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Two Eigenvector Theorems

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Abstract

In this paper, we established a connection between a square matrix "*A*" of order "n" and a matrix $X_{n+1} = (x_1, x_2, \cdots, x_n, x_{n+1})$ defined through a new approach of the recursion relation $x_{i+1} = Ax_i$. (where x_1 is any column matrix with n real elements). Now the new matrix X_{n+1} gives us a characteristic equation of matrix A and we can find the exact determination of Eigenvalues and its Eigenvectors of the matrix A. This new approach was invented by using Two eigenvector theorems along with some examples. In the subsequent paper we apply this approach by considering some examples on this invention.

Keywords

Characteristic Equation, Minimal Polynomial, Eigenvalues, Eigenvectors, Vander Monde Matrix, Jacobi Block Matrix

1. Introduction

In this article, we present results connecting the Eigenvalues and vectors [1]-[5] of a square matrix "$\boldsymbol{A}$" of order "n" and a matrix $\boldsymbol{X}_{n+1} = (x_1, x_2, \cdots, x_n, x_{n+1})$ defined (where x_1 is any column matrix with n elements) through the recursion relation $x_{i+1} = \boldsymbol{A}x_i$. these results will be useful in the context of exact determination of Eigenvectors of a matrix associated with a specific Eigenvalue when the minimal polynomial is known. However this problem, of considerable interest in the field of numerical matrix analysis, is being considered in a separate study.

2 Basic Points

Before presenting these Eigenvector theorems, it is useful to introduce a few notations and some rather obvious lemmas.

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Let $\boldsymbol{A}$ be a matrix with n Eigenvalues λ_i and associated Eigenvectors $u_i(i=1,2,\cdots,n)$ Unless stated otherwise, these roots are assumed to be distinct. Similarly we define $M(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i)$ the minimal (and under the assumption of distinctness, also the characteristic) Polynomial [6] of $\boldsymbol{A}$.

S_k : a set of distinct indices's, a subset of set $\{1,2,\cdots,n\}$.

$\boldsymbol{P}(S_k)$: the vector of n components of the coefficients of $\prod_{j\notin S_k}(\lambda-\lambda_j)$ in reverse order, with trailing zeroes.

$\boldsymbol{P}_r(S_k)$: the same vector as $\boldsymbol{P}(S_k)$ but with leading zeroes; $r\le(n-k)$

When $S=\{\alpha\}$, a singleton, we shall write $\boldsymbol{P}_\alpha=\boldsymbol{P}(S=\{\alpha\})$.

$\boldsymbol{V}=(v_{ij})$: the Vander monde matrix [7], defined by $V_{ij}=\lambda_j^{i-1}$, $i,j=1,2,\cdots,n$.

$\boldsymbol{W}_n=(w_1,w_2,\cdots,w_n)$: an nth order matrix with the following structure. The column w_{j+1}, $(j=0,1,\cdots,n-1)$ has the last element as a^j, successive elements of w_{j+1} from below being obtained by accumulating successive terms in the expansion of $(a+1)^j$.

$\boldsymbol{q}(a,r;n)$: the left justified n-component vector of coefficients of $(1-a)^{n-r}$ in the reverse order.

$\boldsymbol{q}_s(a,r;n)$: same vector as $\boldsymbol{q}$ above, but with S leading zeroes.

$\boldsymbol{J}$: the Jacobi Block matrix [7] [8] with diagonal elements and super diagonal elements1

$\boldsymbol{e}_j$: the j-th column of the identity matrix.

α_j: appropriate scalars as need be.

3. Main Results

The following useful lemmas are rather obvious:

LEMMA 1: $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}_i=\alpha_i\boldsymbol{e}_i$

LEMMA 2: $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}(S_k)=\sum_{i\in S_k}\alpha_i\boldsymbol{e}_i$

LEMMA 3: $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}_r(S_k)=\sum_{i\in S_k}\alpha_i\lambda_i^r\boldsymbol{e}_i$

LEMMA 4: $\boldsymbol{W}_n\boldsymbol{q}(a,r;n)=\sum_{i=1}^{r}\boldsymbol{e}_i$

For clarity we shall illustrate these notations and results by way of illustrations.

