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H-Singular Value of a Positive Tensor

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Abstract

In this paper we study properties of *H*-singular values of a positive tensor $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ and present an iterative algorithm for computing the largest *H*-singular value of the positive tensor. We prove that this method converges for any positive tensors.

Keywords

Singular Value, Positive Tensor, Convergence

1. Introduction

Recently, eigenvalue problems for tensors have gained special attention in the realm of numerical multilinear algebra [1]-[4], and they have a wide range of practical applications [5] [6]. The definition of eigenvalues of square tensors has been introduced in [7]-[9]. Nice properties such as the Perron-Frobenius theorem for eigenvalues of nonnegative square tensors [7] have been discussed. The authors give algorithms to compute the largest eigenvalue of a nonnegative square tensor in [6] [10]. Singular values of rectangular tensors have been introduced in [11]. In [11] [12], properties of singular values of rectangular tensors have been discussed. In particular, Chang, Qi and Zhou [11] established the Perron-Frobenius theorem to singular values of nonnegative rectangular tensors. They also proposed an iterative algorithm to find the largest singular value of a nonnegative rectangular tensor. In [13], the authors studied the convergence of the proposed algorithm.

In this paper, we focus on the tensor $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$, and study properties of *H*-singular values of a positive tensor $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$. For more about the definition of the *H*-singular value of a tensor $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$, one can turn to the paper [14].

The paper is organized as follows. In Section 2, we recall some definitions and define *H*-singular values for a positive tensor, we extend the Perron-Frobenius theorem to *H*-singular values of positive tensors. In Section 3, we give an algorithm to find the largest singular value of a positive tensor, some numerical experiments are given to show that our algorithm is efficient.

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2. *H*-Singular Values for a Tensor

Let $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$. In this paper, we extend the definition of the classical concept of rectangular tensors, the tensors are no need square or rectangular. Consider the optimization problem

$$\max \sum_{i_1}^{n_1} \cdots \sum_{i_m}^{n_m} a_{i_1 \cdots i_m} (x_1)_{i_1} \cdots (x_m)_{i_m}, \quad x_j \in R^{n_j}, \tag{1}$$

under the constraints that

$$\sum_{i=1}^{n_1} (x_1)_i^m = \cdots = \sum_{i=1}^{n_m} (x_m)_i^m = 1.$$

We obtain the following system at a critical point:

$$\begin{cases} \mathcal{A} x_2 \cdots x_m = \lambda x_1^{[m-1]}, \\ \quad \vdots \\ \mathcal{A} x_1 \cdots x_{m-1} = \lambda x_m^{[m-1]}, \end{cases} \tag{2}$$

where

$$\begin{cases} (\mathcal{A} x_2 \cdots x_m)_i = \sum_{i_2=1}^{n_2} \cdots \sum_{i_m}^{n_m} a_{i i_2 \cdots i_m} (x_2)_{i_2} \cdots (x_m)_{i_m}, \\ \quad \vdots \\ (\mathcal{A} x_1 \cdots x_{m-1})_i = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{m-1}}^{n_{m-1}} a_{i_1 \cdots i_{m-1} i} (x_1)_{i_1} \cdots (x_{m-1})_{i_{m-1}}. \end{cases}$$

If $\lambda \in C$, $x_j \in C^{n_j} \setminus \{0\}$ are solutions of (2), then we say that λ is an *H*-singular value of the tensor $\mathcal{A}$, $x_1, \cdots, x_m$ are eigenvectors of $\mathcal{A}$, associated with the *H*-singular value λ.

Let

$$P_k = \{ x \in R^k : x_i \geq 0, i = 1, 2, \cdots, k \},$$

$$\operatorname{int}(P_k) = \{ x \in R^k : x_i \geq 0, i = 1, 2, \cdots, k \}.$$

A vector $x \in R^k$ is called nonnegative if $x \in P_k$ and it is called strongly positive if $x \in \operatorname{int}(P_k)$. For any $j = 1, 2, \cdots, m$, let $x_j \in P_{n_j} \setminus \{0\}$ be a nonnegative vector. We give our main theorems as follows.

