Application of Different $H(x)$ in Homotopy Analysis Methods for Solving Systems of Linear Equations

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Abstract

In this paper, we present homotopy analysis method (HAM) for solving system of linear equations and use of different $H(x)$ in this method. The numerical results indicate that this method performs better than the homotopy perturbation method (HPM) for solving linear systems.

Keywords

HAM, HPM, Linear System

1. Introduction

Approximating the solutions of the system of linear and nonlinear equations has widespread applications in applied mathematics [1]-[11]. Many techniques including homotopy perturbation method (HPM) [12] and iterative methods [13] were suggested to search for the solution of linear systems. In 2009 Keramati [2] and in 2011 Liu [3] in their articles applied HPM to the solution of the system $Ax = b$. In this article we used homotopy analysis method [14] [15] with different $H(x)$ to solve linear system $Ax = b$ and showed that our results were better than the HPM results; then convergence of the method was considered.

Consider a linear system

$$Ax = b, \tag{1}$$

where $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in R^{n \times n}$ is nonsingular and $x, b \in R^n$ is a vector.

First of all, the basic ideas of the homotopy analysis method are being discussed.

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Let x_0 be an initial guess of x, and $q \in [0,1]$ be called the embedding parameter. The homotopy analysis method is based on a kind of continuous mapping $x \to \phi(x;q)$ such that, as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies from the initial guess x_0 to the exact solution x. To ensure this, choose such an auxiliary linear operator as

$$L\big(\phi(x,q)\big) = \phi(x,q) \tag{2}$$

and we define the operator

$$N\big(\phi(x,q)\big) = A\big(\phi(x,q)\big) - b. \tag{3}$$

Let $h \neq 0$ and $H(t) \neq 0$ denote the so-called auxiliary parameter and auxiliary matrix, respectively. Using the embedding parameter $q \in [0,1]$, we construct a family of equations

$$(1-q)L\big(\phi(x,q) - x_0\big) = hqH(x)N\big(\phi(x,q)\big)$$

from (2) and (3) we have

$$(1-q)\big(\phi(x,q) - x_0\big) = hqH(x)\big(A\big(\phi(x,q)\big) - b\big). \tag{4}$$

Obviously, at $q = 0$ and $q = 1$, one has $\phi(x,0) = x_0$ and $A\big(\phi(x,1)\big) = b$ respectively. Thus, as q increases from 0 to 1, $\phi(x;q)$ varies continuously from x_0 to x. Such kind of continuous variation is called deformation in topology [16]. We call the family of equations like (4) the zeroth-order deformation equation. Now we define mth-order deformation derivative

$$x_m = \frac{1}{m!} \frac{\partial^m \phi(x,q)}{\partial q^m}\bigg|_{q=0}, \tag{5}$$

where $m = 1,2,3,\cdots$. Because $\phi(x,q)$ is now a function of the embedding parameter q, by Taylors Theorem, we expand $\phi(x;q)$ in a power series of the embedding parameter q as follows:

$$\phi(x,q) = \phi(x,0) + \sum_{m=1}^{+\infty} \frac{1}{m!} \frac{\partial^m \phi(x,q)}{\partial q^m}\bigg|_{q=0} q^m.$$

By using (5) we have

$$\phi(x,q) = x_0 + \sum_{m=1}^{+\infty} x_m q^m. \tag{6}$$

If the series (6) is convergent at $q = 1$, then using the relationship $\phi(x,1) = x$ one has the series solution

$$x = x_0 + \sum_{m=1}^{+\infty} x_m. \tag{7}$$

Now we have the so-called mth-order deformation equation

$$L\big[x_m - \chi_m x_{m-1}\big] = hH(x)R_m\big(\tilde{x}_{m-1}\big), \tag{8}$$

where

$$R_m\big(\tilde{x}_{m-1}\big) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N\big(\phi(x,q)\big)}{\partial q^{m-1}}\bigg|_{q=0} \tag{9}$$

and

$$\chi_m = \begin{cases} 0 & \text{when } m \leq 1, \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

By using (2) we obtain

$$\begin{aligned} x_m - \chi_m x_{m-1} &= hH(x)R_m\big(\tilde{x}_{m-1}\big), \\ x_m &= \chi_m x_{m-1} + hH(x)R_m\big(\tilde{x}_{m-1}\big). \end{aligned} \tag{11}$$

