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A Generalization of Cramer's Rule

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Abstract

In this paper, we find two formulas for the solutions of the following linear equation $Ax = b$, $x \in IR^m, b \in IR^n, m \geq n$, where $A = \left(a_{j,i}\right)_{n\times m}$ is a $n\times m$ real matrix. This system has been well studied since the 1970s. It is known and simple proven that there is a solution for all $b \in IR^n$ if, and only if, the rows of A are linearly independent, and the minimum norm solution is given by the Moore-Penrose inverse formula, which is often denoted by $A^{\dagger}$; in this case, this solution is given by $A^{\dagger}b = A^{*}\left(AA^{*}\right)^{-1}b$. Using this formula, Cramer's Rule and Burgstahler's Theorem (Theorem 2), we prove the following representation for this solution

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i}b_1 & \langle l_1, l_2\rangle + a_{2i}b_1 & \cdots & \langle l_1, l_n\rangle + a_{ni}b_1 \\ \langle l_2, l_1\rangle + a_{1i}b_2 & \|l_2\|^2 + a_{2i}b_2 & \cdots & \langle l_2, l_n\rangle + a_{ni}b_2 \\ \vdots & \vdots & \ddots & \vdots \\ \langle l_n, l_1\rangle + a_{1i}b_n & \langle l_n, l_2\rangle + a_{2i}b_n & \cdots & \|l_n\|^2 + a_{ni}b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle & \cdots & \langle l_1, l_n\rangle \\ \langle l_2, l_1\rangle & \|l_2\|^2 & \cdots & \langle l_2, l_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle l_n, l_1\rangle & \langle l_n, l_2\rangle & \cdots & \|l_n\|^2 \end{vmatrix}} - 1,$$

$i = 1, 2, 3, \cdots, m$, where $l_1, l_2, \cdots, l_n$ are the row vectors of the matrix A. To the best of our knowledge and looking in to many Linear Algebra books, there is not formula for this solution depending on determinants. Of course, this formula coincides with the one given by Cramer's Rule when $m = n$.

Keywords

Linear Equation, Cramer's Rule, Generalized Formula

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1. Introduction

In this paper, we find a formula depending on determinants for the solutions of the following linear equation

$$Ax = b, \ \ x \in IR^m, \ \ b \in IR^n, \ \ m \geq n \tag{1}$$

or

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = b_n \end{cases}. \tag{2}$$

Now, if we define the column vectors

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix}, l_2 = \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \\ a_{2,m} \end{bmatrix}, \cdots, l_n = \begin{bmatrix} a_{n,1} \\ a_{n,2} \\ \vdots \\ a_{n,m} \end{bmatrix},$$

then the system (2) also can be written as follows:

$$\langle l_i, x \rangle = b_i, \quad i = 1, 2, \cdots, n, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the innerproduct in IR^m and A is $n \times m$ real matrix. Usually, one can apply Gauss Elimination Method to find some solutions of this system, and this method is a systematic procedure for solving systems like (1); it is based on the idea of reducing the augmented matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & b_n \end{bmatrix}, \tag{4}$$

to the form that is simple enough such that the system of equations can be solved by inspection. But, to my knowledge, in general there is not formula for the solutions of (1) in terms of determinants if $m \neq n$.

When $m = n$ and $\det(A) \neq 0$, the system (1) admits only one solution given by $x = A^{-1}b$, and from here one can deduce the well known Cramer Rule which says:

Theorem 1.1. (*Cramer Rule* 1704-1752) *If* A *is* $n \times n$ *matrix with* $\det(A) \neq 0$, *then the solution of the system* (1) *is given by the formula*:

$$x_i = \frac{\det\left((A)_i\right)}{\det(A)}, \quad i = 1, 2, 3, \cdots, n, \tag{5}$$

where $(A)_i$ is the matrix obtained by replacing the entries in the *i*th column of *A* by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

