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On the Norms of *r*-Toeplitz Matrices Involving Fibonacci and Lucas Numbers

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Abstract

Let us define $A = T_r[a_{ij}]$ to be a $n \times n$ r-Toeplitz matrix. The entries in the first row of $A = T_r[a_{ij}]$ are $a_{ij} = F_{i-j}$ or $a_{ij} = L_{i-j}$ where F_n and L_n denote the usual Fibonacci and Lucas numbers, respectively. We obtained some bounds for the spectral norm of these matrices.

Keywords

***r*-Toeplitz Matrix, Fibonacci Numbers, Lucas Numbers, Spectral Norm**

1. Introduction

Toeplitz matrices arise in many different theoretical and applicative fields, in the mathematical modeling of all the problems where some sort of shift invariance occurs in terms of space or of time. As in computation of spline functions, time series analysis, signal and image processing, queueing theory, polynomial and power series computations and many other areas, typical problems modelled by Toeplitz matrices are the numerical solution of certain differential and integral equations [1]-[5].

Lots of article have been written so far, which concern estimates for spectral norms of Toeplitz matrices, which have connections with signal and image processing, time series analysis and many other problems [6]-[8]. Akbulak and Bozkurt found lower and upper bounds for the spectral norms of Toeplitz matrices with classical Fibonacci and Lucas numbers entries in [9]. Shen gave upper and lower bounds for the spectral norms of Toeplitz matrices with *k*-Fibonacci and *k*-Lucas numbers entries in [10].

In this paper, we derive expressions of spectral norms for *r*-Toeplitz matrices. We explain some preliminaries and well-known results. We thicken the identities of estimations for spectral norms of *r*-Toeplitz matrices.

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2. Preliminaries

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

and

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

The rule can be used to extend the sequence backwards. Hence

$$F_{-n} = (-1)^{n+1} F_n$$

and

$$L_{-n} = (-1)^n L_n.$$

If start from $n = 0$, then the Fibonacci and Lucas sequence are given by

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13
L_n	2	1	3	4	7	11	18	29
F_{-n}	0	1	−1	2	−3	5	−8	13
L_{-n}	2	−1	3	−4	7	−11	18	−29

The following sum formulas the Fibonacci and Lucas numbers are well known [11] [12]:

$$\sum_{i=1}^{n-1} F_i^2 = F_n F_{n-1}$$

$$\sum_{i=1}^{n-1} L_i^2 = L_n L_{n-1} - 2$$

$$\sum_{k=1}^{n} F_k F_{k-1} = \begin{cases} F_n^2 - 1 & n \text{ odd} \\ F_n^2 & n \text{ even} \end{cases}$$

$$\sum_{k=1}^{n} L_k L_{k-1} = \begin{cases} L_n^2 + 1 & n \text{ odd} \\ L_n^2 - 4 & n \text{ even} \end{cases}$$

A matrix $T_r = [t_{ij}] \in M_{n,n}(\mathbb{C})$ is called a *r*-Toeplitz matrix if it is of the form

$$t_{ij} = \begin{cases} t_{i-j}, & i \leq j \\ rt_{i-j}, & i > j \end{cases} \tag{1}$$

Obviously, the *r*-Toeplitz matrix *T* is determined by parameter *r* and its first row elements $t_0, t_{-1}, \cdots t_{1-n}$, thus we denote $T = T_r(t_0, t_{-1}, \cdots, t_{1-n})$. Especially, let $r = 1$, the matrix *T* is called a Toeplitz matrix.

A matrix $ST_r = [t_{ij}] \in M_{n,n}(\mathbb{C})$ is called a symmetric *r*-Toeplitz matrix if it is of the form

$$t_{ij} = \begin{cases} rt_{i-j}, & i < j \\ t_{i-j}, & i \geq j \end{cases} \tag{2}$$

Obviously, the symmetric *r*-Toeplitz matrix *T* is determined by parameter *r* and its last row elements $t_{n-1}, t_{n-2}, \cdots, t_0$, thus we denote $T = ST_r \ (t_{n-1}, t_{n-2}, \cdots, t_0)$. Especially, let $r = 1$, the matrix *T* is called a Toeplitz matrix.