Lemma 1. *If a tensor* $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ *is positive, then for any* $x_j \in P_{n_j} \setminus \{0\}$, $j = 1, 2, \cdots, m$,

$$\begin{cases} \mathcal{A} x_2 \cdots x_m > 0, \\ \quad \vdots \\ \mathcal{A} x_1 \cdots x_{m-1} > 0. \end{cases} \tag{3}$$

Proof. If $x_j \in \operatorname{int}(P_{n_j})$, $j = 1, 2, \cdots, m$, suppose $\mathcal{A} x_2 \cdots x_m = 0$, and then $a_{i_1 \cdots i_m} = 0$, a contradiction. If $x_j \notin \operatorname{int}(P_{n_j})$, and $x_j \neq 0$, $j = 1, 2, \cdots, m$, there exists $(x_j)_{i_1^0} \neq 0$, and we can get

$$a_{i_1 \cdots i_m} (x_2)_{i_2^0} \cdots (x_m)_{i_m^0} > 0.$$

Then

$$\mathcal{A} x_2 \cdots x_m > 0.$$

Similarly, we can get $\mathcal{A} x_1 \cdots x_{m-1} > 0$. □

Lemma 2. *Let a tensor* $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ *be positive, and let* $(\lambda, x_1, \cdots, x_m) \in R_+ \times \operatorname{int}(P_{n_1}) \times \cdots \times \operatorname{int}(P_{n_m})$ *be a solution of* (2). *If* $(\sigma, y_1, \cdots, y_m) \in R_+ \times (P_{n_1} \setminus \{0\}) \times \cdots \times (P_{n_m} \setminus \{0\})$ *satisfies*

$$\begin{cases} \mathcal{A} y_2 \cdots y_m \geq \sigma y_1^{[m-1]}, \\ \quad \vdots \\ \mathcal{A} y_1 \cdots y_{m-1} \geq \sigma y_m^{[m-1]}, \end{cases} \tag{4}$$

Then $\sigma \le \lambda$.

Proof. Define $t_0 = \max\left\{s \ge 0 \middle| x_j - sy_j \in P_{n_j}\right\}$. Since $x_j \in \operatorname{int} P_{n_j}$, $t_0 > 0$. We have

$$x_j - ty_j \ge 0,$$

if and only if $t \in [0, t_0]$. Thus

$$\begin{cases} \lambda x_1^{m-1} = \mathcal{A}x_2 \cdots x_m \ge t_0^{m-1}\mathcal{A}y_2 \cdots y_m \ge t_0^{m-1}\sigma y_1^{m-1}, \\ \qquad \vdots \\ \lambda x_m^{m-1} = \mathcal{A}x_1 \cdots x_{m-1} \ge t_0^{m-1}\mathcal{A}y_1 \cdots y_{m-1} \ge t_0^{m-1}\sigma y_m^{m-1}, \end{cases}$$

i.e.,

$$\begin{cases} x_1 \ge t_0 \left(\dfrac{\sigma}{\lambda}\right)^{\frac{1}{m-1}} y_1, \\ \qquad \vdots \\ x_m \ge t_0 \left(\dfrac{\sigma}{\lambda}\right)^{\frac{1}{m-1}} y_m. \end{cases}$$

This implies $\sigma \le \lambda$. □

Remark. If there exists $(\sigma, y_1, \cdots, y_m) \in R_+ \times \operatorname{int}(P_{n_1}) \times \cdots \times \operatorname{int}(P_{n_m})$ such that

$$\begin{cases} \mathcal{A}y_2 \cdots y_m \ge \sigma y_1^{[m-1]}, \\ \qquad \vdots \\ \mathcal{A}y_1 \cdots y_{m-1} \ge \sigma y_m^{[m-1]}, \end{cases} \tag{5}$$

Then σ is the eigenvalue of $\mathcal{A}$ and y_j is the corresponding eigenvectors of σ, $j = 1, 2, \cdots, m$. This remark can be obtained by similar process in [12] [15].

