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Jordan Γ^*-Derivation on Semiprime Γ-Ring *M* with Involution

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Abstract

Let M be a 2-torsion free semiprime Γ-ring with involution satisfying the condition that $a\alpha b\beta c = a\beta b\alpha c$ ($a,b,c \in M$ and $\alpha,\beta \in \Gamma$). In this paper, we will prove that if a non-zero Jordan Γ^*-derivation d on M satisfies $\left[d(x),x\right]_\alpha \in Z(M)$ for all $x \in M$ and $\alpha \in \Gamma$, then $\left[d(x),x\right]_\alpha = 0$.

Keywords

Γ-Ring M with Involution, Jordan Γ^*-Derivation, Commutative Γ-Ring

1. Introduction

The notion of Γ-ring was introduced as a generalized extension of the concept on classical ring. From its first appearance, the extensions and the generalizations of various important results in the theory of classical rings to the theory of Γ-rings have attracted a wider attention as an emerging field of research to enrich the world of algebra. A good number of prominent mathematicians have worked on this interesting area of research to develop many basic characterizations of Γ-rings. Nobusawa [1] first introduced the notion of a Γ-ring and showed that Γ-rings are more general than rings. Barnes [2] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Barnes [2], Luh [3], Kyuno [4], Hoque and Pual [5]-[7], Ceven [8], Dey *et al.* [9] [10], Vukman [11] and others obtained a large number of important basic properties of Γ-rings in various ways and developed more remarkable results of Γ-rings. We start with the following necessary introductory definitions.

Let M and Γ be additive abelian groups. If there exists an additive mapping $(x,\alpha,y) \to (x\alpha y)$ of $M \times \Gamma \times M \to M$ which satisfies the conditions:

1) $x\alpha y \in M$,
2) $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)y = x\alpha y + x\beta y, x\alpha(y+z) = x\alpha y + x\alpha z$,

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3) $(x\alpha y)\beta z = x\alpha(y\beta z)$, then M is called a Γ-ring [2]. Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Let M be a Γ-ring. Then M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be a commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x \text{ for all } \alpha \in \Gamma, y \in M\}$ is called the center of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the following basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \tag{1}$$

$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \tag{2}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Now, we consider the following assumption:

A....... $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

According to assumption (A), the above commutator identities reduce to $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$, which we will extensively used.

During the past few decades, many authors have studied derivations in the context of prime and semiprime rings and Γ-rings with involution [11]-[14]. The notion of derivation and Jordan derivation on a Γ-ring were defined by [15]. Let M be Γ-ring. An additive mapping $d: M \to M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. An additive mapping $d: M \to M$ is called a Jordan derivation if $d(x\alpha x) = d(x)\alpha x + x\alpha d(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

Definition 1 [16]. *An additive mapping* $(x\alpha x) \to (x\alpha x)^*$ *on a Γ-ring M is called an involution if* $(x\alpha y)^* = y^*\alpha x^*$ *and* $(x\alpha x)^{**} = (x\alpha x)$ *for all* $x, y \in M$ *and* $\alpha \in \Gamma$. *A Γ-ring M equipped with an involution is called a Γ-ring M with involution.*

Definition 2. *An element x in a Γ-ring M with involution is said to be hermitian if* $x^* = x$ *and skew-hermitian if* $x^* = -x$. *The sets of all hermitian and skew-hermitian elements of M will be denoted by* $H(M)$ *and* $S(M)$, *respectively.*

Example 1. *Let F be a field, and* $D_2(F)$ *be a set of all diagonal matrices of order* 2, *with respect to the usual operation of addition and multiplication on matrices and the involution * on* $D_2(F)$ *be defined by* $*\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in F$ *with* $*\left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\right) = \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix} : a, -a \in F$, *then we get* $x^* = x$ *and* $x^* = -x$.

Definition 3. *An additive mapping* $d: M \to M$ *is called a* Γ^**-derivation if* $d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y)$ *for all* $x, y \in M$ *and* $\alpha \in \Gamma$.

To further clarify the idea of Γ^*-derivation, we give the following example.