A simple and interested generalization of Cramer Rule is done by Prof. Dr. Sylvan Burgstahler ([1]) from University of Minnesota, Duluth, where he taught for 20 years. This result is given by the following Theorem:

Theorem 1.2. (*Burgstahler* 1983) *If the system of equations*

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases} \tag{6}$$

has(unique) solution $x_1, x_2, \cdots, x_n$, then for all $\lambda_i \in IR, i = 1, 2, \cdots, n$ one has

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \frac{\begin{vmatrix} a_{1,1} + \lambda_1 b_1 & a_{1,2} + \lambda_2 b_1 & \cdots & a_{1,n} + \lambda_n b_1 \\ a_{2,1} + \lambda_1 b_2 & a_{2,2} + \lambda_2 b_2 & \cdots & a_{2,n} + \lambda_n b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + \lambda_1 b_n & a_{n,2} + \lambda_2 b_n & \cdots & a_{n,n} + \lambda_n b_n \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}} - 1. \tag{7}$$

Using Moore-Penrose Inverse Formula and Cramer's Rule, one can prove the following Theorem. But, for better understanding of the reader, we will include here a direct proof of it.

Theorem 1.3. *For all* $b \in IR^n$, *the system* (1) *is solvable if, and only if,*

$$\det\left(AA^*\right) \neq 0. \tag{8}$$

Moreover, one solution for this equation is given by the following formula:

$$x = A^*\left(AA^*\right)^{-1} b, \tag{9}$$

where A^* is the transpose of *A* (or the conjugate transpose of *A* in the complex case).

Also, this solution coincides with the Cramer formula when $n = m$. In fact, this formula is given as follows:

$$x_i = \sum_{j=1}^{n} a_{j,i} \frac{\det\left(\left(AA^*\right)_j\right)}{\det\left(AA^*\right)}, \quad i = 1, 2, 3, \cdots, m, \tag{10}$$

where $\left(AA^*\right)_j$ is the matrix obtained by replacing the entries in the *j*th column of AA^* by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

In addition, this solution has minimum norm, *i.e.*,

$$\|x\| = \inf\left\{\|w\| : Aw = b, w \in IR^m\right\} \tag{11}$$

and $\|x\| = \|w\|$ with $Aw = b \Leftrightarrow x = w$.

The main results of this work are the following Theorems.

Theorem 1.4. *The solutions of* (1)-(3) *given by* (9) *can be written as follows*:

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i} b_1 & \langle l_1, l_2 \rangle + a_{2i} b_1 & \cdots & \langle l_1, l_n \rangle + a_{ni} b_1 \\ \langle l_2, l_1 \rangle + a_{1i} b_2 & \|l_2\|^2 + a_{2i} b_2 & \cdots & \langle l_2, l_n \rangle + a_{ni} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ \langle l_n, l_1 \rangle + a_{1i} b_n & \langle l_n, l_2 \rangle + a_{2i} b_n & \cdots & \|l_n\|^2 + a_{ni} b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2 \rangle & \cdots & \langle l_1, l_n \rangle \\ \langle l_2, l_1 \rangle & \|l_2\|^2 & \cdots & \langle l_2, l_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle l_n, l_1 \rangle & \langle l_n, l_2 \rangle & \cdots & \|l_n\|^2 \end{vmatrix}} - 1, \quad i = 1, 2, 3, \cdots, m. \tag{12}$$

Theorem 1.5. *The system* (1) *is solvable for each* $b \in IR^n$, *if, and only if, the set of vectors* $\{l_1, l_2, \cdots, l_n\}$ *formed by the rows of the matrix A is lineally independent in* IR^m.

