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The Discriminance for *FLDcirc$_r$* Matrices and the Fast Algorithm of Their Inverse and Generalized Inverse

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Abstract

This paper presents a new type of circulant matrices. We call it the first and the last difference *r*-circulant matrix (*FLDcirc$_r$* matrix). We can verify that the linear operation, the matrix product and the inverse matrix of this type of matrices are still *FLDcirc$_r$* matrices. By constructing the basic *FLDcirc$_r$* matrix, we give the discriminance for *FLDcirc$_r$* matrices and the fast algorithm of the inverse and generalized inverse of the *FLDcirc$_r$* matrices.

Keywords

***FLDcirc$_r$* Matrix, Discriminance, Diagonalization, Inverse, Generalized Inverse**

1. Introduction

Circulant matrix plays an important role in the matrix theory, its special structure and properties have been widely used in applied mathematics, physics, modern engineering, and so on [1]-[6]. There have been many new circulant matrices come forward [7]-[12]. In this paper we will firstly put forward the concept of the *FLDcirc$_r$* matrix and the basic *FLDcirc$_r$* matrix. The sum, the difference, the product, the inverse and the adjoint matrix of this type of matrices are still *FLDcirc$_r$* matrices. Then, we will give five discriminance for *FLDcirc$_r$* matrix by constructing the basic *FLDcirc$_r$* matrix. At last, we will discuss the fast algorithm of the inverse and generalized inverse of the *FLDcirc$_r$* matrix and give the numerical example. In this paper, we just study the square matrices in complex field.

2. Definition of the *FLDcirc$_r$* Matrix

Definition 2.1 For a square matrix A of order n, if its form is

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$$A=\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ ra_{n-1} & a_0-ra_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2} \\ ra_{n-2} & ra_{n-1}-ra_{n-2} & a_0-ra_{n-1} & \cdots & a_{n-4} & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ ra_2 & ra_3-ra_2 & ra_4-ra_3 & \cdots & a_0-ra_{n-1} & a_1 \\ ra_1 & ra_2-ra_1 & ra_3-ra_2 & \cdots & ra_{n-1}-ra_{n-2} & a_0-ra_{n-1} \end{pmatrix},$$

We call it the $FLDcirc_r$ matrix, and denote shortly $A=FLDcirc_r(a_0,a_1,\cdots,a_{n-1})$.

Definition 2.2 Let D is the basic $FLDcirc_r$ matrix of order n, that is

$$D=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ r & -r & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We obtain $g(x)=x^n+rx-r$ is the characteristic polynomial of D, $D^n=r(I_n-D)$, we specify $D^0=I_n$.

From the definition of $FLDcirc_r$ matrix, we can prove the following proposition.

Proposition 2.3 If A and B are $FLDcirc_r$ matrices, then $A+B$, $A-B$ and kA are both $FLDcirc_r$ matrices, for any k belongs to the complex field.

Definition 2.4 Let $A\in C^{n\times n}$,the index of A is the least nonnegative integer k such that $rankA^{k+1}=rankA^k$, we note it as $Ind(A)=k$. If A is nonsingular, then $Ind(A)=0$; if A is singular, then $Ind(A)\ge 1$.

Definition 2.5 Let $A\in C^{n\times n}$, if there is $X\in C^{n\times n}$ which satisfies $AXA=A$, $XAX=X$ at the same time, we named X as the reflexive generalize inverse of A, we note it as $A^{\{1,2\}}$.

Definition 2.6 Let $A\in C^{n\times n}$, $Ind(A)=k$, if $X\in C^{n\times n}$ satisfies

$$A^kXA=A^k,$$
$$XAX=X,$$
$$AX=XA,$$

Then we denote X as the Drazin inverse of A, note it as $A^{\{d\}}$.

Lemma 2.7 If polynomial matrix $\begin{pmatrix} f(x) & 1 & 0 \\ g(x) & 0 & 1 \end{pmatrix}$ can transformed into $\begin{pmatrix} d(x) & u(x) & v(x) \\ 0 & s(x) & t(x) \end{pmatrix}$ after elementary row transformation, then we have $(f(x),g(x))=d(x)$, and $u(x)f(x)+v(x)g(x)=d(x)$.

