

# Estimated Bounds for Zeros of Polynomials from Traces of Graeffe Matrices

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## Abstract

**In this paper, we combine Graeffe matrices with the classical numerical method of Dandelin-Graeffe to estimate bounds for the moduli of the zeros of polynomials. Furthermore, we give some examples showing significant gain for the convergence towards the polynomials dominant zeros moduli.**

## Keywords

**Graeffe Matrices, Numerical Approximation of Eigenvalues, Dandelin-Graeffe's Method, Bounds of Zeros of Polynomial**

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## 1. Introduction

Deriving zero bounds for polynomials is a classical problem that has been proven essential in various disciplines such as controlling engineering problems, eigenvalue problems in mathematical physics, and digital audio signal processing problems—to name just a few [1]. Specially, a large number of research papers have given bounds of their moduli, e.g., well-known and classical bounds named after Cauchy, Montel and Kojima [2] [3]; more advanced results can be found in [4]-[8] and references therein. Furthermore, bounds for the zeros of polynomials were needed in achieving some numerical computation methods, as providing information on the location of the zeros can be used to find initial guesses for iterative computation algorithms.

An estimated value of the largest moduli of the roots of a polynomial can be obtained as a limit of a sum of power of these roots using resultants [9]. However, it is well-known that computation of polynomials resultants is tedious and expensive; for this, we proceed by determining the trace of the Graeffe matrix of order  $m$  of the polynomial  $P$ , which is exactly the sum of the  $m$ -th powers of the eigenvalues of the companion matrix of  $P$ . So we get our main result relying on Dandelin-Graeffe method to estimate the largest moduli bounds of the roots of

$P$ . This seems to be a significant progress compared to the use of resultants; moreover, it reveals a substantial convergence gain, teaming up with an approach used in [7]. This finding is rather significant in the large available methods used to improve the location of the zeros of polynomials.

The rest of the paper is organized as follows. In Section 2, we introduce the Graeffe matrix of a complex polynomial  $P$  and derive its trace from the classical Jordan normal form. In Section 3, we extend the Dandelin-Graeffe’s method to find the maximum moduli of the zeros of  $P$ , relying on the limit of Graeffe matrix’s trace of  $P$ . Through some examples, Section 3 illustrates that a few iterations of Graeffe’s matrix trace can give very tight bounds for the absolute value of the zeros of polynomials, with comparatively fast convergence.

## 2. Graeffe Polynomials and Graeffe Matrix’s Trace

For an  $n$ -dimensional matrix  $A$ , we call the spectrum of  $A$  the set of all its eigenvalues, denoted by  $\sigma(A)$ . Its spectral radius  $\rho(A)$  is defined by  $\rho(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\}$  as the largest magnitude attained by any eigenvalue of  $A$ . Setting up the elements of  $\sigma(A)$  with their geometric and algebraic multiplicities, determines the eigenstructure of  $A$ . It’s well-known that all matrices similar to  $A$  have the same eigenstructure, specially we have the useful classical lemma [10]:

**Lemma 1.** *A square complex matrix  $A$  of order  $n$  is similar to a block diagonal matrix*

$$J = \begin{bmatrix} J_1 & & (0) \\ & \ddots & \\ (0) & & J_p \end{bmatrix}$$

for  $i = 1, \dots, p$ , each block  $J_i$  is a square matrix of order  $n_i$  of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ (0) & & & \lambda_i \end{bmatrix} \text{ where } \sum_{i=1}^p n_i = n.$$

For  $i = 1, \dots, p$ ,  $J_i$  is called a “Jordan block” of  $A$  and  $J$  the Jordan normal form of  $A$ .