Also by using (3) and (9) we have

$$R_m\left(\tilde{x}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\left(A\phi(x,q)-b\right)}{\partial q^{m-1}}\bigg|_{q=0}$$

$$= \frac{1}{(m-1)!} \frac{A\partial^{m-1}\left(\phi(x,q)\right)}{\partial q^{m-1}}\bigg|_{q=0} - \frac{1}{(m-1)!} \frac{\partial^{m-1}(b)}{\partial q^{m-1}}\bigg|_{q=0},$$

and then

$$R_m\left(\tilde{x}_{m-1}\right) = Ax_{m-1} - b\left(1-\chi_m\right).$$

Finally by using (11) we obtain

$$x_m = \chi_m x_{m-1} + hH(x)\left(Ax_{m-1} - b\left(1-\chi_m\right)\right). \tag{12}$$

Now with the initial guess $x_0 = 0$ and $h = -1$ we have

$$x_1 = H(x)b,$$
$$x_2 = x_1 - H(x)Ax_1 = \left(I - H(x)A\right)x_1 = \left(I - H(x)A\right)H(x)b,$$
$$x_3 = x_2 - H(x)Ax_2 = \left(I - H(x)A\right)x_2 = \left(I - H(x)A\right)^2 H(x)b,$$
$$\vdots$$
$$x_n = x_{n-1} - H(x)Ax_{n-1} = \left(I - H(x)A\right)x_{n-1} = \left(I - H(x)A\right)^{n-1} H(x)b, \tag{13}$$

hence, by substituting (13) in (7) we obtain

$$x = x_1 + x_2 + x_3 + \cdots + x_n + \cdots$$
$$= H(x)b + \left(I - H(x)A\right)H(x)b + \left(I - H(x)A\right)^2 H(x)b + \cdots \tag{14}$$
$$+ \left(I - H(x)A\right)^{n-1} H(x)b + \cdots,$$

and by factor of $H(x)b$ we have

$$x = \left[I + \left(I - H(x)A\right) + \left(I - H(x)A\right)^2 + \cdots + \left(I - H(x)A\right)^{n-1} + \cdots\right]H(x)b. \tag{15}$$

Now we have to prove the convergence of (15).

Theorem 1. *The sequence* $x^{[m]} = \left[\sum_{k=0}^{m}\left[I - H(x)A\right]^k\right]H(x)b,$ *is a Cauchy sequence if*

$$\left\|I - H(x)A\right\| < 1.$$

Proof: Following ([2], Theorem 1) we have to show that

$$\lim_{m \to \infty}\left\|x^{[m+p]} - x^{[m]}\right\| = 0.$$

Now considering

$$x^{[m+p]} - x^{[m]} = \left[\sum_{k=1}^{p}\left[I - H(x)A\right]^{m+k}\right]H(x)b,$$

then

$$\left\|x^{[m+p]} - x^{[m]}\right\| \le \left\|H(x)b\right\|\sum_{k=1}^{p}\left\|I - H(x)A\right\|^{m+k}$$

let $\gamma = \left\|I - H(x)A\right\|$ then

$$\left\| x^{[m+p]} - x^{[m]} \right\| \leq \left\| H(x)b \right\| \gamma^m \sum_{k=1}^{p} \gamma^k \leq \left\| H(x)b \right\| \gamma^m \left(\frac{\gamma^p - 1}{\gamma - 1} \right),$$

so we have

$$\lim_{m \to \infty} \left\| x^{[m+p]} - x^{[m]} \right\| \leq \left(\frac{\gamma^p - 1}{\gamma - 1} \right) \left\| H(x)b \right\| \left(\lim_{m \to \infty} \gamma^m \right),$$

since $\gamma < 1$ then we obtain

$$\lim_{m \to \infty} \left\| x^{[m+p]} - x^{[m]} \right\| = 0$$

which completes the proof.