3. The Discriminance of the *FLDcirc$_r$* Matrix

Theorem 3.1 A is an $FLDcirc_r$ matrix if and only if A is of the following form

$$A=f(D)=\sum_{i=0}^{n-1}a_iD^i \tag{1}$$

For some polynomial $f(x)=\sum_{i=0}^{n-1}a_ix^i$.

Proof. By the **Definition 2.1** and **Definition 2.2**, we get this result.

Theorem 3.2 A is an $FLDcirc_r$ matrix if and only if $AD=DA$, D is the basic $FLDcirc_r$ matrix.

Proof. $(\Rightarrow)$ For A is an $FLDcirc_r$ matrix, from the definition of A and D, we obtain $AD=DA$.

$(\Leftarrow)$ By the method of undetermined coefficients, let

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}.$$

Due to

$$AD = DA,$$

It follows that

$$a_{21}=ra_{1n},\ a_{22}=a_{11}-ra_{1n},\ a_{23}=a_{12},\cdots,a_{2,n-1}=a_{1,n-2},\ a_{2n}=a_{1,n-1};$$
$$a_{31}=ra_{2n},\ a_{32}=a_{21}-ra_{2n},\ a_{33}=a_{22},\cdots,a_{3,n-1}=a_{2,n-2},\ a_{3n}=a_{2,n-1};$$
$$\cdots.$$

We obtain

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ ra_{1n} & a_{11}-ra_{1n} & a_{12} & \cdots & a_{1,n-2} & a_{1,n-1} \\ ra_{1,n-1} & ra_{1n}-ra_{1,n-1} & a_{11}-ra_{1n} & \cdots & a_{1,n-3} & a_{1,n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ ra_{13} & ra_{14}-ra_{13} & ra_{15}-ra_{14} & \cdots & a_{11}-ra_{1n} & a_{12} \\ ra_{12} & ra_{13}-ra_{12} & ra_{14}-ra_{13} & \cdots & ra_{1n}-ra_{1,n-1} & a_{11}-ra_{1n} \end{pmatrix},$$

So A is an $FLDcirc_r$ matrix.

Corollary 3.3 If A and B are both $FLDcirc_r$ matrices, then AB and BA are $FLDcirc_r$ matrices. Furthermore, we get $AB = BA$.

Proof. Since A and B are $FLDcirc_r$ matrices, by the **Theorem 3.2**, we get

$$AD = DA,\ \ BD = DB,$$

Hence

$$ABD = ADB = DAB,\ \ BAD = BDA = DBA,$$

Then, AB and BA are both $FLDcirc_r$ matrices.

From **Theorem 3.1**, we have

$$AB = BA.$$

4. The Diagonalization of the $FLDcirc_r$ Matrix

First, we consider the diagonalization of the basic $FLDcirc_r$ matrix D.

For the characteristic polynomial $g(x)=x^n+rx-r$ of D has n different roots. So, D has n different eigenvalues:

$$\lambda_0=\omega_0,\ \lambda_1=\omega_1,\cdots,\lambda_{n-1}=\omega_{n-1}.$$

Let

$$V=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \omega_0 & \omega_1 & \omega_2 & \cdots & \omega_{n-2} & \omega_{n-1} \\ \omega_0^2 & \omega_1^2 & \omega_2^2 & \cdots & \omega_{n-2}^2 & \omega_{n-1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_0^{n-2} & \omega_1^{n-2} & \omega_2^{n-2} & \cdots & \omega_{n-2}^{n-2} & \omega_{n-1}^{n-2} \\ \omega_0^{n-1} & \omega_1^{n-1} & \omega_2^{n-1} & \cdots & \omega_{n-2}^{n-1} & \omega_{n-1}^{n-1} \end{pmatrix},$$

Obviously, $V = V(\omega_0, \omega_1, \cdots, \omega_{n-1})$ is a nonsingular *Vandermonde* matrix about $\omega_0, \omega_1, \cdots, \omega_{n-1}$, and

$$D = Vdiag(\omega_0, \omega_1, \cdots, \omega_{n-1})V^{-1}. \tag{2}$$

Next, we study the diagonalization of general $FLDcirc_r$ matrix A.