Example 2. *Let R be a commutative ring with characteristic of R equal* 2. *Define* $M = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in R\right\}$ *and* $\Gamma = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in R\right\}$, *then M and* Γ *are abelian groups under addition of matrices and M is a Γ-ring under multiplication of matrices.*

Define a mapping $d: M \to M$ by $d\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

To show that d is a Γ^*-derivation, let

$$x = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad y = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix}, \quad y^* = \begin{bmatrix} -c & d \\ 0 & -c \end{bmatrix},$$

then

$$\begin{aligned} d(x\alpha y) &= d\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}\right) = d\left(\begin{bmatrix} a\alpha & b\alpha \\ 0 & a\alpha \end{bmatrix}\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a\alpha c & a\alpha d + b\alpha c \\ 0 & a\alpha c \end{bmatrix}\right) = \begin{bmatrix} 0 & a\alpha d + b\alpha c \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now,

$$d(x)\alpha y^* + x\alpha d(y) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} -c & d \\ 0 & -c \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b\alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -c & d \\ 0 & -c \end{bmatrix} + \begin{bmatrix} a\alpha & b\alpha \\ 0 & a\alpha \end{bmatrix}\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -b\alpha c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a\alpha d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a\alpha d - b\alpha c \\ 0 & 0 \end{bmatrix},$$

since $char(R) = 2$, this implies that $d(x)\alpha y^* + x\alpha d(y) = \begin{bmatrix} 0 & a\alpha d + b\alpha c \\ 0 & 0 \end{bmatrix}$ then $d(x\alpha y) = d(x)\alpha y^* + x\alpha d(y)$. Hence, *d* is a Γ^*-derivation.

Definition 4. *An additive mapping* $d: M \to M$ *is called Jordan* Γ^**-derivation if* $d(x\alpha x) = d(x)\alpha x^* + x\alpha d(x)$ *for all* $x \in M$ *and* $\alpha \in \Gamma$.

Every Γ^*-derivation is a Jordan Γ^*-derivation, but the converse in general is not true as shown by the following example

Example 3. *Let M be a* Γ*-ring with involution and let* $a \in M$ *such that* $a\Gamma a = (0)$ *and* $x\alpha a\beta x = 0$ *for all* $x \in M$ *and* $\alpha, \beta \in \Gamma$, *but* $x\alpha a\beta y \neq 0$ *for some* $x, y \in M$ *such that* $x \neq y$.

Define a mapping $d: M \to M$ by $d(x) = x\alpha a + a\alpha x^*$ for all $x \in M$ and $\alpha \in \Gamma$. To show that *d* is a Jordan Γ^*-derivation, we have on the one hand that

$$\begin{aligned} d(x\alpha x) &= d(x)\alpha x^* + x\alpha d(x) = (x\alpha a + a\alpha x^*)\alpha x^* + x\alpha(x\alpha a + a\alpha x^*) \\ &= x\alpha a\alpha x^* + a\alpha x^*\alpha x^* + x\alpha x\alpha a + x\alpha a\alpha x^* \end{aligned} \tag{3}$$

for all $a, x \in M$ and $\alpha \in \Gamma$. On the other hand,

$$d(x\alpha x) = (x\alpha x)\alpha a + a\alpha(x\alpha x)^* = x\alpha x\alpha a + a\alpha x^*\alpha x^* \tag{4}$$

for all $a, x \in M$ and $\alpha \in \Gamma$. If we compare Equations ((3), (4)), we get

$$x\alpha a\alpha x^* + a\alpha x^*\alpha x^* + x\alpha x\alpha a + x\alpha a\alpha x^* = x\alpha x\alpha a + a\alpha x^*\alpha x^*$$

then after reduction we get that *d* is a Jordan Γ^*-derivation. Now to show that *d* is not a Γ^*-derivation, we have on the one hand that

$$\begin{aligned} d(x\alpha y) &= d(x)\alpha y^* + x\alpha d(y) = (x\alpha a + a\alpha x^*)\alpha y^* + x\alpha(y\alpha a + a\alpha y^*) \\ &= x\alpha a\alpha y^* + a\alpha x^*\alpha y^* + x\alpha y\alpha a + x\alpha a\alpha y^* \end{aligned} \tag{5}$$

for all $a, x, y \in M$ and $\alpha \in \Gamma$. On the other hand

$$d(x\alpha y) = (x\alpha y)\alpha a + a\alpha(x\alpha y)^* = x\alpha y\alpha a + a\alpha y^*\alpha x^* \tag{6}$$

for all $a, x, y \in M$ and $\alpha \in \Gamma$. If we compare Equations ((5), (6)), we get

$$x\alpha a\alpha y^* + a\alpha x^*\alpha y^* + x\alpha y\alpha a + x\alpha a\alpha y^* = x\alpha y\alpha a + a\alpha y^*\alpha x^*$$

then after reduction we get that *d* is not a Γ^*-derivation.