Theorem 1. *Assume that a tensor* $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ *is positive, then there exists a solution* $(\lambda_0, x_1^0, \cdots, x_m^0)$ *of system* (1), *satisfying* $\lambda_0 > 0$ *and* $x_j^0 \in \operatorname{int}(P_{n_j})$, *Moreover, if* λ *is a singular value with strongly positive eigenvectors* x_j, $j = 1, 2, \cdots, m$, *then* $\lambda = \lambda_0$, *The strongly positive eigenvectors are unique up to a multiplicative constant,*

Proof. Denote $D_k = \left\{z = (z_1, \cdots, z_k) \in P_k \middle| \sum_{i=1}^{k} z_i = 1\right\}$. Provide by **Lemma 1**, the map F on $D_{n_1} \times \cdots \times D_{n_m}$ into itself:

$$F(\xi_1, \cdots, \xi_m) = \left(\frac{(\mathcal{A}\xi_2 \cdots \xi_m)_i^{\frac{1}{m-1}}}{\sum_{i=1}^{n_1} (\mathcal{A}\xi_2 \cdots \xi_m)_i^{\frac{1}{m-1}}}, \cdots, \frac{(\mathcal{A}\xi_1 \cdots \xi_{m-1})_i^{\frac{1}{m-1}}}{\sum_{i=1}^{n_m} (\mathcal{A}\xi_1 \cdots \xi_{m-1})_i^{\frac{1}{m-1}}} \right).$$

is well defined.

According to the Brouwer Fixed Point Theorem, there exists $(\xi_1^0, \cdots, \xi_m^0) \in D_{n_1} \times \cdots \times D_{n_m}$ such that

$$\begin{cases} \mathcal{A}\xi_2^0 \cdots \xi_m^0 = \sigma_1^0 \xi_1^{[m-1]}, \\ \qquad \vdots \\ \mathcal{A}\xi_1^0 \cdots \xi_{m-1}^0 = \sigma_m^0 \xi_m^{[m-1]}, \end{cases} \tag{6}$$

where

$$\begin{cases} \sigma_1^0 = \left(\displaystyle\sum_{i=1}^{n_1} (\mathcal{A}\xi_2^0 \cdots \xi_m^0)_i^{\frac{1}{m-1}} \right)^{m-1}, \\ \qquad \vdots \\ \sigma_m^0 = \left(\displaystyle\sum_{i=1}^{n_m} (\mathcal{A}\xi_1^0 \cdots \xi_{m-1}^0)_i^{\frac{1}{m-1}} \right)^{m-1}. \end{cases}$$

Let

$$x_1^0 = t_1\xi_1^0, \cdots, x_m^0 = t_m\xi_m^0, \text{ and } \lambda_0 = \left(\sigma_1^0 \cdots \sigma_m^0\right)^{\frac{1}{m}}.$$

Then $\left(\lambda_0, \left(x_1^0, \cdots, x_m^0\right)\right)$ is a solution of (2).

Let us show: $\left(x_1^0, \cdots x_m^0\right) \in \operatorname{int}\left(P_{n_1}\right) \times \cdots \times \operatorname{int}\left(P_{n_m}\right)$. If not, suppose $\left(x_j^0\right)_i = 0$, that is to say,

$$\left(\mathcal{A}x_1^0 \cdots x_{j-1}^0 x_{j+1}^0 \cdots x_m^0\right)_i = 0,$$

this contradicts the result of **Lemma 1**. Therefore,

$$\left(x_1^0, \cdots, x_m^0\right) \in \operatorname{int}\left(P_{n_1}\right) \times \cdots \times \operatorname{int}\left(P_{n_m}\right).$$

The uniqueness of the positive singular value with strongly positive left and right eigenvectors now follows from **Lemma 2** directly. The uniqueness up to a multiplicative constant of the strongly positive left and right eigenvectors is proved in the same way as in [7]. □

Theorem 2. *Assume that* $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ *is a positive tensor, then*

$$\min_{x_j \in P_n \setminus \{0\}} \max_i \left(\frac{\left(\mathcal{A}x_2 \cdots x_m\right)_i}{\left(x_1\right)_i^{m-1}}, \cdots, \frac{\left(\mathcal{A}x_1 \cdots x_{m-1}\right)_i}{\left(x_m\right)_i^{m-1}} \right) = \lambda_0 = \max_{x_j \in P_n \setminus \{0\}} \min_i \left(\frac{\left(\mathcal{A}x_2 \cdots x_m\right)_i}{\left(x_1\right)_i^{m-1}}, \cdots, \frac{\left(\mathcal{A}x_1 \cdots x_{m-1}\right)_i}{\left(x_m\right)_i^{m-1}} \right),$$

where λ_0 is the unique positive singular value corresponding to strongly positive eigenvectors.