From **Theorem 3.1** and Equation (2), we obtain

$$\begin{aligned} A = \sum_{i=0}^{n-1} a_i D^i &= \sum_{i=0}^{n-1} a_i \left(Vdiag(\omega_0, \omega_1, \cdots, \omega_{n-1})V^{-1}\right)^i \\ &= \sum_{i=0}^{n-1} Vdiag\left(a_i\omega_0^i, a_i\omega_1^i, \cdots, a_i\omega_{n-1}^i\right)V^{-1} \\ &= Vdiag\left(f(\omega_0), f(\omega_1), \cdots, f(\omega_{n-1})\right)V^{-1}, \end{aligned}$$

The eigenvalues of A are

$$f(\omega_j) = \sum_{i=0}^{n-1} a_i \omega_j^i.$$

Theorem 4.1 A is an $FLDcirc_r$ matrix if and only if $V^{-1}AV$ is a diagonal matrix.

Proof. $(\Rightarrow)$ If A is an $FLDcirc_r$ matrix, from the above discussion, we have

$$V^{-1}AV = diag\left(f(\omega_0), f(\omega_1), \cdots, f(\omega_{n-1})\right).$$

$(\Leftarrow)$ Let $V^{-1}AV = P_1$, P_1 is a diagonal matrix, then

$$A = VP_1V^{-1}.$$

Let $P_2 = diag(\omega_0, \omega_1, \cdots, \omega_{n-1})$, from Equation (2) we have

$$D = VP_2V^{-1},$$

Thus

$$AD = VP_1V^{-1}VP_2V^{-1} = VP_1P_2V^{-1},$$
$$DA = VP_2V^{-1}VP_1V^{-1} = VP_2P_1V^{-1}.$$

For P_1 and P_2 are both diagonal matrix, so

$$AD = DA,$$

hence, A is an $FLDcirc_r$ matrix.

Theorem 4.2 A is a nonsingular $FLDcirc_r$ matrix if and only if the eigenvalues $f(\omega_i) \neq 0 (i = 0, 1, \cdots, n-1)$, where ω_i $(i = 0, 1, \cdots, n-1)$ are eigenvalues of the basic $FLDcirc_r$ matrix.

Proof. $(\Rightarrow)$ For A is a nonsingular $FLDcirc_r$ matrix, from the above discussion, we have

$$A = Vdiag\left(f(\omega_0), f(\omega_1), \cdots, f(\omega_{n-1})\right)V^{-1},$$

where $f(\omega_i) \neq 0 (i = 0, 1, \cdots, n-1)$ are eigenvalues of A.

So

$$|A| = |V| \prod_{i=0}^{n-1} f(\omega_i) \frac{1}{|V|} = \prod_{i=0}^{n-1} f(\omega_i).$$

Hence, if A is a nonsingular $FLDcirc_r$ matrix, we have $f(\omega_i) \neq 0 (i = 0, 1, \cdots, n-1)$.

$(\Leftarrow)$ Due to $f(\omega_i) \neq 0 (i = 0, 1, \cdots, n-1)$,

Then

$$|A| = |V| \prod_{i=0}^{n-1} f(\omega_i) \frac{1}{|V|} = \prod_{i=0}^{n-1} f(\omega_i) \neq 0,$$

So A is nonsingular.

5. The Fast Algorithm of the Inverse and Generalized Inverse of the *FLDcirc*$_r$ Matrix

Theorem 5.1 If *A* is a nonsingular matrix, then *A* is an *FLDcirc*$_r$ matrix if and only if A^{-1} is an *FLDcirc*$_r$ matrix.

Proof. $(\Rightarrow)$ From *A* is nonsingular and **Theorem 3.2**, we obtain

$$AD = DA,$$

Hence

$$DA^{-1} = A^{-1}D,$$

That is to say A^{-1} is an *FLDcirc*$_r$ matrix.

$(\Leftarrow)$ Clearly, the nonsingular matrix *A* is an *FLDcirc*$_r$ matrix.

Corollary 5.2 If *A* is a nonsingular *FLDcirc*$_r$ matrix, then A^* is a nonsingular *FLDcirc*$_r$ matrix.