In this paper we will prove that if a non-zero Jordan Γ^*-derivation *d* of a 2-torsion free semiprime Γ-ring *M* with involution satisfies $[d(x), x]_\alpha \in Z(M)$ for all $x \in M$ and $\alpha \in \Gamma$, then $[d(x), x]_\alpha = 0$.

2. The Relation between Jordan Γ*-Derivation and $[d(x), x]_\alpha \in Z(M)$ on Semiprime Γ-Ring *M* with Involution

To prove our main results we need the following lemmas.

Lemma 1. *Let M be a* 2*-torsion free semiprime* Γ*-ring with involution and* $d: M \to M$ *be a Jordan* Γ^**-derivation which satisfies* $[d(x), x]_\alpha \in Z(M)$ *for all* $x \in M$ *and* $\alpha \in \Gamma$, *then* $[d(h), h]_\alpha = 0$ *for all*

$h \in H(M)$ *and* $\alpha \in \Gamma$.

Proof. We have

$$[d(x), x]_\alpha \in Z(M) \tag{7}$$

for all $x \in M$ and $\alpha \in \Gamma$. Putting $x\alpha x$ for x in (7), we get

$$[d(x\alpha x), x\alpha x]_\alpha \in Z(M) \tag{8}$$

for all $x \in M$ and $\alpha \in \Gamma$. Therefore,

$$[d(x)\alpha x^* + x\alpha d(x), x\alpha x]_\alpha \in Z(M)$$

for all $x \in M$ and $\alpha \in \Gamma$. Setting $x = h \in H(M)$ in the above relation, we get

$$[d(h)\alpha h + h\alpha d(h), h\alpha h]_\alpha \in Z(M) \tag{9}$$

for all $h \in H(M)$ and $\alpha \in \Gamma$, because of

$$d(h)\alpha h + h\alpha d(h) = 2h\alpha d(h) - [h, d(h)]_\alpha \tag{10}$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. According to (9) and (10), we get

$$\left[2h\alpha d(h) - [h, d(h)]_\alpha, h\alpha h\right]_\alpha \in Z(M) \tag{11}$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. Then from relation (11)

$$[2h\alpha d(h), h\alpha h]_\alpha - \left[[h, d(h)]_\alpha, h\alpha h\right]_\alpha \in Z(M)$$

$$[2h\alpha d(h), h\alpha h]_\alpha + \left[[h, d(h)]_\alpha, h\alpha h\right]_\alpha \in Z(M)$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. Since $[d(h), h]_\alpha \in Z(M)$, then $\left[[d(h), h]_\alpha, h\alpha h\right]_\alpha = 0$, and hence from the above relation

$$[2h\alpha d(h), h\alpha h]_\alpha = 2\left(h\alpha [d(h), h\alpha h]_\alpha + [h, h\alpha h]_\alpha \alpha d(h)\right) \in Z(M)$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. Therefore,

$$\begin{aligned} 2h\alpha [d(h), h\alpha h]_\alpha &= 2h\alpha\left(h\alpha [h, d(h)]_\alpha + [h, d(h)]_\alpha \alpha h\right) \\ &= 2h\alpha h\alpha [h, d(h)]_\alpha + 2h\alpha [h, d(h)]_\alpha \alpha h \\ &= 2h\alpha h\alpha [h, d(h)]_\alpha + 2h\alpha h\alpha [h, d(h)]_\alpha \in Z(M) \end{aligned}$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. Then by using assumption (A), we obtain