Moreover, a solution for the system (1) is given by the following formula:

$$x_i = \upsilon_{1i} \|\upsilon_1\|^2 b_1 + \upsilon_{2i} \|\upsilon_2\|^2 \left(b_2 - \langle l_2, \upsilon_1\rangle \|\upsilon_1\|^2 c_1\right), \tag{13}$$

$$+ \cdots + \upsilon_{ni} \|\upsilon_n\|^2 \left(b_n - \sum_{i=1}^{n-1} \langle l_n, \upsilon_i\rangle \|\upsilon_i\|^2 c_i\right), \quad i = 1, 2, \cdots, m, \tag{14}$$

where the set of vectors $\{\upsilon_1, \upsilon_2, \cdots, \upsilon_n\}$ is obtain by the Gram-Schmidt process and the numbers $c_1, c_2, \cdots, c_n$ are given by

$$\begin{aligned} c_1 &= b_1 \\ c_2 &= b_2 - \langle l_2, \upsilon_1\rangle \|\upsilon_1\|^2 c_1 \\ c_3 &= b_3 - \langle l_3, \upsilon_1\rangle \|\upsilon_1\|^2 c_1 - \langle l_3, \upsilon_2\rangle \|\upsilon_2\|^2 c_2 \\ &\vdots \\ c_n &= b_n - \sum_{i=1}^{n-1} \langle l_n, \upsilon_i\rangle \|\upsilon_i\|^2 c_i \end{aligned} \tag{15}$$

and $\upsilon_i = [\upsilon_{i1}, \upsilon_{i2}, \upsilon_{i3}, \cdots, \upsilon_{im}]^{\mathrm{T}}$, $i = 1, 2, \cdots, n$.

2. Proof of the Main Theorems

In this section we shall prove **Theorems 1.3**, **1.4**, **1.5** and more. To this end, we shall denote by $\langle x, y\rangle$ the Euclidian innerproduct in IR^k and the associated norm by $\|x\| = \sqrt{\langle x, x\rangle}$. Also, we shall use some ideas from [2] and the following result from [3], pp 55.

Lemma 2.1. *Let W and Z be Hilbert space,* $G \in L(W, Z)$ *and* $G^* \in L(Z, W)$ *the adjoint operator, then the following statements holds,*

[(i)] $\mathrm{Rang}(G) = Z \Leftrightarrow \exists \gamma > 0$ *such that*

$$\|G^* z\|_W \geq \gamma \|z\|_Z, \quad z \in Z.$$

[(ii)] $\overline{\mathrm{Rang}(G)} = Z \Leftrightarrow Ker(G^*) = \{0\} \Leftrightarrow G^*$ is $1-1$.

We will include here a direct proof of **Theorem 1.3** just for better understanding of the reader.

Proof of Theorem 1.3. The matrix A may also viewed as a linear operator $A: IR^m \to IR^n$; therefore $A \in L(IR^m, IR^n)$ and its adjoint operator A^* is the transpose of A and $A^*: IR^n \to IR^m$.

Then, system (1) is solvable for all $b \in IR^n$ if, and only if, the operator A is surjective. Hence, from the **Lemma 2.1** there exists $\gamma > 0$ such that

$$\|A^* z\|_{IR^m} \geq \gamma \|z\|_{IR^n}, \quad z \in IR^n.$$

Therefore,

$$\langle AA^* z, z\rangle \geq \gamma^2 \|z\|^2_{IR^n}, \quad z \in IR^n.$$

This implies that AA^* is one to one. Since AA^* is a $n \times n$ matrix, then $\det(AA^*) \neq 0$.

Suppose now that $\det(AA^*) \neq 0$. Then $(AA^*)^{-1}$ exists and given $b \in IR^n$ we can see that $x = A^*(AA^*)^{-1} b$ is a solution of $Az = b$.