The Euclidean norm of the matrix *A* is defined as

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

The singular values of the matrix A is

$$\sigma_i = \sqrt{\lambda_i\left(A^*A\right)}$$

where λ_i is an eigenvalue of A^*A and A^* is conjugate transpose of matrix A. For a square matrix A, the square roots of the maximum eigenvalues of A^*A are called the spectral norm of A. The spectral norm of the matrix A is

$$\|A\|_2 = \max\left(\sigma_i\right).$$

The following inequality holds,

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E .$$

Define the maximum column lenght norm c_1, and the maximum row lenght norm r_1 of any matrix A by

$$r_1(A) = \max_i \sqrt{\sum_j \left|a_{ij}\right|^2}$$

and

$$c_1(A) = \max_j \sqrt{\sum_i \left|a_{ij}\right|^2}$$

respectively. Let A, B and C be $m \times n$ matrices. If $A = B \circ C$ then

$$\|A\|_2 \le r_1(B)c_1(C) \quad [13].$$

Theorem 1 [9]. *Let* $A = T\left[a_{ij}\right]$ *be a Toeplitz matrix satisfying* $a_{ij} = F_{i-j}$, *then*

$$\begin{cases} \sqrt{\dfrac{2}{n}\left(F_n^2\right)} \le \|A\|_2 \le \sqrt{\left(1+F_nF_{n-1}\right)\left(F_nF_{n-1}\right)} & n \text{ even} \\ \sqrt{\dfrac{2}{n}\left(F_n^2-1\right)} \le \|A\|_2 \le \sqrt{\left(1+F_nF_{n-1}\right)\left(F_nF_{n-1}\right)} & n \text{ odd} \end{cases}$$

where $\|.\|_2$ is the spectral norm and F_n denotes the nth Fibonacci number.

Theorem 2 [9]. *Let* $A = T\left[a_{ij}\right]$ *be a Toeplitz matrix satisfying* $a_{ij} = L_{i-j}$, *then*

$$\begin{cases} \sqrt{\dfrac{2}{n}\left(L_n^2-4\right)} \le \|A\|_2 \le \sqrt{\left(L_nL_{n-1}-1\right)\left(L_nL_{n-1}+2\right)} & n \text{ even} \\ \sqrt{\dfrac{2}{n}\left(L_n^2+1\right)} \le \|A\|_2 \le \sqrt{\left(L_nL_{n-1}-1\right)\left(L_nL_{n-1}+2\right)} & n \text{ odd} \end{cases}$$

where $\|.\|_2$ is the spectral norm and L_n denotes the nth Lucas number.

3. Result and Discussion

Theorem 3. *Let* $A = T_r\left[a_{ij}\right]$ *be a r-Toeplitz matrix satisfying* $a_{ij} = F_{i-j}$, *where* $r \in \mathbb{C}$.

- $|r| \ge 1,\ \begin{cases} \sqrt{\dfrac{2}{n}\left(F_n^2-1\right)} \le \|A\|_2 \le |r|\sqrt{(n-1)F_nF_{n-1}} & n \text{ odd} \\ \sqrt{\dfrac{2}{n}\left(F_n^2\right)} \le \|A\|_2 \le |r|\sqrt{(n-1)F_nF_{n-1}} & n \text{ even} \end{cases}$

- $|r| < 1,\ \begin{cases} \sqrt{\dfrac{2}{n}|r|^2\left(F_n^2-1\right)} \le \|A\|_2 \le \sqrt{(n-1)F_nF_{n-1}} & n \text{ odd} \\ \sqrt{\dfrac{2}{n}|r|^2\left(F_n^2\right)} \le \|A\|_2 \le \sqrt{(n-1)F_nF_{n-1}} & n \text{ even} \end{cases}$

where $\|.\|_2$ is the spectral norm and F_n denotes the *n*th Fibonacci number.

Proof. The matrix *A* is of the form

$$A=\begin{bmatrix} F_0 & F_{-1} & \cdots & F_{2-n} & F_{1-n} \\ rF_1 & F_0 & \cdots & F_{3-n} & F_{2-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rF_{n-2} & rF_{n-3} & \cdots & F_0 & F_{-1} \\ rF_{n-1} & rF_{n-2} & \cdots & rF_1 & F_0 \end{bmatrix}$$