2. Main Results

In this section For solving the linear system (1) we apply different $H(x)$ and the convergence of the method is checked. At first assume that A is a nonsingular diagonally dominate matrix and $a_{ii} \neq 0, i = 1, 2, \cdots, n.$ Dividing (1) by a_{ii} and without loss of generality we can obtain

$$Bx = d. \tag{16}$$

where $B = \left[b_{i,j} \right]$, such that

$$b_{i,j} = \begin{cases} 1 & \text{for } i = j, i = 1, 2, \cdots, n \\ \dfrac{a_{i,j}}{a_{i,i}} & \text{for } i \neq j, i, j = 1, 2, \cdots, n, \end{cases} \tag{17}$$

and

$$d_i = \frac{b_i}{a_{i,i}} \quad i = 1, 2, \cdots, n.$$

Now we apply different $H(x)$ and the convergence of the method is tested.
1) we propose $H(x) = I + S$ with

$$S = (s_{ij}) = \begin{cases} -b_{ii+1} & \text{for } i = 1, 2, \cdots, n-1, j = i+1 \\ 0 & \text{otherwise}, \end{cases} \tag{18}$$

and show that

$$\left\| I - H(x)B \right\| < 1.$$

Theorem 2. *If A is diagonally dominated and* $B = \left[b_{i,j} \right]$*, where* $b_{i,j}$ *is defined in (17) then*

$$\left\| I - H(x)B \right\|_{\infty} < 1$$

Proof: By direct calculation we have

$$I - H(x)B = \begin{bmatrix} b_{12}b_{21} & 0 & -b_{13} + b_{12}b_{23} & \cdots & -b_{1n} + b_{12}b_{2n} \\ -b_{21} + b_{23}b_{31} & b_{23}b_{32} & 0 & \cdots & -b_{1n} + b_{12}b_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_{n-1,1} + b_{n-1,n}b_{n,1} & -b_{n-1,2} + b_{n-1,n}b_{n,2} & \cdots & b_{n-1,n}b_{n,n-1} & 0 \\ -b_{n,1} & -b_{n,2} & \cdots & -b_{n,n-1} & 0 \end{bmatrix}$$

and first row is satisfied:

$$\left| b_{12}b_{21} \right| + \left| -b_{13} + b_{12}b_{23} \right| + \left| -b_{14} + b_{12}b_{24} \right| + \cdots + \left| -b_{1n} + b_{12}b_{2n} \right|$$
$$\leq \left| b_{12} \right| \left| b_{21} \right| + \left| b_{13} \right| + \left| b_{12} \right| \left| b_{23} \right| + \left| b_{14} \right| + \left| b_{12} \right| \left| b_{24} \right| + \cdots + \left| b_{1n} \right| + \left| b_{12} \right| \left| b_{2n} \right|$$
$$= \left| b_{12} \right| \left(\left| b_{21} \right| + \left| b_{23} \right| + \left| b_{24} \right| + \cdots + \left| b_{2n} \right| \right) + \left(\left| b_{13} \right| + \left| b_{14} \right| + \cdots + \left| b_{1n} \right| \right).$$

Since A is diagonally dominated, B is diagonally dominated and we have

$$|b_{21}| + |b_{23}| + |b_{24}| + \cdots + |b_{2n}| \leq 1. \tag{19}$$

Now by using (19) we obtain

$$|b_{12}| \left(\underbrace{|b_{21}| + |b_{23}| + |b_{24}| + \cdots + |b_{2n}|}_{\leq 1} \right) + \left(|b_{13}| + |b_{14}| + \cdots + |b_{1n}| \right)$$

$$\leq |b_{12}| + |b_{13}| + |b_{14}| + \cdots + |b_{1n}| \leq 1.$$

This relation satisfis for other rows also and

$$\left\| I - H(x)B \right\|_{\infty} < 1.$$

2) We propose $H(x) = I + R$ with

$$R = \begin{cases} -b_{nj} & \text{for } j = 1, 2, \cdots, n-1 \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

and show that

$$\left\| I - H(x)B \right\| < 1.$$

Theorem 3. *If A is diagonally dominated and $B = \left[b_{i,j} \right]$, where $b_{i,j}$ is defined in (17) then*