$$4h\alpha h\gamma [h, d(h)]_\alpha \in Z(M) \tag{12}$$

for all $h \in H(M)$ and $\alpha, \gamma \in \Gamma$. And

$$4\left[h\alpha h\gamma [h, d(h)]_\alpha, d(h)\right]_\beta = 0 \tag{13}$$

for all $h \in H(M)$ and $\alpha, \beta, \gamma \in \Gamma$. Then from (13),

$$\begin{aligned} &4\left(h\alpha h\gamma \left[[h, d(h)]_\alpha, d(h)\right]_\beta + [h\alpha h, d(h)]_\alpha \gamma [h, d(h)]_\beta\right) \\ &= 4\left([h\alpha h, d(h)]_\alpha \gamma [h, d(h)]_\beta\right) \\ &= 4\left(h\alpha [h, d(h)]_\alpha + [h, d(h)]_\alpha \alpha h\right)\gamma [h, d(h)]_\beta \\ &= 4h\alpha [h, d(h)]_\alpha \gamma [h, d(h)]_\beta + 4[h, d(h)]_\alpha \alpha h\gamma [h, d(h)]_\beta \\ &= 4h\alpha [h, d(h)]_\alpha \gamma [h, d(h)]_\beta + 4[h, d(h)]_\alpha \alpha h\gamma [h, d(h)]_\beta = 0 \end{aligned}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Since $[h,d(h)]_\alpha \in Z(M)$, then from the above relation

$$4h\alpha[h,d(h)]_\alpha \gamma [h,d(h)]_\beta + 4h\alpha[d(h),h]_\alpha \gamma [d(h),h]_\beta$$
$$= 4h\alpha[d(h),h]_\alpha \gamma [d(h),h]_\beta + 4h\alpha[h,d(h)]_\alpha \gamma [h,d(h)]_\beta = 0,$$

hence by using assumption (A), we obtain

$$8h\gamma[h,d(h)]_\alpha \beta [h,d(h)]_\alpha = 0 \tag{14}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Therefore,

$$8\left[h\gamma[h,d(h)]_\alpha \beta [h,d(h)]_\alpha, d(h)\right]_\alpha = 0 \tag{15}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Then from relation (15),

$$8h\gamma\left[\left([h,d(h)]_\alpha \beta [h,d(h)]_\alpha\right), d(h)\right]_\alpha + 8[h,d(h)]_\alpha \gamma\left([h,d(h)]_\alpha \beta [h,d(h)]_\alpha\right) = 0$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. By using relation (15) and assumption (A), we get

$$8\left([h,d(h)]_\alpha \beta [h,d(h)]_\alpha\right)\delta[h,d(h)]_\alpha = 0 \tag{16}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Since M is 2-torsion free, we get

$$\left([h,d(h)]_\alpha \beta [h,d(h)]_\alpha\right)\delta[h,d(h)]_\alpha = 0 \tag{17}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Right multiplication of (17) by $z\delta[h,d(h)]_\alpha$ and using assumption (A), we get

$$\left([h,d(h)]_\alpha \beta [h,d(h)]_\alpha\right)\delta z \delta [h,d(h)]_\alpha \beta [h,d(h)]_\alpha = 0 \tag{18}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. By semiprimeness of M, we have

$$[h,d(h)]_\alpha \beta [h,d(h)]_\alpha = 0 \tag{19}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Left multiplication of (19) by z yields

$$[d(h),h]_\alpha \beta z \beta [d(h),h]_\alpha = 0 \tag{20}$$

for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. By semiprimeness of M again, we get $[d(h),h]_\alpha = 0$ for all $h \in H(M)$ and $\alpha,\beta,\gamma \in \Gamma$. □

Lemma 2 *Let M be a 2-torsion free semiprime Γ-ring with involution and $d: M \to M$ be a Jordan Γ^*-derivation which satisfies $[d(x),x]_\alpha \in Z(M)$ for all $x \in M$ and $\alpha \in \Gamma$, then $[d(s),s]_\alpha = 0$ for all $s \in S(M)$ and $\alpha \in \Gamma$.*

