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Canonical Form Associated with an *r*-Jacobi Algebra

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Abstract

In this paper, we denote by *A* a commutative and unitary algebra over a commutative field *K* of characteristic 0 and *r* an integer ≥1. We define the notion of *r*-Jacobi algebra *A* and we construct the canonical form associated with the *r*-Jacobi algebra *A*.

Keywords

Module of Kähler Differential, Lie Algebra of Order *r*, Jacobi Algebra of Order *r*

1. Introduction

The concept of *n*-Lie algebra over a field *K*, *n* an integer ≥2, introduced by Fillipov [1], is a generalization of the concept of Lie algebra over a field *K*, which corresponds to the case where *n* = 2. A structure of *n*-Lie algebra over a *K*-vector space *W*, is the given of an alternating multilinear mapping of degree *n*

$$\{,\cdots,\}: W^n = W \times \cdots \times W \to W, (x_1, x_2, \cdots, x_n) \mapsto \{x_1, x_2, \cdots, x_n\}$$

verifying the identity

$$\{x_1, x_2, \cdots, x_{n-1}, \{y_1, y_2, \cdots, y_n\}\} = \sum_{i=1}^{n} \{y_1, y_2, \cdots, y_{i-1}, \{x_1, x_2, \cdots, x_{n-1}, y_i\}, y_{i+1}, \cdots, y_n\}$$

for all $x_1, x_2, \cdots, x_{n-1}, y_1, y_2, \cdots, y_n \in W$. This identity is called Jacobi identity of *n*-Lie algebra *W* [1] [2].

A derivation of an *n*-Lie algebra $(W, \{,\cdots,\})$ is a *K*-linear map

$$D: W \to W$$

such that

$$D\{x_1,x_2,\cdots,x_n\}=\sum_{i=1}^{n}\{x_1,\cdots,D(x_i),\cdots,x_n\}$$

for all $x_1,x_2,\cdots,x_n\in W$.

The set of all derivations of a n-Lie algebra W is a K-Lie algebra denoted by $Der_K(W)$.

If $(W,\{,\cdots,\})$ is a n-Lie algebra, then for all $x_1,x_2,\cdots,x_{n-1}\in W$, the map

$$ad(x_1,x_2,\cdots,x_{n-1}):W\to W,y\mapsto\{x_1,x_2,\cdots,x_{n-1},y\}$$

is a derivation of $(W,\{,\cdots,\})$.

When A is a commutative algebra, with unit 1_A over a commutative field K of characteristic zero, and when M is a A-module, a linear map

$$\partial:A\to M$$

is a differential operator of order ≤ 1 [3] [4] if, for all a and b belonging to A,

$$\partial(ab)=\partial(a)\cdot b+a\cdot\partial(b)-ab\cdot\partial(1_A).$$

When $\partial(1_A)=0$, we have the usual notion of derivation from A into M.

We denote by $\mathrm{Diff}_K(A,M)$ the A-module of differential operator of order ≤ 1 from A into M and by $\mathrm{Diff}_K(A)$ the A-module of differential operator of order ≤ 1 on A ($M=A$).

The aim of this work is to define the notion of r-Jacobi algebra and to construct the canonical form associated with this r-Jacobi algebra.

In the following, A denotes a unitary commutative algebra over a commutative field K of characteristic zero with unit 1_A and $\Omega_K(A)$ the module of Kähler differential of A and

$$d_{A/K}:A\to\Omega_K(A),a\mapsto d_{A/K}(a)$$

the canonical derivation [3] [4].

2. Structure of Jacobi Algebra of Order $r\ge 1$

A-Module $A\times\Omega_K(A)$

Proposition 1 [3] *The map* $D_{A/K}:A\to A\times\Omega_K(A),a\mapsto(a,d_{A/K}(a))$ *is a differential operator of order* ≤ 1. *Moreover the image of* $D_{A/K}$ *generates the A-module* $A\times\Omega_K(A)$.