$$\left\| I - H(x)B \right\|_{\infty} < 1$$

***Proof*:** Following Theorem (2)

$$I - H(X)B = \begin{bmatrix} 0 & -b_{12} & -b_{13} & \cdots & -b_{1n} \\ -b_{21} & 0 & -b_{23} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_{n-1,1} & -b_{n-1,2} & \cdots & 0 & -b_{n-1,n} \\ R_{n1} & R_{n2} & \cdots & R_{n,n-1} & R_{n,n} \end{bmatrix},$$

such that

$$R_{n1} = b_{n,2}b_{21} + b_{n,3}b_{31} + \cdots + b_{n,n-1}b_{n-1,1},$$
$$R_{n2} = b_{n,1}b_{12} + b_{n,3}b_{32} + \cdots + b_{n,n-1}b_{n-1,2},$$
$$\vdots$$
$$R_{n,n-1} = b_{n,1}b_{1,n-1} + b_{n,2}b_{2,n-1} + \cdots + b_{n,n-2}b_{n-2,n-1},$$
$$R_{n,n} = b_{n,1}b_{1,n} + b_{n,2}b_{2,n} + \cdots + b_{n,n-1}b_{n-1,n}$$

and last row is satisfied:

$$\left| b_{n,2}b_{21} + b_{n,3}b_{31} + \cdots + b_{n,n-1}b_{n-1,1} \right| + \left| b_{n,1}b_{12} + b_{n,3}b_{32} + \cdots + b_{n,n-1}b_{n-1,2} \right| + \cdots$$

$$+ \left| b_{n,1}b_{1,n-1} + b_{n,2}b_{2,n-1} + \cdots + b_{n,n-2}b_{n-2,n-1} \right| + \left| b_{n,1}b_{1,n} + b_{n,2}b_{2,n} + \cdots + b_{n,n-1}b_{n-1,n} \right|$$

$$\leq |b_{n,1}| \left(\underbrace{|b_{12}| + |b_{13}| + \cdots + |b_{1,n}|}_{\leq 1} \right) + |b_{n,2}| \left(\underbrace{|b_{21}| + |b_{23}| + \cdots + |b_{2,n}|}_{\leq 1} \right)$$

$$+ |b_{n,3}| \left(\underbrace{|b_{31}| + |b_{32}| + \cdots + |b_{3,n}|}_{\leq 1} \right) + \cdots + |b_{n,n-1}| \left(\underbrace{|b_{n-1,1}| + |b_{n-1,2}| + \cdots |b_{n-1,n}|}_{\leq 1} \right)$$

$$\leq |b_{n,1}| + |b_{n,2}| + \cdots |b_{n,n-1}| \leq 1$$

This relation satisfis for other rows also

$$\left\| I - H(x)B \right\|_\infty < 1.$$

3) We propose $H(x) = I + S + R$ such that S and R was explained in (18) and (20) respectively and show that

$$\left\| I - H(x)B \right\| < 1.$$

Theorem 4. *If A is diagonally dominated and* $B = \left[b_{i,j} \right]$, *where* $b_{i,j}$ *is defined in* (17) *then*

$$\left\| I - H(x)B \right\|_\infty < 1$$

Proof: Similar to proof of Theorems (2) and (3).

4) We propose $H(x) = I + S(m)$ with

$$S(m) = \left(S_{ij} \right) = \begin{cases} -b_{ik_i} & \text{for } 1 \leq i < n, j > i \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

$$k_i = \left\{ \min j \left| \max_j \left| a_{ij} \right| \right. \right\}, \quad 1 \leq i < n$$

and show that

$$\left\| I - H(x)B \right\| < 1.$$

Theorem 5. *If A is diagonally dominated and* $B = \left[b_{i,j} \right]$, *where* $b_{i,j}$ *is defined in* (17) *then*

$$\left\| I - H(x)B \right\|_\infty < 1$$

Proof: Following Theorem (2) after expanding $I - H(x)B$ according to the first row we have

$$\left| b_{1k_1} b_{k_1 1} \right| + \left| -b_{12} + b_{1k_1} b_{k_1 2} \right| + \left| -b_{13} + b_{1k_1} b_{k_1 3} \right| + \cdots + \left| -b_{1k_1 - 1} + b_{1k_1} b_{k_1 k_1 - 1} \right|$$

$$+ 0 + \left| -b_{1k_1 + 1} + b_{1k_1} b_{k_1 k_1 + 1} \right| + \cdots + \left| -b_{1n} + b_{1k_1} b_{k_1 n} \right|$$