Proof. Putting $x+y$ for x in (7),

$$[d(x+y),x+y]_\alpha = [d(x),x]_\alpha + [d(x),y]_\alpha + [d(y),x]_\alpha + [d(y),y]_\alpha \in Z(M)$$

for all $x,y \in M$ and $\alpha \in \Gamma$. By using Lemma (1), we get

$$\left([d(x),y]_\alpha + [d(y),x]_\alpha\right) \in Z(M) \tag{21}$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Replacing x by $x\alpha x$ and y by x^* yields

$$\left([d(x\alpha x),x^*]_\alpha + [d(x^*),x\alpha x]_\alpha\right) \in Z(M) \tag{22}$$

for all $x \in M$ and $\alpha \in \Gamma$. Setting $x = s \in S(M)$, we get

$$\left([d(s\alpha s),s^*]_\alpha + [d(s^*),s\alpha s]_\alpha\right) \in Z(M) \tag{23}$$

for all $s \in S(M)$ and $\alpha \in \Gamma$. But

$$d(s\alpha s)=d(s)\alpha s^{*}+s\alpha d(s)=s\alpha d(s)-d(s)\alpha s=[s,d(s)]_{\alpha}\in Z(M) \quad (24)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. So then from (23) and (24), we get

$$\left([[s,d(s)]_{\alpha},s^{*}]_{\alpha}+[d(s^{*}),s\alpha s]_{\alpha}\right)\in Z(M)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Hence,

$$\left([[d(s),s]_{\alpha},s]_{\alpha}+[s\alpha s,d(s)]_{\alpha}\right)\in Z(M)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Since $[d(s),s]_{\alpha}\in Z(M)$, then $[[d(s),s]_{\alpha},s]_{\alpha}=0$, hence from the above relation,

$$[s\alpha s,d(s)]_{\alpha}\in Z(M) \quad (25)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Therefore,

$$[s\alpha s,d(s)]_{\alpha}=s\alpha[s,d(s)]_{\alpha}+[s,d(s)]_{\alpha}\alpha s=2s\alpha[s,d(s)]_{\alpha}\in Z(M) \quad (26)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Since $[s,d(s)]_{\alpha}\in Z(M)$, we obtain

$$2[s\alpha[s,d(s)]_{\alpha},d(s)]_{\alpha}=0 \quad (27)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Then from relation (27),

$$2\left(s\alpha[[s,d(s)]_{\alpha},d(s)]_{\alpha}+[s,d(s)]_{\alpha}\gamma[s,d(s)]_{\alpha}\right)=0$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. By using relation (27) again,

$$2[s,d(s)]_{\alpha}\gamma[s,d(s)]_{\alpha}=0 \quad (28)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Since M is 2-torsion free, we get

$$[d(s),s]_{\alpha}\gamma[d(s),s]_{\alpha}=0 \quad (29)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. Right multiplication by z yields

$$[d(s),s]_{\alpha}\gamma z\gamma[d(s),s]_{\alpha}=0 \quad (30)$$

for all $s\in S(M)$ and $\alpha\in\Gamma$. By semiprimeness of M, we therefore get $[d(s),s]_{\alpha}=0$ for all $s\in S(M)$ and $\alpha\in\Gamma$. □

Remark 1 [17]. *A* Γ*-ring M is called a simple* Γ*-ring if* $M\Gamma M\neq 0$ *and its ideals are* 0 *and M.*

Remark 2. *Let M be a* 2*-torsion free simple* Γ*-ring with involution, then every* $x\in M$ *can be uniquely represented in the form* $2x=h+s$ *where* $h\in H(M)$ *and* $s\in S(M)$.

Proof. Define $H(M)=\{x\in M; x^{*}=x\}$, $S(M)=\{x\in M; x^{*}=-x\}$, since $2M$ is an ideal of M and M is simple, it implies that $2M=M$. So for every $x\in M$, $x/2$ makes sense and so we can write

$$x=\frac{x+x^{*}}{2}+\frac{x-x^{*}}{2}$$

Now

$$\left(\frac{x+x^{*}}{2}\right)^{*}=\frac{1}{2}(x+x^{*})^{*}=\frac{1}{2}\left(x^{*}+(x^{*})^{*}\right)=\frac{1}{2}(x^{*}+x)=\frac{1}{2}(x+x^{*})=\frac{x+x^{*}}{2}$$

hence

$$\frac{x+x^{*}}{2}\in H(M)$$

and

$$\left(\frac{x-x^*}{2}\right)^* = \frac{1}{2}\left(x-x^*\right)^* = \frac{1}{2}\left(x^*-\left(x^*\right)\right)^* = \frac{1}{2}\left(x^*-x\right) = -\frac{1}{2}\left(x-x^*\right) = -\frac{x-x^*}{2}$$

hence

$$\frac{x-x^*}{2} \in S(M)$$

Therefore

$$x = \frac{x+x^*}{2} + \frac{x-x^*}{2} \in H(M) + S(M)$$

hence $M = H(M)+S(M)$. Let $x \in H(M)\cap S(M)$, then $x \in H(M)$ and $x \in S(M)$, so $x^* = x$ and $x^* = -x$. Therefore $x = -x$ which implies that $2x = 0$, so $x = 0$. Thus $H(M)\cap S(M) = 0$. Hence $2M = M = H(M)+S(M)$ implies that $2x = h+s$ where $h \in H(M)$ and $s \in S(M)$. □