ILLUSTRATION 0:

let $n=4$; $\lambda_1=5,\lambda_2=8,\lambda_3=7,\lambda_4=9,S_1=\{2\}$ and $S_2=\{3,1\}$.

$$\boldsymbol{V}=\begin{bmatrix}1&1&1&1\\5&8&7&9\\25&64&49&81\\125&512&343&729\end{bmatrix}$$

$$\boldsymbol{W}_4=\begin{bmatrix}1&a+1&a^2+2a+1&a^3+3a^2+3a+1\\1&a+1&a^2+2a+1&a^3+3a^2+3a\\1&a+1&a^2+2a&a^3+3a^2\\1&a&a^2&a^3\end{bmatrix}$$

$$\begin{aligned}\boldsymbol{P}_2&=\boldsymbol{P}(S_1=\{2\})\\&=\{\text{Coefficient vector of }(\lambda-\lambda_1)(\lambda-\lambda_3)(\lambda-\lambda_4)\}\\&=\{\text{Coefficient vector of }(\lambda-5)(\lambda-7)(\lambda-9)=\lambda^3-21\lambda^2+143\lambda-315\}\\&=(-315,143,-21,1)^{\mathrm{T}}\end{aligned}$$

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}_2=(0,-3,0,0)^{\mathrm{T}}=-3\boldsymbol{e}_2$$

$$\begin{aligned}\boldsymbol{P}(S_2)&=\{\text{Coefficient vector of }(\lambda-\lambda_2)(\lambda-\lambda_4)\}\\&=\{\text{Coefficient vector of }(\lambda-8)(\lambda-9)=\lambda^2-12\lambda+72\}=(72,-17,1,0)^{\mathrm{T}}\end{aligned}$$

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}(S_2)=(12,0,2,0)^{\mathrm{T}}=12\boldsymbol{e}_1-2\boldsymbol{e}_3$$

$$\boldsymbol{P}_1(S_2)=(0,72,-17,1)^{\mathrm{T}}$$

$$\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}_1(S_2)=(60,0,14,0)^{\mathrm{T}}=12(5\boldsymbol{e}_1)+2(7\boldsymbol{e}_3)$$

$$\boldsymbol{q}(a,1;4)=\left\{\text{coefficient vector of }(1-a)^{4-1}\right\}=\left(-a^3,3a^2,-3a,1\right)^{\mathrm{T}}$$

$$\boldsymbol{W}_4\boldsymbol{q}(a,1;4)=(1,0,0,0)^{\mathrm{T}}=\boldsymbol{e}_1$$

$$\boldsymbol{q}(a,3;4)=\left\{\text{coefficient vector of }(1-a)^{4-3}\right\}=(-a,1,0,0)^{\mathrm{T}};$$

$$\boldsymbol{W}_4\boldsymbol{q}=(1,1,1,0)^{\mathrm{T}}=\sum\nolimits_{i=1}^{3}\boldsymbol{e}_i$$

$$\boldsymbol{q}_1(a,3;4)=(0,-a,1,0)^{\mathrm{T}};$$

$$\boldsymbol{W}_4\boldsymbol{q}_1=(a+1,a+1,a,0)^{\mathrm{T}}$$

$$\boldsymbol{q}_2(a,3;4)=(0,0,-a,1)^{\mathrm{T}};$$

$$\boldsymbol{W}_4\boldsymbol{q}_2=\left(a^2+2a+1,a^2+2a,a^2,0\right)^{\mathrm{T}}$$

$$\boldsymbol{J}_3=\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}$$

$$\boldsymbol{J}_3^2=\begin{bmatrix}\lambda^2 & 2\lambda & 1\\ 0 & \lambda^2 & 2\lambda\\ 0 & 0 & \lambda^2\end{bmatrix}$$

$$\boldsymbol{J}_3^3=\begin{bmatrix}\lambda^3 & 3\lambda^2 & 3\lambda\\ 0 & \lambda^3 & 3\lambda^2\\ 0 & 0 & \lambda^3\end{bmatrix}$$

$$\boldsymbol{J}_3^m=\begin{bmatrix}\lambda^m & mC_1\lambda^{m-1} & mC_2\lambda^{m-2}\\ 0 & \lambda^m & mC_1\lambda^{m-1}\\ 0 & 0 & \lambda^m\end{bmatrix}$$

