tilde{X}_4^{(82)} = \begin{pmatrix} 0.6809 & -0.3258 & 0.3206 & -0.1912 & 0.7079 & 0.1287 & 0.2288 & 0.1271 \\ -0.3258 & 1.5242 & -0.7005 & -0.7838 & 0.3594 & -0.2261 & 0.1465 & 0.2288 \\ 0.3206 & -0.7005 & 2.6931 & -0.4311 & -0.1002 & -0.4047 & -0.2261 & 0.1287 \\ -0.1912 & -0.7838 & -0.4311 & 2.4751 & -0.0815 & -0.1002 & 0.3594 & 0.7079 \\ 0.7079 & 0.3594 & -0.1002 & -0.0815 & 2.4751 & -0.4311 & -0.7838 & -0.1912 \\ 0.1287 & -0.2261 & -0.4047 & -0.1002 & -0.4311 & 2.6931 & -0.7005 & 0.3206 \\ 0.2288 & 0.1465 & -0.2261 & 0.3594 & -0.7838 & -0.7005 & 1.5242 & -0.3258 \\ 0.1271 & 0.2288 & 0.1287 & 0.7079 & -0.1912 & 0.3206 & -0.3258 & 0.6809 \end{pmatrix},$$

with

$$\sum_{r=1}^{4}\left\|\tilde{P}_{82,r}\right\|^2 = 5.9073e-011, \text{ and } \sum_{r=1}^{4}\left\|\tilde{X}_i^{(82)}\right\| = 31.2518 > \sum_{r=1}^{4}\left\|X_i^{(74)}\right\| = 14.0628.$$

And

$$\min_{[X_1,X_2,X_3,X_4]\in K}\|A_1X_1B_1+A_2X_2B_2+A_3X_3B_3+A_4X_4B_4-C\|=57.0635.$$

Example 2. Suppose that the matrices $A_1,B_1,A_2,B_2,A_3,B_3,A_4,B_4$ are the same as Example 1, let $C=A_1\tilde{X}_1B_1+A_2\tilde{X}_2B_2+A_3\tilde{X}_3B_3+A_4\tilde{X}_4B_4$, where $\tilde{X}_1=-2I_6$, $\tilde{X}_2=5I_8$, $\tilde{X}_3=I_7$, $\tilde{X}_4=3I_8$, that is to say, Equation (5) is consistent over set *K*. Then similarly Algorithm 2.1 in Peng [14] we can conduct another iteration algorithm as follows:

Algorithm 2. For an arbitrary initial matrix group $\left[X_1^{(0)},X_2^{(0)},X_3^{(0)},X_4^{(0)}\right]\in K$, compute

Step 1. $R_0=C-\sum_{r=1}^{4}A_rX_r^{(0)}B_r$;

$$P_{0,1}=A_1^{\mathrm{T}}R_0B_1^{\mathrm{T}};$$

$$P_{0,2}=\frac{1}{2}\left(A_2^{\mathrm{T}}R_0B_2^{\mathrm{T}}+B_2R_0^{\mathrm{T}}A_2\right);$$

$$P_{0,3}=\frac{1}{2}\left(A_3^{\mathrm{T}}R_0B_3^{\mathrm{T}}+S_{n_3}A_3^{\mathrm{T}}RB_3^{\mathrm{T}}S_{n_3}\right);$$

$$P_{0,4}=\frac{1}{4}\left(A_4^{\mathrm{T}}R_0B_4^{\mathrm{T}}+B_4R_0^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}R_0B_4^{\mathrm{T}}+B_4R_0^{\mathrm{T}}A_4\right)S_{n_4}\right);$$

$$Q_{0,r}=P_{0,r},\quad r=1,2,3,4.$$

Step 2. If $R_k=0$, then stop; else, $k:=k+1$, and compute

Step 3. $M_k=\sum_{r=1}^{4}A_rQ_{k,r}B_r$;

$$\alpha_k=\frac{\|R_k\|^2}{\left\|\sum_{r=1}^{4}Q_{k,r}\right\|^2},\quad X_r^{(k+1)}=X_r^{(k)}+\alpha_kQ_{k,r},\quad r=1,2,3,4;$$

$$R_{k+1}=R_k-\alpha_kM_k;$$

$$P_{k+1,1}=A_1^{\mathrm{T}}R_{k+1}B_1^{\mathrm{T}}=P_{k,1}-\alpha_kA_1^{\mathrm{T}}M_kB_1^{\mathrm{T}};$$

$$P_{k+1,2}=\frac{1}{2}\left(A_2^{\mathrm{T}}R_{k+1}B_2^{\mathrm{T}}+B_2R_{k+1}^{\mathrm{T}}A_2\right)=P_{k,2}-\frac{\alpha_k}{2}\left[A_2^{\mathrm{T}}M_kB_2^{\mathrm{T}}+B_2M_k^{\mathrm{T}}A_2\right];$$