Proof. Let $x_j \in P_{n_j} \setminus \{0\}$, $j = 1, 2, \cdots, m$. We define

$$u_*\left(x_1, \cdots, x_m\right) = \min_i \left(\frac{\left(\mathcal{A}x_2 \cdots x_m\right)_i}{\left(x_1\right)_i^{m-1}}, \cdots, \frac{\left(\mathcal{A}x_1 \cdots x_{m-1}\right)_i}{\left(x_m\right)_i^{m-1}} \right).$$

Since it is a positively 0-homogeneous function, it can be restricted on $D_{n_1} \times \cdots \times D_{n_m}$. Let

$$r_* = u_*\left(x_1^*, \cdots, x_m^*\right) = \max_{x_i \in D_{n_i}} u_*\left(x_1, \cdots, x_m\right).$$

Let $\left(\lambda_0, \left(x_1^0, \cdots, x_m^0\right)\right)$ is a solution of (2). On one hand, we have

$$\lambda_0 = u_*\left(x_1^0, \cdots, x_m^0\right) \le u_*\left(x_1^*, \cdots, x_m^*\right) = r_*.$$

On the other hand, by the definition of $u_*\left(x_1, \cdots, x_m\right)$, we get

$$r_* = u_*\left(x_1^*, \cdots, x_m^*\right) = \min_i \left(\frac{\left(\mathcal{A}x_2^* \cdots x_m^*\right)_i}{\left(x_1^*\right)_i^{m-1}}, \cdots, \frac{\left(\mathcal{A}x_1^* \cdots x_{m-1}^*\right)_i}{\left(x_m^*\right)_i^{m-1}} \right).$$

This means

$$\begin{cases} \mathcal{A}x_2^* \cdots x_m^* \ge r_* x_1^{*[m-1]}, \\ \quad\vdots \\ \mathcal{A}x_1^* \cdots x_{m-1}^* \ge r_* x_m^{*[m-1]}. \end{cases} \tag{7}$$

According to **Lemma 2**, we have $r_* \le \lambda_0$, and the we get

$$\lambda_0 = r_*.$$

Similarly, we prove the other equality. □

Theorem 3. *Assume that* $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ *is a positive tensor, and* λ_0 *is the positive singular value with strongly positive eigenvectors. Then* $|\lambda| \le \lambda_0$ *for all singular values* λ *of* $\mathcal{A}$.

Proof. Let $x_j \in C_{n_j} \setminus \{0\}$ for some $\lambda \in C$, $j = 1, 2, \cdots, m$. We wish to show $|\lambda| \le \lambda_0$. Let $y_i = |x_i|$. We get

$$|\lambda|\left(y_j\right)_i^{m-1} = |\lambda|\left|x_j\right|_i^{m-1} = \left|\left(\mathcal{A}x_1 \cdots x_{j-1}x_j \cdots x_m\right)_i\right| \le \left(\mathcal{A}y_1 \cdots y_{j-1}y_j \cdots y_m\right)_i.$$

Apply **Theorem 2**, we can get

$$|\lambda| \le \max_{y_j \in P_n \setminus \{0\}} \min_i \left(\frac{(\mathcal{A}y_2 \cdots y_m)_i}{(y_1)_i^{m-1}}, \cdots, \frac{(\mathcal{A}y_1 \cdots y_{m-1})_i}{(y_m)_i^{m-1}} \right) = \lambda_0. \qquad \square$$

Theorem 4. *Suppose that* $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$ *is a positive tensor satisfying*

$$\sum_{i_2}^{n_2} \cdots \sum_{i_m}^{n_m} a_{i_1 \cdots i_m} = \cdots = \sum_{i_1}^{n_1} \cdots \sum_{i_{m-1}}^{n_{m-1}} a_{i_1 \cdots i_m} = C,$$

where $C \ge 0$ is a constant. Then $\lambda_0(\mathcal{A}) = C$.