Then we have,

$$\|A\|_F^2 = nF_0^2 + \sum_{i=1}^{n-1} |r|^2 (n-i) F_i^2 + \sum_{i=1}^{n-1} (n-i) F_{-i}^2$$

hence, when $|r| \geq 1$ we obtain

$$\|A\|_F^2 \geq nF_0^2 + 2\sum_{i=1}^{n-1} (n-i) F_i^2 = nF_0^2 + 2\sum_{i=1}^{n-1}\sum_{k=1}^{i} F_k^2 = 2\sum_{k=1}^{n} F_k F_{k-1}$$

that is

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{2}{n}\left(F_n^2 - 1\right)} & n \text{ odd} \\ \sqrt{\frac{2}{n}\left(F_n^2\right)} & n \text{ even} \end{cases}$$

On the other hand, let the matrices *B* and *C* as

$$B=\begin{bmatrix} F_0 & 1 & \cdots & 1 & 1 \\ r & F_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & F_0 & 1 \\ r & r & \cdots & r & F_0 \end{bmatrix}$$

and

$$C=\begin{bmatrix} F_0 & F_{-1} & \cdots & F_{2-n} & F_{1-n} \\ F_1 & F_0 & \cdots & F_{3-n} & F_{2-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-2} & F_{n-3} & \cdots & F_0 & F_{-1} \\ F_{n-1} & F_{n-2} & \cdots & F_1 & F_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{|r|^2 (n-1)} = |r|\sqrt{n-1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} F_i^2} = \sqrt{F_n F_{n-1}}.$$

We have

$$\|A\|_2 \leq |r|\sqrt{(n-1) F_n F_{n-1}}$$

when $|r| < 1$ we also obtain

$$\|A\|_F^2 \geq nF_0^2 + 2|r|^2 \sum_{i=1}^{n-1} (n-i) F_i^2 = nF_0^2 + 2|r|^2 \sum_{i=1}^{n-1}\sum_{k=1}^{i} F_k^2 = 2|r|^2 \sum_{k=1}^{n} F_k F_{k-1}$$

that is

$$\|A\|_2 \ge \begin{cases} \sqrt{\frac{2}{n}|r|^2\left(F_n^2-1\right)} & n \text{ odd} \\ \sqrt{\frac{2}{n}|r|^2\left(F_n^2\right)} & n \text{ even} \end{cases}$$

On the other hand, let the matrices B and C as

$$B = \begin{bmatrix} F_0 & 1 & \cdots & 1 & 1 \\ r & F_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & F_0 & 1 \\ r & r & \cdots & r & F_0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} F_0 & F_{-1} & \cdots & F_{2-n} & F_{1-n} \\ F_1 & F_0 & \cdots & F_{3-n} & F_{2-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-2} & F_{n-3} & \cdots & F_0 & F_{-1} \\ F_{n-1} & F_{n-2} & \cdots & F_1 & F_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{n-1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} F_i^2} = \sqrt{F_n F_{n-1}}.$$

We have

$$\|A\|_2 \le \sqrt{(n-1)F_n F_{n-1}}.$$

■

Thus, the proof is completed.

Corollary 4. *Let* $A = ST_r(F_{n-1}, F_{n-2}, \cdots, F_0)$ *be a symmetric r-Toeplitz matrix, where r C, then*

- $|r| \ge 1, \begin{cases} \sqrt{\frac{2}{n}\left(F_n^2-1\right)} \le \|A\|_2 \le |r|\sqrt{(n-1)F_n F_{n-1}} & n \text{ odd} \\ \sqrt{\frac{2}{n}\left(F_n^2\right)} \le \|A\|_2 \le |r|\sqrt{(n-1)F_n F_{n-1}} & n \text{ even} \end{cases}$
- $|r| < 1, \begin{cases} \sqrt{\frac{2}{n}|r|^2\left(F_n^2-1\right)} \le \|A\|_2 \le \sqrt{(n-1)F_n F_{n-1}} & n \text{ odd} \\ \sqrt{\frac{2}{n}|r|^2\left(F_n^2\right)} \le \|A\|_2 \le \sqrt{(n-1)F_n F_{n-1}} & n \text{ even} \end{cases}$

where $\|.\|_2$ is the spectral norm and F_n denotes the nth Fibonacci number.

Proof. Owing to the fact that the sum of all elements squares are equal in matrices (1) and (2), the proof is concluded analogously in the proof of previous theorem. ■

Theorem 5. *Let* $A = T_r[a_{ij}]$ *be a r-Toeplitz matrix satisfying* $a_{ij} = L_{i-j}$, *where* $r \in \mathbb{C}$.