Proof. For *A* is an *FLDcirc*$_r$ matrix, we have $AD = DA$, so

$$D^*A^* = A^*D^*.$$

Due to

$$DD^* = |D|I, D^* = |D|D^{-1},$$

Thus

$$|D|D^{-1}A^* = |D|A^*D^{-1},$$

Hence

$$A^*D = DA^*,$$

Then A^* is an *FLDcirc*$_r$ matrix.

Theorem 5.3 If *A* is an *FLDcirc*$_r$ matrix, then *A* is nonsingular if and only if $(f(x), g(x)) = 1$.

Proof. If *A* is a nonsingular *FLDcirc*$_r$ matrix, from **Theorem 4.2**, we have $f(\omega_i) \neq 0 (i = 0,1,\cdots,n-1)$, so $f(x)$ and $g(x)$ don't have the same solutions, thus $(f(x), g(x)) = 1$.

Otherwise, if $(f(x), g(x)) = 1$, there exist $u(x)$, $v(x)$ such that $u(x)f(x) + v(x)g(x) = 1$, $u(D)f(D) + v(D)g(D) = E$. For $g(D) = 0$, $f(D) = A$, we have $u(D)A = E$. So, *A* is nonsingular and $A^{-1} = u(D)$. From **Theorem 3.1**, we have A^{-1} is an *FLDcirc*$_r$ matrix.

Corollary 5.4 If *A* is a nonsingular *FLDcirc*$_r$ matrix, there exits $A^{-1} = u(D)$.

Corollary 5.5 *A* is a singular *FLDcirc*$_r$ matrix, there exists an *FLDcirc*$_r$ matrix *H* that satisfies $A^{\{1,2\}} = A^{\{d\}} = H$.

Proof. For A is singular, we get $(f(x), g(x)) \neq 1$. Suppose $(f(x), g(x)) = d(x)$, $g(x) = d(x)g_1(x)$, $f(x) = d(x)f_1(x)$, then $(f_1(x), g_1(x)) = 1$. Furthermore, $g(x)$ doesn't have repeated root, thus, $(d(x), g_1(x)) = 1$, $(d(x)\mathrm{f}_1(x), g_1(x)) = 1$, $(d(x)d(x)\mathrm{f}_1(x), g_1(x)) = 1$. So, $(d(x)\mathrm{f}(x), g_1(x)) = 1$.

Hence, there exist $u(x)$, $v(x)$ such that

$$u(x)d(x)f(x) + v(x)g_1(x) = 1. \tag{3}$$

Equation (3) both sides multiplied by $f(x)$, then

$$f(x)u(x)d(x)f(x) + f_1(x)v(x)g(x) = f(x).$$

For $g(D) = 0$, $f(D) = A$, we have

$$Au(D)d(D)A = A. \tag{4}$$

Equation (3) both sides multiplied by $u(x)d(x)$. Similarly, we get

$$u(D)d(D)Au(D)d(D) = u(D)d(D). \tag{5}$$

If $H = u(D)d(D)$, then *H* is the polynomial of *D*, from **Theorem 3.1**, we get *H* is an *FLDcirc*$_r$ matrix, and from Equation (4), Equation (5) we have $H = A^{\{1,2\}}$.

Due to

$$A^k HA = A^{k-1}(AHA) = A^{k-1}A = A^k,$$

$$AH = Au(D)d(D) = f(D)u(D)d(D) = u(D)d(D)f(D) = HA,$$

Hence $H = A^{\{d\}}$.

From **Lemma 2.7** and the proof of **Theorem 5.3**, **Corollary 5.5**, we can get the fast algorithm of the inverse and generalized inverse of the *FLDcirc_r* matrix. The general steps are as follows:

Step 1 get the greatest common factor $d(x)$ of $f(x)$, $g(x)$;

Step 2 If $d(x)=1$, the polynomial matrix $\begin{pmatrix} f(x) & 1 \\ g(x) & 0 \end{pmatrix}$ can transformed into $\begin{pmatrix} 1 & u(x) \\ 0 & s(x) \end{pmatrix}$ after elementary row transformation, then $A^{-1}=u(D)$;

Step 3 If $d(x)\neq 1$, divide $g(x)$ by $d(x)$, get $g_1(x)$, then the polynomial matrix $\begin{pmatrix} f(x)d(x) & 1 \\ g_1(x) & 0 \end{pmatrix}$ can transformed into $\begin{pmatrix} 1 & u(x) \\ 0 & s(x) \end{pmatrix}$ after elementary row transformation, hence $A^{\{1,2\}} = A^{\{d\}} = u(D)d(D)$.