$$\leq \left| b_{1k_1} \right| \left| b_{k_1 1} \right| + \left| -b_{12} \right| + \left| b_{1k_1} \right| \left| b_{k_1 2} \right| + \left| -b_{13} \right| + \left| b_{1k_1} \right| \left| b_{k_1 3} \right| + \cdots + \left| -b_{1k_1 - 1} \right|$$

$$+ \left| b_{1k_1} \right| \left| b_{k_1 k_1 - 1} \right| + \left| -b_{1k_1 + 1} \right| + \left| b_{1k_1} \right| \left| b_{k_1 k_1 + 1} \right| + \cdots + \left| -b_{1n} \right| + \left| b_{1k_1} \right| \left| b_{k_1 n} \right|$$

$$= \left| b_{1k_1} \right| \left(\underbrace{\left| b_{k_1 1} \right| + \left| b_{k_1 2} \right| + \cdots + \left| b_{k_1 k_1 - 1} \right| + \left| b_{k_1 k_1 + 1} \right| + \cdots + \left| b_{k_1 n} \right|}_{\leq 1} \right)$$

$$+ \left(\left| -b_{12} \right| + \left| -b_{13} \right| + \cdots + \left| -b_{1k_1 - 1} \right| + \left| -b_{1k_1 + 1} \right| + \cdots + \left| -b_{1n} \right| \right)$$

$$\leq \left| b_{1k_1} \right| + \left| -b_{12} \right| + \left| -b_{13} \right| + \cdots + \left| -b_{1k_1 - 1} \right| + \left| -b_{1k_1 + 1} \right| + \cdots + \left| -b_{1n} \right| \leq 1.$$

This relation satisfis for other rows also

$$\left\| I - H(x)B \right\|_\infty < 1.$$

5) We propose $H(x) = I + S(m) + R$ such that $S(m)$ and R was explained in (21) and (20) respectively and show that

$$\left\| I - H(x)B \right\| < 1.$$

Theorem 6. *If A is diagonally dominated and* $B = \left[b_{i,j} \right]$, *where* $b_{i,j}$ *is defined in* (17) *then*

$$\left\| I - H(x)B \right\|_\infty < 1$$

Proof: Similar to proof of Theorems (3) and (5).

6) We propose $H(x) = I - U$ such that U is the strictly upper triangular part of A and show that

$$\|I - H(x)B\| < 1.$$

Theorem 7. *If A is diagonally dominated and* $B = \begin{bmatrix} b_{i,j} \end{bmatrix}$, *where* $b_{i,j}$ *is defined in* (17) *then*

$$\|I - H(x)B\|_{\infty} < 1$$

Proof: Following Theorem (2) after expanding $I - H(x)B$ according to the first row we have

$$\left| b_{12}b_{21} + b_{13}b_{31} + \cdots + b_{1n}b_{n1} \right| + \left| b_{13}b_{32} + b_{14}b_{42} + \cdots + b_{1n}b_{n2} \right|$$

$$+ \left| b_{12}b_{23} + b_{14}b_{43} + \cdots + b_{1n}b_{n3} \right| + \cdots + \left| b_{12}b_{2n} + b_{13}b_{3n} + \cdots + b_{1n-1}b_{n-1,n} \right|$$

$$\leq \left| b_{12} \right| \left| b_{21} \right| + \left| b_{13} \right| \left| b_{31} \right| + \cdots + \left| b_{1n} \right| \left| b_{n1} \right| + \left| b_{13} \right| \left| b_{32} \right| + \left| b_{14} \right| \left| b_{42} \right| + \cdots + \left| b_{1n} \right| \left| b_{n2} \right|$$

$$+ \left| b_{12} \right| \left| b_{23} \right| + \left| b_{14} \right| \left| b_{43} \right| + \cdots + \left| b_{1n} \right| \left| b_{n3} \right| + \cdots + \left| b_{12} \right| \left| b_{2n} \right| + \left| b_{13} \right| \left| b_{3n} \right| + \cdots + \left| b_{1n-1} \right| \left| b_{n-1,n} \right|$$