Proof. Let $\left(\lambda_0, \left(x_1^0, \cdots, x_m^0\right)\right)$ is a solution of (2). Without loss of generality, we suppose that $\left(x_1^0\right)_1 = \max_i \left(x_j^0\right)_i$, $j = 1, \cdots, m$. Then

$$\begin{aligned} \lambda_0 &= \frac{\left(\mathcal{A}x_2^0 \cdots x_m^0\right)_1}{\left(x_1^0\right)_1^{m-1}} = \frac{\left(\sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m} (x_2^0)_{i_2} \cdots (x_m^0)_{i_m}\right)_1}{\left(x_1^0\right)_1^{m-1}} \\ &\le \frac{\left(\sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m} \left(x_1^0\right)_1 \cdots \left(x_1^0\right)_1\right)_1}{\left(x_1^0\right)_1^{m-1}} \\ &= \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m} = C. \end{aligned}$$

On the other hand, it is easy to check that C is an eigenvalue of A with corresponding eigenvectors $\left(x_j\right)_i = 1$, $j = 1, \cdots, m$. So $C \le \lambda_0$. Thus we have $\lambda_0 = C$. $\square$

3. An Iterative Algorithm

In this section, we propose an iterative algorithm to calculate the largest H-singular value of a positive tensor based on **Theorem 2** and **Theorem 3**. This algorithm is a modified version of the one given in [11] [13], and we will show the convergence of the proposed algorithm for any positive tensor. In this section, we always suppose that $\mathcal{A}$ is a positive tensor.

For a positive tensor $\mathcal{A} \in R^{n_1 \times \cdots \times n_m}$, $\rho > 0$, $x_j \in P_{n_j}$, let

$$\begin{cases} \eta_1(x_1, \cdots, x_m) = \mathcal{A}x_2 \cdots x_m + \rho x_1^{[m-1]}, \\ \qquad \vdots \\ \eta_m(x_1, \cdots, x_m) = \mathcal{A}x_1 \cdots x_{m-1} + \rho x_m^{[m-1]}. \end{cases} \tag{8}$$

Algorithm 3.1

Step 0 Choose $x_1^{(1)} > 0, \cdots, x_m^{(1)} > 0$. Set $k = 1$;

Step 1 Compute

$$\begin{cases} \eta_1^{(k)} = \eta_1\left(x_1^k, \cdots, x_m^k\right), \\ \qquad \vdots \\ \eta_m^{(k)} = \eta_m\left(x_1^k, \cdots, x_m^k\right). \end{cases} \tag{9}$$

Let

$$\underline{u}_k = \min_{\left(x_j^{(k)}\right)_i > 0} \left\{ \frac{\left(\eta_1^{(k)}\right)_i}{\left(x_1^{(k)}\right)_i^{m-1}}, \cdots, \frac{\left(\eta_m^{(k)}\right)_i}{\left(x_m^{(k)}\right)_i^{m-1}} \right\}, \tag{10}$$

$$\overline{u}_k = \max_{\left(x_j^{(k)}\right)_i > 0} \left\{ \frac{\left(\eta_1^{(k)}\right)_i}{\left(x_1^{(k)}\right)_i^{m-1}}, \cdots, \frac{\left(\eta_m^{(k)}\right)_i}{\left(x_m^{(k)}\right)_i^{m-1}} \right\}; \tag{11}$$

Step 2 If $\underline{u}_k = \overline{u}_k$, then stop. Otherwise, compute

$$\begin{cases} x_1^{(k+1)} = \dfrac{\left(\eta_1^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|}, \\ \vdots \\ x_m^{(k+1)} = \dfrac{\left(\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|}, \end{cases} \tag{12}$$

and replace k by $k+1$ and go to Step 1.

In the following, we will give a convergence result for **Algorithm 3.1**.