Theorem 1. *Let M be a* 2*-torsion free semiprime* Γ*-ring with involution and* $d: M \to M$ *be a Jordan* Γ^**-derivation which satisfies* $\left[d(x),x\right]_\alpha \in Z(M)$ *for all* $x \in M$ *and* $\alpha \in \Gamma$ *and* $\left[d(h),s\right]_\alpha \in Z(M)$ *or* $\left[d(s),h\right]_\alpha \in Z(M)$ *for all* $h \in H(M)$, $s \in S(M)$ *and* $\alpha \in \Gamma$, *then* $[d(x),x]_\alpha = 0$ *for all* $x \in M$ *and* $\alpha \in \Gamma$.

Proof. Assume that $\left[d(h),s\right]_\alpha \in Z(M)$ for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. By using Lemma (1), we have

$$\left[d(h),h\right]_\alpha = 0 \tag{31}$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. For $h_1,h_2 \in H(M)$, putting h_1+h_2 for *h* in (31) yields

$$\left[d(h_1+h_2),h_1+h_2\right]_\alpha = \left[d(h_1),h_1\right]_\alpha + \left[d(h_1),h_2\right]_\alpha + \left[d(h_2),h_1\right]_\alpha + \left[d(h_2),h_2\right]_\alpha = 0$$

for all $h \in H(M)$ and $\alpha \in \Gamma$. By using relation (31), we obtain

$$\left[d(h_1),h_2\right]_\alpha + \left[d(h_2),h_1\right]_\alpha = 0 \tag{32}$$

for all $h_1,h_2 \in H(M)$ and $\alpha \in \Gamma$. Since $s\beta s \in H(M)$ for all $s \in S(M)$, then replace h_2 by $s\beta s$ in (32), to get

$$\left[d(h_1),s\beta s\right]_\alpha + \left[d(s\beta s),h_1\right]_\alpha = 0 \tag{33}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. By using Lemma (2), we have

$$d(s\beta s) = d(s)\beta s^* + s\beta d(s) = s\beta d(s) - d(s)\beta s = \left[s,d(s)\right]_\beta = 0 \tag{34}$$

for all $s \in S(M)$ and $\beta \in \Gamma$. According to relations (33) and (34), we get

$$\left[d(h_1),s\beta s\right]_\alpha + \left[\left[s,d(s)\right]_\beta,h_1\right]_\alpha = \left[d(h_1),s\beta s\right]_\alpha - \left[\left[d(s),s\right]_\beta,h_1\right]_\alpha = 0$$

for all $s \in S(M)$ and $\beta \in \Gamma$. By using Lemma (2), we get

$$\left[d(h_1),s\beta s\right]_\alpha = 0 \tag{35}$$

for all $s \in S(M)$ and $\beta \in \Gamma$. Then from relation (35),

$$s\beta\left[d(h_1),s\right]_\alpha + \left[d(h_1),s\right]_\alpha \beta s = 0$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Therefore since $\left[d(h_1),s\right]_\alpha \in Z(M)$, we obtain

$$2s\beta\left[d(h_1),s\right]_\alpha = 0 \tag{36}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Hence,

$$2\left[d(h_1),s\beta\left[d(h_1),s\right]_\alpha\right]_\gamma = 0 \tag{37}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Therefore,

$$2\left[d(h_1), s\beta\left[d(h_1),s\right]_\alpha\right]_\gamma = 2\left(s\beta\left[d(h_1),\left[d(h_1),s\right]_\alpha\right]_\gamma + \left[d(h_1),s\right]_\beta \gamma\left[d(h_1),s\right]_\beta\right) = 0$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Then by using (37), we get

$$2\left[d(h_1),s\right]_\beta \gamma\left[d(h_1),s\right]_\beta = 0 \tag{38}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\beta,\gamma \in \Gamma$. Since M is 2-torsion free, we get