Now, since $z = (AA^*)^{-1} b$ is the only solution of the equation

$$(AA^*) w = b,$$

then from **Theorem 1.1** (Cramer Rule) we obtain that:

$$z_1 = \frac{\det\left(\left(AA^*\right)_1\right)}{\det\left(AA^*\right)}, z_2 = \frac{\det\left(\left(AA^*\right)_2\right)}{\det\left(AA^*\right)}, \cdots, z_n = \frac{\det\left(\left(AA^*\right)_n\right)}{\det\left(AA^*\right)},$$

where $\left(AA^*\right)_i$ is the matrix obtained by replacing the entries in the ith column of AA^* by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then, the solution $x = A^*\left(AA^*\right)^{-1} b$ of (1) can be written as follows

$$x = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,m} & a_{2,m} & \cdots & a_{n,m} \end{bmatrix} \begin{bmatrix} \frac{\det\left(\left(AA^*\right)_1\right)}{\det\left(AA^*\right)} \\ \frac{\det\left(\left(AA^*\right)_2\right)}{\det\left(AA^*\right)} \\ \vdots \\ \frac{\det\left(\left(AA^*\right)_n\right)}{\det\left(AA^*\right)} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{j,1} \frac{\det\left(\left(AA^*\right)_j\right)}{\det\left(AA^*\right)} \\ \sum_{j=1}^{n} a_{j,2} \frac{\det\left(\left(AA^*\right)_j\right)}{\det\left(AA^*\right)} \\ \vdots \\ \sum_{j=1}^{n} a_{j,m} \frac{\det\left(\left(AA^*\right)_j\right)}{\det\left(AA^*\right)} \end{bmatrix}.$$

Now, we shall see that this solution has minimum norm. In fact, consider w in IR^m such that $Aw = b$ and

$$\|w\|^2 = \|x + (w - x)\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, w - x\rangle + \|w - x\|^2.$$

On the other hand,

$$\langle x, w - x\rangle = \left\langle A^*\left(AA^*\right)^{-1} b, w - x\right\rangle = \left\langle \left(AA^*\right)^{-1} b, Aw - Ax\right\rangle = \left\langle \left(AA^*\right)^{-1} b, b - b\right\rangle = 0.$$

Hence, $\|w\|^2 - \|x\|^2 = \|w - x\|^2 \geq 0$.

Therefore, $\|x\| \leq \|w\|$, and $\|x\| = \|w\|$ if $x = w$.

Proof of Theorem 1.5. Suppose the system is solvable for all $b \in IR^n$. Now, assume the existence of real numbers $c_i, i = 1, 2, \cdots, n$ such that

$$c_1 l_1 + c_2 l_2 + c_3 l_3 + \cdots + c_n l_n = 0.$$

Then, there exists $x \in IR^m$ such that

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = c_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = c_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = c_n \end{cases}.$$

In other words,

$$\langle l_i, x\rangle = c_i, \quad i = 1, 2, \cdots, n.$$

Hence,

$$\langle c_i l_i, x\rangle = c_i^2, \; i = 1, 2, \cdots, n.$$

So,

$$\left\langle c_1 l_1 + c_2 l_2 + c_3 l_3 + \cdots + c_n l_n, x \right\rangle = c_1^2 + c_2^2 + c_3^2 + \cdots + c_n^2 = 0.$$

Therefore, $c_1 = c_2 = \cdots = c_n = 0$, which prove the independence of $\{l_1, l_2, \cdots, l_n\}$.

Now, suppose that the set $\{l_1, l_2, \cdots, l_n\}$ is linearly independent in IR^m. Using the Gram-Schmidt process we can find a set $\{\upsilon_1, \upsilon_2, \cdots, \upsilon_n\}$ of orthogonal vectors in IR^m given by the formula:

$$\begin{aligned}
\upsilon_1 &= l_1 \\
\upsilon_2 &= l_2 - \frac{\langle l_2, \upsilon_1 \rangle}{\|\upsilon_1\|^2} \upsilon_1 \\
\upsilon_3 &= l_3 - \frac{\langle l_3, \upsilon_1 \rangle}{\|\upsilon_1\|^2} \upsilon_1 - \frac{\langle l_3, \upsilon_2 \rangle}{\|\upsilon_2\|^2} \upsilon_2 \, . \\
&\vdots \\
\upsilon_n &= l_n - \sum_{i=1}^{n-1} \frac{\langle l_n, \upsilon_i \rangle}{\|\upsilon_i\|^2} \upsilon_i
\end{aligned} \tag{16}$$