Theorem 5. *Assume that* $\left(u_0, \left(x_1^0, \cdots, x_m^0\right)\right)$ *is a solution of* (2). *Then,*

$$0 \le \underline{u}_1 \le \underline{u}_2 \le \cdots \le u_0 \le \cdots \le \overline{u}_2 \le \overline{u}_1.$$

Proof. By (8), $0 < \underline{u}_1$. From **Theorem 2**, for $k = 1, 2, \cdots$,

$$\underline{u}_k \le u_0 \le \overline{u}_k.$$

We now prove for any $k \ge 1$,

$$\underline{u}_k \le \underline{u}_{k+1}, \quad \overline{u}_k \le \overline{u}_{k+1}.$$

For each $k = 1, 2, \cdots$, by the definition of $\underline{u}_k$ and **Lemma 1**, we have

$$\eta_1^{(k)} \ge \underline{u}_k \left(x_1^{(k)}\right)^{[m-1]} > 0, \cdots, \eta_m^{(k)} \ge \underline{u}_k \left(x_m^{(k)}\right)^{[m-1]} > 0.$$

Then,

$$\left(\eta_1^{(k)}\right)^{\left[\frac{1}{m-1}\right]} \ge \left(\underline{u}_k\right)^{\frac{1}{m-1}} x_1^{(k)} > 0, \cdots, \left(\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]} \ge \left(\underline{u}_k\right)^{\frac{1}{m-1}} x_m^{(k)} > 0.$$

So,

$$x_1^{(k+1)} = \frac{\left(\eta_1^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|} \ge \frac{\left(\underline{u}_k\right)^{\frac{1}{m-1}} x_1^{(k)}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|} > 0,$$

$$\vdots$$

$$x_m^{(k+1)} = \frac{\left(\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|} \ge \frac{\left(\underline{u}_k\right)^{\frac{1}{m-1}} x_m^{(k)}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|} > 0.$$

Hence, we get

$$\eta_1^{(k+1)} \ge \frac{\underline{u}_k \eta_1^{(k)}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\frac{1}{m-1}}\right\|^{m-1}} = \underline{u}_k \left(x_1^{(k+1)}\right)^{m-1},$$

$$\vdots$$

$$\eta_m^{(k+1)} \ge \frac{\underline{u}_k \eta_1^{(k)}}{\left\|\left(\eta_1^{(k)},\cdots,\eta_m^{(k)}\right)^{\frac{1}{m-1}}\right\|^{m-1}} = \underline{u}_k \left(x_m^{(k+1)}\right)^{m-1},$$

which means for $j = 1, 2, \cdots, m$

$$\underline{u}_k \le \frac{\left(\eta_j^{(k+1)}\right)_i}{\left(x_j^{(k+1)}\right)_i^{m-1}}.$$

Therefore, we get

$$\underline{u}_k \le \underline{u}_{k+1}.$$

Similarly, we can prove that

$$\overline{u}_k \le \overline{u}_{k+1}.$$ □

From **Theorem 5**, $\{\underline{u}_k\}$ is a monotonic increasing sequence and it has an upper bound, so the limit exists. Since $\{\overline{u}_k\}$ is monotonic decreasing sequence and it has a lower bound, the limit exists as well. We suppose

$$\underline{u} = \lim_{k\to\infty} \underline{u}_k, \quad \overline{u} = \lim_{k\to\infty} \overline{u}_k.$$

By **Theorem 5**, we have

$$0 < \underline{u} \le u_0 \le \overline{u}. \tag{13}$$

The argument used in the following proof is parallel to that in [13]. We proceed the proof for completeness.

Theorem 6. *Let* $\{x_j^{(k)}\}$, $\{\eta_j^{(k)}\}$ $(j = 1, 2, \cdots, m)$ *be the sequences produced by* ***Algorithm*** **3.1**. *Then*

a) $\{x_j^{(k)}\}$ have convergent subsequences which converge to x_j^*, respectively. Moreover, $x_j^* \in P_{n_j} \setminus \{0\}$, $j = 1, 2, \cdots, m$.

b) $\eta_1\left(x_2^*, \cdots, x_m^*\right) \ge \underline{u}\left(x_1^*\right)^{m-1}, \cdots, \eta_m\left(x_1^*, \cdots, x_{m-1}^*\right) \ge \underline{u}\left(x_m^*\right)^{m-1}$.

c) $\underline{u} = u_0 = \overline{u}$.