Now for the

3.1. First Eigenvector Theorem

Let $\boldsymbol{A}=\boldsymbol{U}\Lambda\boldsymbol{U}^{-1}$
where $\Lambda=diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$, then $\boldsymbol{X}_n=\boldsymbol{U}\boldsymbol{V}^{\mathrm{T}}$. Proof is obvious once it is noted that if $x_1=\sum\boldsymbol{u}_i$, then $x_k=\sum\lambda_i^k\boldsymbol{u}_i$

Since eigenvectors are unique up to scale, it is obvious that, by proper scaling one can always have, For arbitrary x_1, the relation $x_1=\sum\boldsymbol{u}_i$ (provided of-course tha x_1 lies in the full-space, but in no Proper subspace) with the set $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\cdots,\boldsymbol{u}_n\}$ as basis.

COROLLARY 1.1: $\boldsymbol{X}_n\boldsymbol{P}_i=\alpha_i\boldsymbol{u}_i$

COROLLARY 1.2: $\boldsymbol{X}_n\boldsymbol{P}(S_k)=\sum_{i\in S_k}\alpha_i\boldsymbol{u}_i$

COROLLARY 1.3: $\boldsymbol{X}_n\boldsymbol{P}_r(S_k)=\sum_{i\in S_k}\alpha_i\lambda_i^r\boldsymbol{u}_i$

COROLLARY 1.4: Let $\alpha_1, \alpha_2 = a \pm ib$ be a pair of complex conjugate Eigenvalues of $\boldsymbol{A}$ and Let $(\boldsymbol{u} \pm i\boldsymbol{v})$ be the associated Eigenvectors where $\boldsymbol{u}$, $\boldsymbol{v}$ are real vectors.

Let $S_2 = \{1,2\}$.

Then $\boldsymbol{X}_n \boldsymbol{P}(S_2) = 2\boldsymbol{u}$ and $\boldsymbol{X}_n \boldsymbol{P}_1(S_2) = 2(a\boldsymbol{u} - b\boldsymbol{v})$.

Another analogous corollary, in respect of Eigenvalues $(a \pm \sqrt{b})$ where $\sqrt{b}$ is a surd is obvious.

ILLUSTRATION 1.1:

let $\boldsymbol{A} = \begin{bmatrix} 7 & 0 & -3 & 1 \\ 4 & 5 & -6 & 2 \\ 4 & 4 & -6 & 3 \\ 4 & 4 & -8 & 5 \end{bmatrix}$ and $\boldsymbol{X}_1 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$

Then $\boldsymbol{X}_4 = \begin{bmatrix} 2 & 7 & 31 & 145 \\ 3 & 9 & 37 & 165 \\ 3 & 8 & 34 & 158 \\ 2 & 6 & 30 & 150 \end{bmatrix}$;

$$M(\lambda) = \lambda^4 - 11\lambda^3 + 41\lambda^2 - 61\lambda - 30 = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-5).$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 5$$

Hence $\boldsymbol{P} = (\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_4) = \begin{bmatrix} -30 & -15 & -10 & -6 \\ 31 & 23 & 17 & 11 \\ -10 & -9 & -8 & -6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

and

U. diag $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \boldsymbol{X}_4 \boldsymbol{P} = \begin{bmatrix} -8 & -3 & -4 & 24 \\ -16 & -6 & -8 & 24 \\ -24 & -9 & -8 & 24 \\ -24 & -12 & -8 & 24 \end{bmatrix}$

Let $S_2 = \{1,4\}$.

Then $\boldsymbol{P}(S_2) = \begin{bmatrix} 6 \\ -5 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{P}_1(S_2) = \begin{bmatrix} 0 \\ 6 \\ -5 \\ 1 \end{bmatrix}$

And $\boldsymbol{X}_4(\boldsymbol{P}(S_2), \boldsymbol{P}_1(S_2)) = \begin{bmatrix} 8 & 32 \\ 10 & 34 \\ 12 & 36 \\ 12 & 36 \end{bmatrix} = (\boldsymbol{u}_1 + \boldsymbol{u}_4, \boldsymbol{u}_1 + 5\boldsymbol{u}_4)$

where $\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 6 \end{bmatrix}$, $\boldsymbol{u}_4 = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$

ILLUSTRATION 1.2:

Let $\boldsymbol{A} = \begin{bmatrix} 4 & 6 & 10 \\ 12 & 2 & -6 \\ -6 & 2 & 6 \end{bmatrix}$ and $\boldsymbol{X}_3 = \begin{bmatrix} 1 & 20 & 148 \\ 1 & 8 & 244 \\ 1 & 2 & -92 \end{bmatrix}$

We have $M(\lambda)=\lambda^3-12\lambda^2+44\lambda-240=(\lambda-10)(\lambda^2-2\lambda+24)$

A has $\lambda_1=10$ as one real root and $\lambda_{2,3}=1\pm i\sqrt{23}$ as two complex conjugate roots.