The pair $(A\times\Omega_K(A),D_{A/K})$ has the following universal property [3] [5] [6]: for all A-module M and for all differential operator of order ≤ 1

$$\varphi:A\to M$$

there exists an unique A-linear map

$$\tilde{\varphi}:A\times\Omega_K(A)\to M$$

such that

$$\tilde{\varphi}\circ D_{A/K}=\varphi.$$

Moreover, the map

$$Hom_A(A\times\Omega_K(A),M)\to\mathrm{Diff}_K(A,M),\psi\mapsto\psi\circ D_{A/K}$$

is an isomorphism of A-modules.

For all integer $p\ge 1$, we say that an alternating K-multilinear map

$$\varphi:A^p=A\times A\times\cdots\times A\to M$$

is a alternating p-differential operator if for all $a_1,a_2,\cdots,a_p\in A$, the map

$$\varphi^i : A \to M, a_i \mapsto \varphi\left(a_1, a_2, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_p\right)$$

is a alternating differential operator of order ≤ 1 for all $i = 1, 2, \cdots, p$.

We denote by $\mathcal{L}_{alt}^p\left(A \times \Omega_K(A), M\right)$, the A-module of alternating A-multilinear maps of degree p from $A \times \Omega_K(A)$ into M and $\mathrm{Diff}_{alt}^p(A, M)$, the A-module of alternating p-differential operators from A into M.

One notes

$$D_{A/K}^{(p)} = D_{A/K} \times D_{A/K} \times \cdots \times D_{A/K} : A^p \to \left[A \times \Omega_K(A)\right]^p$$

such that

$$D_{A/K}^{(p)}\left(a_1, a_2, \cdots, a_p\right) = \left(D_{A/K}(a_1), D_{A/K}(a_2), \cdots, D_{A/K}(a_p)\right)$$

for all $a_1, a_2, \cdots, a_p \in A$.

When $\Lambda_A\left[A \times \Omega_K(A)\right] = \bigoplus_{n \in \mathbb{N}} \Lambda_A^n\left[A \times \Omega_K(A)\right]$ is the A-exterior algebra of the A-module $A \times \Omega_K(A)$, the differential operator

$$D_{A/K} : A \to A \times \Omega_K(A)$$

can be extended into a differential operator again noted

$$D_{A/K} : \Lambda_A\left[A \times \Omega_K(A)\right] \to \Lambda_A\left[A \times \Omega_K(A)\right]$$

of degree +1 and of square 0. Thus, the pair $\left(\Lambda_A\left[A \times \Omega_K(A)\right], D_{A/K}\right)$ is a differential complex [3].

For all A-module M and for all alternating p-differential operator

$$\varphi : A^p \to M,$$

there exists an unique alternating A-multilinear map of degree p

$$\tilde{\varphi} : \left[A \times \Omega_K(A)\right]^p \to M$$

such that

$$\tilde{\varphi} \circ D_{A/K}^{(p)} = \varphi.$$

Thus, the existence of an unique A-linear map

$$\bar{\varphi} : \Lambda_A^p\left[A \times \Omega_K(A)\right] \to M$$

such that

$$\bar{\varphi}\left(D_{A/K}(a_1) \Lambda D_{A/K}(a_2) \Lambda \cdots \Lambda D_{A/K}(a_p)\right) = \varphi\left(a_1, a_2, \cdots, a_p\right)$$

for all $a_1, a_2, \cdots, a_p$ elements of A when the map

$$\varphi : A^p \to M,$$

is a alternating p-differential operator. Moreover, the map

$$\mathcal{L}_{alt}^p\left(A \times \Omega_K(A), M\right) \to \mathrm{Diff}_{alt}^p(A, M), f \mapsto f \circ D_{A/K}^{(p)}$$

is an isomorphism of A-modules [3].