$$\left[d(h_1),s\right]_\beta \gamma\left[d(h_1),s\right]_\beta = 0 \tag{39}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\beta,\gamma \in \Gamma$. Right multiplication of relation (39) by z yields

$$\left[d(h_1),s\right]_\beta \gamma z\gamma\left[d(h_1),s\right]_\beta = 0 \tag{40}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\beta,\gamma \in \Gamma$. By semiprimeness of M, we get

$$\left[d(h_1),s\right]_\alpha = 0 \tag{41}$$

for all $h_1 \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Putting s for x and h for y in relation (21), we get

$$\left[d(s),h\right]_\alpha + \left[d(h),s\right]_\alpha \in Z(M) \tag{42}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Comparing relations (41) and (42), we get

$$\left[d(s),h\right]_\alpha \in Z(M) \tag{43}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Since $h\gamma h \in H(M)$ for all $h \in H(M)$, then from relation (43), we obtain

$$\left[d(s),h\gamma h\right]_\alpha \in Z(M) \tag{44}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Then

$$\left[d(s),h\gamma h\right]_\alpha = h\gamma\left[d(s),h\right]_\alpha + \left[d(s),h\right]_\alpha \gamma h \in Z(M) \tag{45}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Since $\left[d(s),h\right]_\alpha \in Z(M)$ for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$, then from relation (45), we get

$$2h\gamma\left[d(s),h\right]_\alpha \in Z(M) \tag{46}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha,\gamma \in \Gamma$. Hence

$$2\left[d(s),h\gamma\left[d(s),h\right]_\alpha\right]_\beta = 0 \tag{47}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Then from relation (47),

$$2\left(h\gamma\left[d(s),\left[d(s),h\right]_\alpha\right]_\beta + \left[d(s),h\right]_\alpha \gamma\left[d(s),h\right]_\beta\right) = 0$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha,\beta,\gamma \in \Gamma$. Then by using (47), we get

$$2\left[d(s),h\right]_\beta \gamma\left[d(s),h\right]_\beta = 0 \tag{48}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta,\gamma \in \Gamma$. Since M is 2-torsion free, we get

$$\left[d(s),h\right]_\beta \gamma\left[d(s),h\right]_\beta = 0 \tag{49}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta,\gamma \in \Gamma$. Right multiplication of the relation (49) by z we get

$$\left[d(s),h\right]_\beta \gamma z\gamma\left[d(s),h\right]_\beta = 0 \tag{50}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta,\gamma \in \Gamma$. By semiprimeness of M, we get

$$\left[d(s),h\right]_{\beta}=0 \tag{51}$$

for all $h\in H(M)$, $s\in S(M)$ and $\beta\in\Gamma$. To prove $\left[d(x),x\right]_{\alpha}=0$, since *M* is 2-torsion free, we only show

$$4\left[d(x),x\right]_{\alpha}=0 \tag{52}$$

for all $x\in M$ and $\alpha\in\Gamma$. By using Remark 2, we have for all $x\in M$ and $\alpha\in\Gamma$ that $2x=s+h$ for all $s\in S(M)$ and $h\in H(M)$. Therefore

$$4\left[d(x),x\right]_{\alpha}=\left[d(2x),2x\right]_{\alpha}=\left[d(s+h),s+h\right]_{\alpha}=0$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$. Hence

$$4\left[d(x),x\right]_{\alpha}=\left[d(s),s\right]_{\alpha}+\left[d(h),s\right]_{\alpha}+\left[d(s),h\right]_{\alpha}+\left[d(h),h\right]_{\alpha}=0 \tag{53}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$. By using Lemma (1), Lemma (2) and relation (41), (51), we get

$$\left[d(x),x\right]_{\alpha}=0.$$

Now assume $\left[d(s),h\right]_{\alpha}\in Z(M)$ for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$. Since $h\gamma h\in H(M)$ for all $h\in H(M)$, then we get

$$\left[d(s),h\gamma h\right]_{\alpha}\in Z(M) \tag{54}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha,\gamma\in\Gamma$. Then from relation (54), we obtain