$$P_{k+1,3}=\frac{1}{2}\left(A_3^{\mathrm{T}}R_{k+1}B_3^{\mathrm{T}}+S_{n_3}A_3^{\mathrm{T}}R_{k+1}B_3^{\mathrm{T}}S_{n_3}\right)$$
$$=P_{k,3}-\frac{\alpha_k}{2}\left[A_3^{\mathrm{T}}M_kB_3^{\mathrm{T}}+S_{n_3}A_3^{\mathrm{T}}M_kB_3^{\mathrm{T}}S_{n_3}\right];$$

$$P_{k+1,4}=\frac{1}{4}\left(A_4^{\mathrm{T}}R_{k+1}B_4^{\mathrm{T}}+B_4R_{k+1}^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}R_{k+1}B_4^{\mathrm{T}}+B_4R_{k+1}^{\mathrm{T}}A_4\right)S_{n_4}\right)$$
$$=P_{k,4}-\frac{\alpha_k}{4}\left[A_4^{\mathrm{T}}M_kB_4^{\mathrm{T}}+B_4M_k^{\mathrm{T}}A_4+S_{n_4}\left(A_4^{\mathrm{T}}M_kB_4^{\mathrm{T}}+B_4M_k^{\mathrm{T}}A_4\right)S_{n_4}\right];$$

$$\beta_k=\frac{\|R_{k+1}\|^2}{\|R_k\|^2},\quad Q_{k+1,r}=P_{k+1,r}+\beta_kQ_{k,r},\quad r=1,2,3,4;$$

Step 4. Go to step 2.

The main differences of Algorithm 1 and Algorithm 2 are: in Algorithm 1 the selection of coefficient α_k make $\|R_{k+1}\|=\min$, and β_k such that $\left\langle\sum_{r=1}^{4}A_rQ_{k+1,r}B_r,\sum_{r=1}^{4}A_rQ_{k,r}B_r\right\rangle=0$, but in Algorithm 2, the choosing of α_k such that $\langle R_{k+1},R_k\rangle=0$, and β_k such that $\sum_{r=1}^{4}\langle Q_{k+1,r},Q_{k,r}\rangle=0$. Noting that Algorithm 2 satisfies the Galerkin condition, but lacks of minimization property. Choosing the initial matrix $X_i^{(0)}=0\,(i=1,2,3,4)$ where 0 denotes zero matrix in appropriate dimension, by making use of Algorithm 1 and Algorithm 2, we can

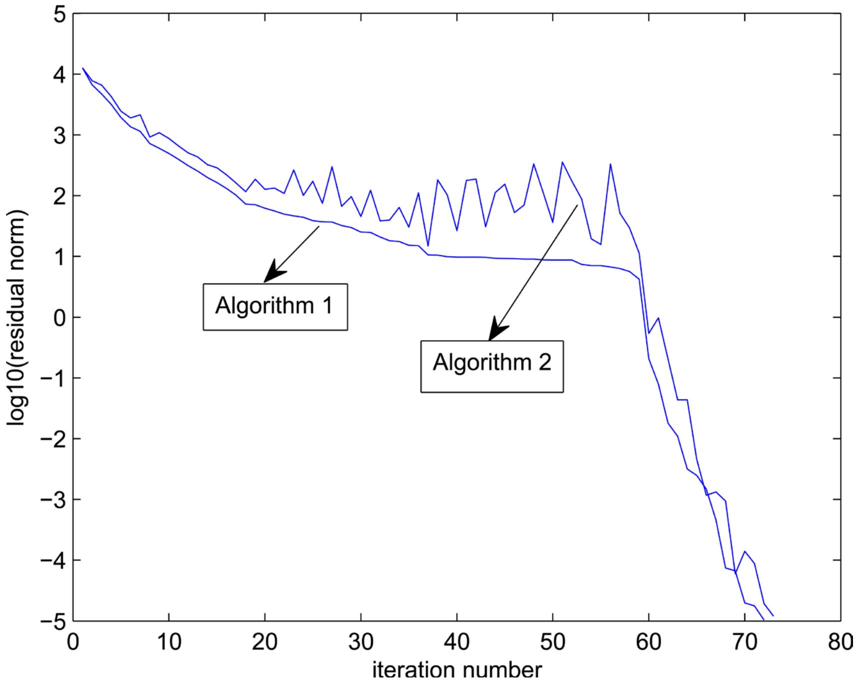

Figure 1. The comparison of residual norm between these two algorithm.

obtain the same least norm solution group, and we also obtain the convergence curves of residual norm shown in **Figure 1**. The results in this figure show clearly that the residual norm of Algorithm 1 is monotonically decreasing, which is in accordance with the theory established in this paper, and the convergence curve is more smooth than that in Algorithm 2.

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