- $|r|\geq 1, \begin{cases} \sqrt{\dfrac{2\left(L_n^2+1\right)}{n}} \leq \|A\|_2 \leq \sqrt{\left[|r|^2(n-1)+1\right]\left[L_nL_{n-1}+2\right]} & n \text{ odd} \\ \sqrt{\dfrac{2\left(L_n^2-4\right)}{n}} \leq \|A\|_2 \leq \sqrt{\left[|r|^2(n-1)+1\right]\left[L_nL_{n-1}+2\right]} & n \text{ even} \end{cases}$
- $|r|<1, \begin{cases} \sqrt{\dfrac{2|r|^2\left(L_n^2+1\right)+4n\left(1-|r|^2\right)}{n}} \leq \|A\|_2 \leq \sqrt{n\left(L_nL_{n-1}+2\right)} & n \text{ odd} \\ \sqrt{\dfrac{2|r|^2\left(L_n^2-4\right)+4n\left(1-|r|^2\right)}{n}} \leq \|A\|_2 \leq \sqrt{n\left(L_nL_{n-1}+2\right)} & n \text{ even} \end{cases}$

where $\|\cdot\|_2$ is the spectral norm and L_n denotes the nth Lucas number.

Proof. The matrix A is of the form

$$A=\begin{bmatrix} L_0 & L_{-1} & \cdots & L_{2-n} & L_{1-n} \\ rL_1 & L_0 & \cdots & L_{3-n} & L_{2-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rL_{n-2} & rL_{n-3} & \cdots & L_0 & L_{-1} \\ rL_{n-1} & rL_{n-2} & \cdots & rL_1 & L_0 \end{bmatrix}$$

then we have

$$\|A\|_F^2 = nL_0^2 + \sum_{i=1}^{n-1}|r|^2(n-i)L_i^2 + \sum_{i=1}^{n-1}(n-i)L_{-i}^2$$

hence when $|r|\geq 1$ we obtain

$$\|A\|_2^2 \geq nL_0^2 + 2\sum_{i=1}^{n-1}(n-i)L_i^2 = nL_0^2 + 2\sum_{i=1}^{n-1}\sum_{k=1}^{i}L_k^2 = nL_0^2 + 2\sum_{k=1}^{n}L_kL_{k-1}$$

that is

$$\|A\|_2 \geq \begin{cases} \sqrt{\dfrac{2\left(L_n^2+1\right)}{n}} & n \text{ odd} \\ \sqrt{\dfrac{2\left(L_n^2-4\right)}{n}} & n \text{ even} \end{cases}$$

On the other hand let matrices B and C be as

$$B=\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{bmatrix}$$

and

$$C=\begin{bmatrix} L_0 & L_{-1} & \cdots & L_{2-n} & L_{1-n} \\ L_1 & L_0 & \cdots & L_{3-n} & L_{2-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{n-2} & L_{n-3} & \cdots & L_0 & L_{-1} \\ L_{n-1} & L_{n-2} & \cdots & L_1 & L_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{|r|^2(n-1)} = |r|\sqrt{n-1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} L_i^2} = \sqrt{F_n F_{n-1} + 2}.$$

We have

$$\|A\|_2 \le \sqrt{\left[|r|^2(n-1)+1\right]\left[L_n L_{n-1} + 2\right]}$$

when $|r| < 1$ we also obtain

$$\|A\|_2^2 \ge nL_0^2 + 2|r|^2 \sum_{i=1}^{n-1}(n-i)L_i^2 = nL_0^2 + 2|r|^2 \sum_{i=1}^{n-1}\sum_{k=1}^{i} L_k^2 = 4n + 2|r|^2 \sum_{k=1}^{n} L_k L_{k-1}$$

that is

$$\|A\|_2 \ge \begin{cases} \sqrt{\dfrac{2|r|^2(L_n^2+1)+4n\left(1-|r|^2\right)}{n}} & n \text{ odd} \\ \sqrt{\dfrac{2|r|^2(L_n^2-4)+4n\left(1-|r|^2\right)}{n}} & n \text{ even} \end{cases}$$

On the other hand, let matrices B and C be as

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} L_0 & L_{-1} & \cdots & L_{2-n} & L_{1-n} \\ L_1 & L_0 & \cdots & L_{3-n} & L_{2-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{n-2} & L_{n-3} & \cdots & L_0 & L_{-1} \\ L_{n-1} & L_{n-2} & \cdots & L_1 & L_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{n}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} L_i^2} = \sqrt{F_n F_{n-1} + 2}.$$

We have

$$\|A\|_2 \le \sqrt{n(L_n L_{n-1} + 2)}$$

■

Thus, the proof is completed.