3. Structure of *r*-Jacobi Algebra

We say that a commutative algebra with unit A on a commutative field K of characteristic zero, is a r-Jacobi algebra, $r \ge 1$ an integer, if A is provided with a structure of $2r$-Lie algebra over K of bracket $\{, \cdots, \}$, such that for all $(a_1,a_2,\cdots,a_{2r-1}) \in A^{2r-1}$, the map

$$ad(a_1,a_2,\cdots,a_{2r-1}): A \to A, b \mapsto \{a_1,a_2,\cdots,a_{2r-1},b\}$$

is a differential operator of order ≤ 1.

Proposition 2 *When A is a r-Jacobi algebra, then there exist an unique A-linear map*

$$\overline{ad}: \Lambda_A^{2r-1}[A \times \Omega_K(A)] \to \mathrm{Diff}_K(A)$$

such that, for all $(a_1,a_2,\cdots,a_{2r-1}) \in A$

$$\overline{ad}(D_{A/K}(a_1) \Lambda D_{A/K}(a_2) \Lambda \cdots \Lambda D_{A/K}(a_{2r-1})) = ad(a_1,a_2,\cdots,a_{2r-1}).$$

Proof. The map

$$ad: A^{2r-1} \to \mathrm{Diff}_K(A), (a_1,a_2,\cdots,a_{2r-1}) \mapsto ad(a_1,a_2,\cdots,a_{2r-1})$$

is an alternating $(2r-1)$-differential operator. Thus deduced the existence and the uniqueness of the A-linear map

$$\overline{ad}: \Lambda_A^{2r-1}[A \times \Omega_K(A)] \to \mathrm{Diff}_K(A)$$

such that

$$\overline{ad}(D_{A/K}(a_1) \Lambda D_{A/K}(a_2) \Lambda \cdots \Lambda D_{A/K}(a_{2r-1})) = ad(a_1,a_2,\cdots,a_{2r-1}).$$

That ends the proof.

Canonical form Associated with a *r*-Jacobi Algebra

In what follows, A is a r-Jacobi algebra.

Theorem 3 *The map*

$$A^{2r} \to A, (a_1,a_2,\cdots,a_{2r-1},a_{2r}) \mapsto (1-2r) \cdot \{a_1,a_2,\cdots,a_{2r-1},a_{2r}\}$$

is an alternating $2r$-differential operator and induces an alternating A-multilinear mapping and only one of degree $2r$

$$\omega_{2r}: [A \times \Omega_K(A)]^{2r} \to A$$

such that

$$\omega_{2r}(D_{A/K}(a_1),\cdots,D_{A/K}(a_{2r-1}),D_{A/K}(a_{2r})) = (1-2r) \cdot \{a_1,a_2,\cdots,a_{2r-1},a_{2r}\}.$$

Proof. As the map

$$ad(a_1,a_2,\cdots,a_{2r-1}): A \to A, b \mapsto \{a_1,a_2,\cdots,a_{2r-1},b\}$$

is a A-differential operator of order ≤ 1 and the map

$$ad: A^{2r-1} \to \mathrm{Diff}_K(A), (a_1,a_2,\cdots,a_{2r-1}) \mapsto ad(a_1,a_2,\cdots,a_{2r-1})$$

is an alternating $(2r-1)$-differential operator.

The unique A-alternating multinear map of degree $2r$

$$\omega_{2r}: [A \times \Omega_K(A)]^{2r} \to A$$

induce an unique A-linear map

$$\omega: \Lambda_A^{2r}[A \times \Omega_K(A)] \to A$$

such that

$$\omega\left(D_{A/K}(a_1)\Lambda\cdots\Lambda D_{A/K}(a_{2r-1})\Lambda D_{A/K}(a_{2r})\right)=(1-2r)\cdot\{a_1,a_2,\cdots,a_{2r-1}\}$$

for all $a_1,a_2,\cdots,a_{2r}\in A$.

We say that ω is the canonical form associated with the *r*-Jacobi algebra *A*.

Corollary 1 *For all* $u\in\Lambda_A^{2r-1}\left[A\times\Omega_K(A)\right]$,

$$\left[\overline{ad}(u)\right](a)=(1-2r)^{-1}\cdot\omega\left(u\Lambda D_{A/K}(a)\right)$$

for any $a\in A$.

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