$$\left[d(s),h\gamma h\right]_{\alpha}=h\gamma\left[d(s),h\right]_{\alpha}+\left[d(s),h\right]_{\alpha}\gamma h\in Z(M) \tag{55}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$. Since $\left[d(s),h\right]_{\alpha}\in Z(M)$ for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$, then from relation (55), we get

$$2h\gamma\left[d(s),h\right]_{\alpha}\in Z(M) \tag{56}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha,\gamma\in\Gamma$. Hence

$$2\left[d(s),h\gamma\left[d(s),h\right]_{\alpha}\right]_{\beta}=0 \tag{57}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha,\beta,\gamma\in\Gamma$. Then from relation (57),

$$2\left(h\gamma\left[d(s),\left[d(s),h\right]_{\alpha}\right]_{\beta}+\left[d(s),h\right]_{\alpha}\gamma\left[d(s),h\right]_{\beta}\right)=0$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha,\beta,\gamma\in\Gamma$. Then by using relation (57) we get

$$2\left[d(s),h\right]_{\beta}\gamma\left[d(s),h\right]_{\beta}=0 \tag{58}$$

for all $h\in H(M)$, $s\in S(M)$ and $\beta,\gamma\in\Gamma$. Since *M* is 2-torsion free, we get

$$\left[d(s),h\right]_{\beta}\gamma\left[d(s),h\right]_{\beta}=0 \tag{59}$$

for all $h\in H(M)$, $s\in S(M)$ and $\beta,\gamma\in\Gamma$. Right multiplication of relation (59) by *z* yields

$$\left[d(s),h\right]_{\beta}\gamma z\gamma\left[d(s),h\right]_{\beta}=0 \tag{60}$$

for all $h\in H(M)$, $s\in S(M)$ and $\beta,\gamma\in\Gamma$. By semiprimeness of *M*, we get

$$\left[d(s),h\right]_{\alpha}=0 \tag{61}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$. Comparing relations (42) and (61), we get

$$\left[d(h),s\right]_{\alpha}\in Z(M) \tag{62}$$

for all $h\in H(M)$, $s\in S(M)$ and $\alpha\in\Gamma$. Since $s\gamma s\in H(M)$ for all $s\in S(M)$, then from (62), we obtain

$$\left[d(h), s\gamma s\right]_{\alpha} \in Z(M) \tag{63}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Then,

$$\left[d(h), s\gamma s\right]_{\alpha} = s\gamma\left[d(h), s\right]_{\alpha} + \left[d(h), s\right]_{\alpha}\gamma s \in Z(M) \tag{64}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Since $\left[d(h), s\right]_{\alpha} \in Z(M)$ for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$, then from relation (64), we get

$$2s\gamma\left[d(h), s\right]_{\alpha} \in Z(M) \tag{65}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha, \gamma \in \Gamma$. Hence

$$2\left[d(h), s\gamma\left[d(h), s\right]_{\alpha}\right]_{\beta} = 0 \tag{66}$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha, \beta, \gamma \in \Gamma$. Then from relation (66)

$$2\left(s\gamma\left[d(h), \left[d(h), s\right]_{\alpha}\right]_{\beta} + \left[d(h), s\right]_{\alpha}\gamma\left[d(h), s\right]_{\beta}\right) = 0$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha, \beta, \gamma \in \Gamma$. Then by using (66) we get

$$2\left[d(h), s\right]_{\beta}\gamma\left[d(h), s\right]_{\beta} = 0 \tag{67}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta, \gamma \in \Gamma$. Since M is 2-torsion free, we get

$$\left[d(h), s\right]_{\beta}\gamma\left[d(h), s\right]_{\beta} = 0 \tag{68}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta, \gamma \in \Gamma$. Right multiplication of relation (68) by z yields

$$\left[d(h), s\right]_{\beta}\gamma z\gamma\left[d(h), s\right]_{\beta} = 0 \tag{69}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta, \gamma \in \Gamma$. By semiprimeness of M, we get

$$\left[d(h), s\right]_{\beta} = 0 \tag{70}$$

for all $h \in H(M)$, $s \in S(M)$ and $\beta \in \Gamma$. Therefore, by using Lemma (1), Lemma (2) and relation (61), (70), we get a similar result as the first assumption $\left[d(x), x\right]_{\alpha} = 0$ for all $x \in M$ and $\alpha \in \Gamma$, and hence the proof of the theorem is complete. □

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