$$= \left| b_{12} \right| \left(\underbrace{\left| b_{21} \right| + \left| b_{23} \right| + \cdots + \left| b_{2n} \right|}_{\leq 1} \right) + \left| b_{13} \right| \left(\underbrace{\left| b_{31} \right| + \left| b_{32} \right| + \left| b_{34} \right| + \cdots + \left| b_{3n} \right|}_{\leq 1} \right) + \cdots$$

$$+ \left| b_{1n} \right| \left(\underbrace{\left| b_{n1} \right| + \left| b_{n2} \right| + \cdots + \left| b_{n,n-1} \right|}_{\leq 1} \right) \leq \underbrace{\left| b_{12} \right| + \left| b_{13} \right| + \cdots + \left| b_{1n} \right|}_{\leq 1} \leq 1$$

This relation satisfis for other rows also

$$\|I - H(x)B\|_{\infty} < 1.$$

7) We propose $H(x) = I - U + R$ such that U is the strictly upper triangular part of A and R was explained in (20) and show that

$$\|I - H(x)B\| < 1.$$

Theorem 8. *If A is diagonally dominated and* $B = \begin{bmatrix} b_{i,j} \end{bmatrix}$, *where* $b_{i,j}$ *is defined in* (17) *then*

$$\|I - H(x)B\|_{\infty} < 1$$

Proof: Similar to proof of Theorems (3) and (7).

Now in the next section we apply $H(x)$ for solving numerical examples.

3. Numerical Results

In this section, we present some numerical examples to apply HAM and HPM methods for solving linear system. We used of Matlab 2013 for numerical results.

Example 1. *Consider the linear system* $Ax = b$, *that* $A = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 6 & 2 \\ 0 & 1 & -3 \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ 9 \\ 5 \end{bmatrix}$ *and the exact solution is*

$$x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Table 1 shows the iteration number, error, spectral radius of iteration matrix and computation time.

According to **Table 1** we obtain the desirable result for solving this system by seven iterations with HAM and $H(x) = I - U + R$ while by HPM method we used of fourteen iteration.

In this example the matrices S and $S(m)$ are same and the results are same too.

Example 2. *In this example we apply HAM method for solving the linear system*

$$Ax = b$$

where A is a 1000×1000 matrix, b is a 1000×1 vector that its components are sum of the row components of the corresponding matrix and the exact solution is $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$. The numerical results are in **Table 2**.

Table 1. Camparision between HPM and HAM for 3×3 system.

Method	Iteration	Error	Spectral radius	Times (s)
HAM $(H(x) = I)$ (HPM)	14	10^{-5}	0.4004	0.013
HAM $(H(x) = I + S)$	11	10^{-5}	0.3057	0.010
HAM $(H(x) = I + R)$	9	10^{-5}	0.2168	0.010
HAM $(H(x) = I + R + S)$	8	10^{-5}	0.1806	0.014
HAM $(H(x) = I + S(m))$	11	10^{-5}	0.2918	0.010
HAM $(H(x) = I + S(m) + R)$	8	10^{-5}	0.1806	0.014
HAM $(H(x) = I - U)$	8	10^{-5}	0.2918	0.011
HAM $(H(x) = I - U + R)$	7	10^{-5}	0.1667	0.011

Table 2. Camparision between HPM and HAM for 1000×1000 system.

Method	Iteration	Error	Spectral radius	Times (s)
HAM $(H(x) = I)$ (HPM)	36	10^{-4}	0.5258	16.378
HAM $(H(x) = I + S)$	32	10^{-4}	0.5167	19.141
HAM $(H(x) = I + R)$	32	10^{-4}	0.5157	17.176
HAM $(H(x) = I + R + S)$	32	10^{-4}	0.5169	18.010
HAM $(H(x) = I + S(m))$	32	10^{-4}	0.5162	20.112
HAM $(H(x) = I + S(m) + R)$	32	10^{-4}	0.5261	19.200
HAM $(H(x) = I - U)$	25	10^{-4}	0.4143	11.731
HAM $(H(x) = I - U + R)$	20	10^{-4}	0.4092	9.175

4. Conclusion

From the numerical results, we have seen that the HAM method with different $H(x)$ produces a spectral radius smaller than the HPM and with the less iteration we obtain the desirable result.

Acknowledgements

References

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