Proof. As $\left\|\left(x_1^{(k)}, \cdots, x_m^{(k)}\right)\right\| = 1$ for all $k \ge 2$. Hence, there exists a convergent subsequence by the compactness of the unit ball in R^n and x_j^* must not be a zero vector.

By the continuity of η_j, (8) and (9), we get the result (b).

If $\underline{u} \ne \overline{u}$, we get that someone of the follow inequations exists:
$\eta_1\left(x_2^*, \cdots, x_m^*\right) \ne \underline{u}\left(x_1^*\right)^{m-1}, \cdots \eta_m\left(x_1^*, \cdots, x_{m-1}^*\right) \ne \underline{u}\left(x_m^*\right)^{m-1}$. By Theorem 2.5 in [13], there exists a positive integer s such that

$$\underline{u}\eta_j^s\left(x_1^*, \cdots, x_m^*\right) < \eta_j^s\left(\left(\eta_1^*\right)^{\frac{1}{m-1}}, \cdots, \left(\eta_m^*\right)^{\frac{1}{m-1}}\right).$$

By (a) and the continuity of η_j, for any sufficiently large k_j, we obtain

$$\underline{u}\eta_j^s\left(x_1^{k_j}, \cdots, x_m^{k_j}\right) < \eta_j^s\left(\left(\eta_1^{k_j}\right)^{\frac{1}{m-1}}, \cdots, \left(\eta_m^{k_j}\right)^{\frac{1}{m-1}}\right).$$

Then we obtain $\underline{u}_{k_j+s} > \underline{u}$, which contradicts with **Theorem 5**. So (c) holds. □

By **Theorem 6**, we can get the largest *H*-singular value of $\mathcal{A}$ is

$$\lambda_0 = u_0 - \rho.$$

In the following, in order to show the viability of **Algorithm 3.1**, we used Matlab 7.1 to test it with some randomly generated rectangular tensors. For these randomly generated tensors, the value of each entry is between 0 and 10. we set $x_1^{(1)} = [1,1,\cdots,1]^{\mathrm{T}} \in R^{n_1}, \cdots, x_m^{(1)} = [1,1,\cdots,1]^{\mathrm{T}} \in R^{n_m}$. We terminated our iteration when $\overline{\lambda}_k - \underline{\lambda}_k \le 10^{-6}$.

Our numerical results are shown in **Table 1**. In this table, Ite denotes the number of iterations, $\overline{\lambda} - \underline{\lambda}$ and λ denote the values of $\overline{\lambda}_k - \underline{\lambda}_k$ and $0.5\left(\overline{\lambda}_k + \underline{\lambda}_k\right)$ at the final iteration, respectively. NormX_j $(j = 1, 2, \cdots, m)$ denote the values of at the final iteration, respectively. The results in **Table 1** show that the proposed algorithm is promising. The algorithm is able to produce the largest singular values for all these randomly generated posi-

Table 1. Numerical results of Algorithm 3.1 for randomly generated tensors.

(n_1, n_2, n_3)	Ite	$\bar{\lambda} - \underline{\lambda}$	λ	NormX_1	NormX_2	NormX_3
$2\times3\times3$	26	8.95e−007	36.78	2.42e−008	1.96e−008	1.87e−008
$2\times3\times4$	27	7.68e−007	41.08	1.18e−008	8.30e−009	8.86e−009
$2\times3\times5$	28	6.10e−007	46.39	2.82e−009	2.44e−009	1.87e−009
$2\times3\times10$	29	9.24e−007	77.87	2.16e−009	1.71e−009	8.89e−010
$3\times6\times10$	30	7.27e−007	165.51	6.59e−009	4.04e−009	3.57e−009

tive tensors.

4. Conclusion

In this paper, we give some eigenvalues properties about the *H*-singular value of a positive tensor $\mathcal{A} \in R^{n_1\times\cdots\times n_m}$ introduced in [6]. We find that the Perron-Frobenius like theorem for nonnegative square tensors can not be extended to the nonnegative tensor $\mathcal{A} \in R^{n_1\times\cdots\times n_m}$, so here we limit the tensor $\mathcal{A} \in R^{n_1\times\cdots\times n_m}$ to the positive case. An algorithm is given to compute the largest *H*-singular value of the positive tensor.

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