Hence $\boldsymbol{X}_3\boldsymbol{P}_1=\begin{bmatrix}132\\252\\-72\end{bmatrix}$ is the Eigenvector for $\lambda=10$.

$$\boldsymbol{X}_3\boldsymbol{P}(S_2=\{2,3\})=\boldsymbol{X}_3\begin{bmatrix}-10\\1\\0\end{bmatrix}=\begin{bmatrix}10\\-2\\-8\end{bmatrix}=2U$$

$\boldsymbol{X}_3\boldsymbol{P}_1(S_2)=\boldsymbol{X}_3\begin{bmatrix}0\\-10\\1\end{bmatrix}=\begin{bmatrix}-52\\164\\-112\end{bmatrix}=2(U-\sqrt{2}3V)$ from which we get the complex conjugate Eigen vectors $\boldsymbol{U}\pm i\boldsymbol{V}$

where $\boldsymbol{U}=\begin{bmatrix}5\\-1\\-4\end{bmatrix}$; $\boldsymbol{V}=\frac{1}{\sqrt{23}}\begin{bmatrix}31\\-83\\52\end{bmatrix}$.

We shall now state

3.2. The Second Eigenvector Theorem (The Generalized Eigenvector Theorem)

Let $\boldsymbol{A}=\boldsymbol{UJU}^{-1}$; then $\boldsymbol{X}_n=\boldsymbol{UW}_n$ and hence $\sum_{i=1}^{r}\alpha_i\boldsymbol{u}_i=\boldsymbol{X}_n q(\lambda,r;n)$.

Proof is obvious once it is observed that $\boldsymbol{W}n_q(\lambda,r;n)=\sum_1^r\boldsymbol{e}_i$.

ILLUSTRATION 2:

Let $\boldsymbol{A}=\begin{bmatrix}2&1&0\\-1&3&1\\-1&0&4\end{bmatrix}$ and $\boldsymbol{X}_3=\begin{bmatrix}4&10&23\\2&3&-1\\1&0&-10\end{bmatrix}$

We have the minimal polynomial of degree 3, as $M(\lambda)=(\lambda-3)^3$,

This is also the characteristic polynomial.

Hence $\boldsymbol{A}=\boldsymbol{UJU}^{-1}$

where $\boldsymbol{J}=\begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}$

$\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ are such that $\boldsymbol{Au}_1=3\boldsymbol{u}_1$;

$\boldsymbol{Au}_2=3\boldsymbol{u}_2+\boldsymbol{u}_1$, and $\boldsymbol{Au}_3=3\boldsymbol{u}_3+\boldsymbol{u}_2$.

taking $\boldsymbol{Q}=(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)=\begin{bmatrix}9&-3&1\\-6&1&0\\1&0&0\end{bmatrix}$

where $\boldsymbol{q}_i=q(3,i;3)$,

We get $(\boldsymbol{u}_1,\boldsymbol{u}_1+\boldsymbol{u}_2,\boldsymbol{u}_1+\boldsymbol{u}_2+\boldsymbol{u}_3)=\boldsymbol{X}_3\boldsymbol{Q}=\begin{bmatrix}-1&-2&4\\-1&-3&2\\-1&-3&1\end{bmatrix}$

where $(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3)=\begin{bmatrix}-1&-1&6\\-1&-2&5\\-1&-2&4\end{bmatrix}$

4. Summary

Extensions of these two theorems and their corollaries to cases where the minimal polynomial is a proper factor of the characteristic polynomial and hence, for some of the multiple Eigenvalues at least, the associated Eigenspace is of dimension more than one is obvious though explicit proof is slightly cumbersome.

The proposed method can be used in many mathematical subsequence applications viz., in most of the big data analysis, image processing and multivariate data analysis.

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