Then, system (1) will be equivalent to the following system:

$$\begin{cases}
\langle \upsilon_1, x \rangle = c_1 \\
\langle \upsilon_2, x \rangle = c_2 \\
\langle \upsilon_3, x \rangle = c_3 \\
\qquad \vdots \\
\langle \upsilon_n, x \rangle = c_n
\end{cases}, \tag{17}$$

where

$$\begin{aligned}
c_1 &= b_1 \\
c_2 &= b_2 - \frac{\langle l_2, \upsilon_1 \rangle}{\|\upsilon_1\|^2} c_1 \\
c_3 &= b_3 - \frac{\langle l_3, \upsilon_1 \rangle}{\|\upsilon_1\|^2} c_1 - \frac{\langle l_3, \upsilon_2 \rangle}{\|\upsilon_2\|^2} c_2 \, . \\
&\vdots \\
c_n &= b_n - \sum_{i=1}^{n-1} \frac{\langle l_n, \upsilon_1 \rangle}{\|\upsilon_i\|^2} c_i
\end{aligned} \tag{18}$$

If we denote the vectors υ_i's by

$$\upsilon_i = \begin{bmatrix} \upsilon_{i1} \\ \upsilon_{i2} \\ \upsilon_{i3} \\ \vdots \\ \upsilon_{im} \end{bmatrix}, \quad i = 1, 2, \cdots, n,$$

and the $n \times m$ matrix Υ by

$$\Upsilon = \begin{bmatrix} \upsilon_{1,1} & \upsilon_{1,2} & \cdots & \upsilon_{1,m} \\ \upsilon_{2,1} & \upsilon_{2,2} & \cdots & \upsilon_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \upsilon_{n,1} & \upsilon_{n,2} & \cdots & \upsilon_{n,m} \end{bmatrix},$$

then, applying **Theorem 1.3** we obtain that system (17) has solution for all $C \in IR^n$ if, and only if, $\det\left(\Upsilon\Upsilon^*\right) \neq 0$. But,

$$\Upsilon\Upsilon^* = \begin{bmatrix} \|\upsilon_1\|^2 & \langle \upsilon_1, \upsilon_2 \rangle & \cdots & \langle \upsilon_1, \upsilon_n \rangle \\ \langle \upsilon_2, \upsilon_1 \rangle & \|\upsilon_2\|^2 & \cdots & \langle \upsilon_2, \upsilon_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \upsilon_n, \upsilon_1 \rangle & \langle \upsilon_n, \upsilon_2 \rangle & \cdots & \|\upsilon_n\|^2 \end{bmatrix} = \begin{bmatrix} \|\upsilon_1\|^2 & 0 & 0 & \cdots & 0 \\ 0 & \|\upsilon_2\|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|\upsilon_n\|^2 \end{bmatrix}.$$

So,

$$\det\left(\Upsilon\Upsilon^*\right) = \|\upsilon_1\|^2 \|\upsilon_2\|^2 \cdots \|\upsilon_n\|^2 \neq 0.$$

From here and using the formula (9) we complete the proof of this Theorem.

Examples and Particular Cases

In this section we shall consider some particular cases and examples to illustrate the results of this work.

Example 2.1. *Consider the following particular case of system* (1)

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b. \tag{19}$$

In this case $n = 1$ and $A = \left[a_{1,1}, a_{1,2}, \cdots, a_{1,m}\right]$. Then, if we define the column vector

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix},$$

$$AA^* = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \end{bmatrix} \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix} = \|l_1\|^2.$$

Then, $\left(AA^*\right)^{-1} b = b\|l_1\|^{-2}$ and

$$x = A^*\left(AA^*\right)^{-1} b = \begin{bmatrix} a_{1,1}b\|l_1\|^{-2} \\ a_{1,2}b\|l_1\|^{-2} \\ \vdots \\ a_{1,m}b\|l_1\|^{-2} \end{bmatrix}.$$