Corollary 6. *Let* $A = ST_r\left(L_{n-1}, L_{n-2}, \cdots, L_0\right)$ *be a symmetric r-Toeplitz matrix, where* $r \in \mathbb{C}$, *then*

- $$|r| \geq 1, \begin{cases} \sqrt{\dfrac{2\left(L_n^2+1\right)}{n}} \leq \|A\|_2 \leq \sqrt{\left[|r|^2(n-1)+1\right]\left[L_n L_{n-1}+2\right]} & n \text{ odd} \\ \sqrt{\dfrac{2\left(L_n^2-4\right)}{n}} \leq \|A\|_2 \leq \sqrt{\left[|r|^2(n-1)+1\right]\left[L_n L_{n-1}+2\right]} & n \text{ even} \end{cases}$$
- $$|r| < 1, \begin{cases} \sqrt{\dfrac{2|r|^2\left(L_n^2+1\right)+4n\left(1-|r|^2\right)}{n}} \leq \|A\|_2 \leq \sqrt{n\left(L_n L_{n-1}+2\right)} & n \text{ odd} \\ \sqrt{\dfrac{2|r|^2\left(L_n^2-4\right)+4n\left(1-|r|^2\right)}{n}} \leq \|A\|_2 \leq \sqrt{n\left(L_n L_{n-1}+2\right)} & n \text{ even} \end{cases}$$

where $\|.\|_2$ is the spectral norm and L_n denotes the *n*th Lucas number.

Proof. Owing to the fact that the sum of all elements squares are equal in matrices (1) and (2), the proof is concluded analogously in the proof of previous theorem. ■

4. Numarical Examples

Example 7. *Let* $A = T_r\left(F_0, F_{-1}, \cdots, F_{1-n}\right)$ *be a r-Toeplitz matrix, in which* F_i $\left(i = 0,1,\cdots,n-1\right)$ *denotes the Fibonacci number, where* $r \in \mathbb{C}$. *From* **Table 1**, *it is easy to find that upper bounds for the spectral norm, of Theorem* 3 *are more sharper than Theorem* 1 (*see* **Table 1**).

Table 1. Numerical results of $a_{ij} = F_{i-j}$, $r = 1$.

n	**Theorem 1**	**Theorem 3**
2	$\sqrt{2}$	$\sqrt{1}$
3	$\sqrt{6}$	$\sqrt{4}$
4	$\sqrt{42}$	$\sqrt{18}$
5	$\sqrt{240}$	$\sqrt{60}$
6	$\sqrt{1640}$	$\sqrt{200}$
$\cdots$	$\cdots$	$\cdots$
n	$\sqrt{(1+F_nF_{n-1})(F_nF_{n-1})}$	$\sqrt{(n-1)F_nF_{n-1}}$

Table 2. Numerical results of $a_{ij} = L_{i-j}$, $r = 1$.

n	**Theorem 2**	**Theorem 5**
2	$\sqrt{10}$	$\sqrt{10}$
3	$\sqrt{154}$	$\sqrt{42}$
4	$\sqrt{810}$	$\sqrt{120}$
5	$\sqrt{6004}$	$\sqrt{395}$
6	$\sqrt{39400}$	$\sqrt{1200}$
$\cdots$	$\cdots$	$\cdots$
n	$\sqrt{(L_nL_{n-1}-1)(L_nL_{n-1}+2)}$	$\sqrt{(n-1)(L_nL_{n-1}+2)}$

Example 8. *Let* $A = T_r(L_0, L_{-1}, \cdots, L_{1-n})$ *be a r-Toeplitz matrix, in which* L_i $(i = 0,1,\cdots,n-1)$ *denotes the Lucas number, where* $r \in \mathbb{C}$. *From* **Table 2**, *it is easy to find that upper bounds for the spectral norm, of Theorem* 5 *are more sharper than Theorem* 2, *when* $n \geq 2$ (*see* **Table 2**).

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