Example 5.1 If the 3 order matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \\ -2 & 4 & -1 \end{pmatrix}$, then whether *A* is a nonsingular matrix? If *A* is nonsingular, solving A^{-1}.

From **Definition 2.1** we get $A = FLDcirc_2(1,-1,1)$, $f(x) = 1-x+x^2$, $g(x) = x^3+2x-2$, $D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$. Because of $(f(x), g(x)) = 1$, so *A* is nonsingular.

After a series of elementary row transformation of the following polynomial matrix, we obtain

$$\begin{pmatrix} x^2-x+1 & 1 \\ x^3+2x-2 & 0 \end{pmatrix} \to \begin{pmatrix} x^2-x+1 & 1 \\ x^2+x-2 & -x \end{pmatrix} \to \begin{pmatrix} -2x+3 & x+1 \\ x^2+x-2 & -x \end{pmatrix}$$

$$\to \begin{pmatrix} -2x+3 & x+1 \\ 7 & 2x^2+3x+5 \end{pmatrix} \to \begin{pmatrix} 0 & x^3+2x-2 \\ 1 & \frac{2}{7}x^2+\frac{3}{7}x+\frac{5}{7} \end{pmatrix}.$$

So

$$u(x) = \frac{2}{7}x^2 + \frac{3}{7}x + \frac{5}{7}.$$

Therefore

$$A^{-1} = u(D) = \frac{2}{7}D^2 + \frac{3}{7}D + \frac{5}{7}E = \begin{pmatrix} \frac{5}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{4}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix},$$

That is

$$A^{-1} = FLDcirc_2\left(\frac{5}{7}, \frac{3}{7}, \frac{2}{7}\right).$$

Example 5.2 If the 3 order matrix $A=\begin{pmatrix}1+\sqrt{3} & 1\\ 2 & \sqrt{3}-1\end{pmatrix}$, solving $A^{\{1,2\}}$.

From **Definition 2.1** we have $A=FLDcirc_2\left(1+\sqrt{3},1\right)$, $f\left(x\right)=x+1+\sqrt{3}$, $g\left(x\right)=x^2+2x-2$, $D=\begin{pmatrix}0 & 1\\ 2 & -2\end{pmatrix}$.

Then $d\left(x\right)=\left(f\left(x\right),g\left(x\right)\right)=x+1+\sqrt{3}$, so, A is singular and $g_1\left(x\right)=x+1-\sqrt{3}$.
From Step 3, we get

$$\begin{pmatrix}x^2+2\left(1+\sqrt{3}\right)x+4+2\sqrt{3} & 1\\ x+1-\sqrt{3} & 0\end{pmatrix}\to\begin{pmatrix}\left(1+3\sqrt{3}\right)x+4+2\sqrt{3} & 1\\ x+1-\sqrt{3} & 0\end{pmatrix}$$

$$\to\begin{pmatrix}1 & \frac{1}{12}\\ x+1-\sqrt{3} & 0\end{pmatrix}\to\begin{pmatrix}1 & \frac{1}{12}\\ 0 & x+1-\sqrt{3}\end{pmatrix}.$$

Then

$$u\left(x\right)=\frac{1}{12},$$

So

$$A^{\{1,2\}}=A^{\{d\}}=u\left(D\right)d\left(D\right)=\frac{1}{12}\left[D+\left(1+\sqrt{3}\right)E\right]=\begin{pmatrix}\frac{1+\sqrt{3}}{12} & \frac{1}{12}\\ \frac{1}{6} & \frac{\sqrt{3}-1}{12}\end{pmatrix},$$

That is

$$A^{\{1,2\}}=A^{\{d\}}=FLDcirc_2\left(\frac{1+\sqrt{3}}{12},\frac{1}{12}\right).$$

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