Therefore, a solution of the system (19) is given by:

$$x_i = \frac{a_{1i}b}{\|l_1\|^2} = \frac{a_{1i}b}{\sum_{j=1}^{m} a_{1j}^2}, \quad i = 1, 2, \cdots, m. \tag{20}$$

Example 2.2. *Consider the following particular case of system* (1)

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m &= b_2 \end{aligned} \tag{21}$$

In this case $n = 2$ and

$$A=\begin{bmatrix} a_{1,1} & a_{1,2}x_2 & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \end{bmatrix},$$

Then, if we define the column vectors

$$l_1=\begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix}, \quad l_2=\begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \\ a_{2,m} \end{bmatrix},$$

then

$$AA^*=\begin{bmatrix} a_{1,1} & a_{1,2}x_2 & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \end{bmatrix}\begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ \vdots & \vdots \\ a_{1,m} & a_{12,m} \end{bmatrix}=\begin{bmatrix} \|l_1\|^2 & \langle l_1,l_2\rangle \\ \langle l_2,l_1\rangle & \|l_2\|^2 \end{bmatrix}.$$

Hence, from the formula (10) we obtain that:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}=A^*\left(AA^*\right)^{-1}b=\begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ \vdots & \vdots \\ a_{1,m} & a_{12,m} \end{bmatrix}\begin{bmatrix} \dfrac{\det\left(\left(AA^*\right)_1\right)}{\det\left(AA^*\right)} \\ \dfrac{\det\left(\left(AA^*\right)_2\right)}{\det\left(AA^*\right)} \end{bmatrix}.$$

Therefore, a solution of the system (21) is given by:

$$x_1=a_{11}\frac{b_1\|l_2\|^2-b_2\langle l_1,l_2\rangle}{\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2}+a_{21}\frac{b_2\|l_1\|^2-b_1\langle l_2,l_1\rangle}{\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2} \tag{22}$$

$$x_2=a_{12}\frac{b_1\|l_2\|^2-b_2\langle l_1,l_2\rangle}{\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2}+a_{22}\frac{b_2\|l_1\|^2-b_1\langle l_2,l_1\rangle}{\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2} \tag{23}$$

$$\vdots \tag{24}$$

$$x_m=a_{1m}\frac{b_1\|l_2\|^2-b_2\langle l_1,l_2\rangle}{\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2}+a_{2m}\frac{b_2\|l_1\|^2-b_1\langle l_2,l_1\rangle}{\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2}. \tag{25}$$

Now, we shall apply the foregoing formula or (12) to find the solution of the following system

$$\begin{cases} x_1+x_2=1 \\ -x_1+x_2+x_3=-1 \end{cases}. \tag{26}$$

If we define the column vectors

$$l_1=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad l_2=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

then $\det\left(AA^*\right)=\|l_1\|^2\|l_2\|^2-|\langle l_1,l_2\rangle|^2=\|l_1\|^2\|l_2\|^2=6$ and $x_1=\frac{5}{6}$, $x_2=\frac{1}{6}$, and $x_3=\frac{-2}{6}$.

Example 2.3. *Consider the following general case of system* (1)

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = b_n \end{cases}. \quad (27)$$

Then, if $\{l_1, l_2, \cdots, l_n\}$ is an orthogonal set in IR^m, we get

$$AA^* = \begin{bmatrix} \|l_1\|^2 & 0 & 0 & \cdots & 0 \\ 0 & \|l_2\|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|l_n\|^2 \end{bmatrix},$$

and the solution of the system (1) is very simple and given by:

$$x_i = \sum_{j=1}^{n} a_{j,i} b_j \|l_j\|^{-2}, \quad i = 1, 2, \cdots, m. \quad (28)$$

Now, we shall apply the formula (28) or (12) to find solution of the following system:

$$\begin{cases} -x_1 - x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 1 \\ x_1 - x_2 - x_3 + x_4 = 1 \end{cases}. \quad (29)$$

If we define the column vectors

$$l_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad l_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Then, $\{l_1, l_2, L_3\}$ is an orthogonal set in IR^3 and the solution of this system is given by: $x_1 = \frac{-1}{4}$, $x_2 = \frac{-1}{4}$, $x_3 = \frac{-1}{4}$ and $x_4 = \frac{3}{4}$.

3. Variational Method to Obtain Solutions

Theorems 1.3, **1.4** and **1.5** give a formula for one solution of the system (1) which has minimum norma. But it is not the only way allowing to build solutions of this equation. Next, we shall present a variational method to obtain solutions of (1) as a minimum of the quadratic functional $j : IR^n \to IR$,

$$j(\xi) = \frac{1}{2}\|A^*\xi\|^2 - \langle b, \xi \rangle, \quad \forall \xi \in IR^n. \quad (30)$$

Proposition 3.1. *For a given* $b \in IR^n$ *the Equation* (1) *has a solution* $x \in IR^m$ *if, and only if,*

$$\langle x, A^*\xi \rangle - \langle b, \xi \rangle = 0, \quad \forall \xi \in IR^n. \quad (31)$$

It is easy to see that (31) is in fact an optimality condition for the critical points of the quadratic functional *j* define above.

Lemma 3.1. *Suppose the quadratic functional j has a minimizer* $\xi_b \in IR^n$. *Then,*

$$x_b = A^* \xi_b \tag{32}$$

is a solution of (1).

Proof. First, we observe that *j* has the following form:

$$j(\xi) = \frac{1}{2}\langle AA^*\xi, \xi\rangle - \langle b, \xi\rangle, \quad \forall \xi \in IR^n.$$

Then, if ξ_b is a point where *j* achieves its minimum value, we obtain that:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\{j\}(\xi_b) = AA^*\xi_b - b = 0.$$

So, $AA^*\xi_b = b$ and $x_b = A^*\xi_b$ is a solution of (1).

Remark 3.1. *Under the condition of **Theorem** 1.3, the solution given by the formulas* (32) *and* (9) *coincide*.

Theorem 3.1. *The system* (1) *is solvable if, and only if, the quadratic functional j defined by* (30) *has a minimum for all* $b \in IR^n$.

Proof. Suppose (8) is solvable. Then, the matrix *A* viewed as an operator from IR^m to IR^n is surjective. Hence, from **Lemma 2.1.** there exists $\gamma > 0$ such that

$$\|A^*\xi\|^2 \geq \gamma^2 \|\xi\|^2, \ \xi \in IR^n.$$

Then,

$$j(\xi) \geq \frac{\gamma^2}{2}\|\xi\|^2 - \|b\|\|\xi\|, \quad \xi \in IR^n.$$

Therefore,

$$\lim_{\|\xi\| \to \infty} j(\xi) = \infty.$$

Consequently, *j* is coercive and the existence of a minimum is ensured.

The other way of the proof follows as in proposition 3.1.

Now, we shall consider an example where **Theorems 1.3**, **1.4** and **1.5** can not be applied, but proposition 3.1 does.

Example 3.1. *It considers the system with linearly independent rows*

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + 2x_3 = 2 \end{cases}.$$

In this case $n = 2$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then

$$AA^* = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}.$$

Therefore, the critical points of the quadratic functional *j* given by (30) satisfy the equation:

$$AA^*\xi = b,$$

i.e.,

$$\begin{matrix} 3\xi_1 + 6\xi_2 = 1 \\ 6\xi_1 + 12\xi_2 = 2 \end{matrix}.$$

So, there are infinitely many critical points given by

$$\xi = \begin{bmatrix} \frac{1}{3} - 2a \\ a \end{bmatrix}, \quad a \in IR.$$

Hence, a solution of the system is given by

$$x = A^{*}\xi = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} - 2a \\ a \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

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