**Remark:** The spectrum of a matrix  $A$  is identical with the spectrum of its Jordan normal form.

**Lemma 2.** *If  $J_\alpha$  is a Jordan block of order  $l_\alpha$  of  $A$  with eigenvalue  $\lambda_\alpha$  and  $m$  a positive integer, we have*

$$(J_\alpha^m)_{ij} = \begin{cases} \binom{m}{j-i} \times \lambda_\alpha^{m-j+i} & \text{when } j \geq i \\ 0 & \text{otherwise} \end{cases}; i, j = 1, \dots, l_\alpha,$$

$$\text{where } \binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!} & \text{when } k = 0, 1, \dots, l_{\alpha-1}. \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* For arbitrary indices  $i$  and  $j$ , it is direct to verify the claim by using a proof by induction.

From the previous result, we deduce the two following corollaries:

**Corollary 1.** *The  $m$ -th power of an  $n$ -dimensional matrix  $A$  is similar to the  $m$ -th power of its Jordan normal form as a direct sum of upper triangular matrices. Besides, each triangular block will consist of  $m$ -th power of its associate eigenvalue on the main diagonal.*

**Corollary 2.** *The trace of the  $m$ -th power of  $A$  is the sum of the  $m$ -th powers of the eigenvalues of  $A$ .*

**Definition 1.** For  $P(X) = X^d + a_1X^{d-1} + \dots + a_{d-1}X + a_d, (a_d \neq 0)$  a polynomial of degree  $d$  with complex coefficients, we will use the Frobenius companion matrix of  $P$  defined as

$$C(P) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_d & -a_{d-1} & -a_{d-2} & \dots & -a_2 & -a_1 \end{pmatrix}.$$

Let  $k$  be a positive integer and  $m$ , then the Graeffe matrix of  $P$  of order  $m$  is defined as the  $m$ -th power of  $C(P)$ , and denoted by  $G_m(P) := C(P)^m$ .

**Definition 2.** The characteristic polynomial of  $G_m(P)$  is called the Graeffe polynomial of  $P$  of order  $m$ , denoted by  $\mathcal{G}_m(P)$ .

**Remarks:**

1) It can be computed also using resultants, as quoted in the following lemma ([9], Thèreme 3.1): assuming that  $P$  be a polynomial of degree  $d$  and  $m \geq 2$ , then  $\mathcal{G}_m(P)(Y) = Res_X(P(X), Y - X^m)$ .

2) Relying on Proposition 1, it follows that for all  $k \in \mathbb{N}$ , there exists  $N_k = o(\log k)$  products of square-matrices of order  $d$ ; moreover, for  $k + d$  consecutive index, this number is  $N_{k+d}$ . With comparatively using resultants, such computing time gain is shown through the following example (Table 1) done via the computer algebra system Maple (see Maple code in Appendix).

**Table 1.** Computing time for  $\mathcal{G}_m(P)$ ,  $P(X) = X^5 + X^4 + 3X^3 + 5X^2 - 4X + 4$  and  $2^{13} \leq m \leq 2^{27}$ .

m	$2^{13}$	$2^{15}$	$2^{20}$	$2^{25}$	$2^{27}$
Graeffegen	0.234 s	1.216 s	262.284 s	3971.235 s	4854.183 s
GraeffeMatrix	0.093 s	0.171 s	0.592 s	28.126 s	137.421 s

**Proposition 1.** Let  $P$  be a polynomial of degree  $d$  with complex coefficients, then the trace of  $G_m(P)$  is exactly the sum of the  $m$ -th power of the zeros of  $P$ .

*Proof.* It is well-known that the eigenvalues are the zeros of the polynomial  $P$ ; therefore, the result follows immediately from Corollary 2.

### 3. Estimation Method of Dandelin-Graeffe

**Proposition 2.** Let  $P$  be a polynomial of degree  $d$  with complex coefficients and  $\lambda_j, 1 \leq j \leq d$  as its zeros, such that

$$P(X) = X^d + a_1X^{d-1} + \dots + a_{d-1}X + a_d = \prod_{j=1}^d (X - \lambda_j), \quad (a_d \neq 0)$$

There exists a sequence of polynomials  $P_k$  such that

$$P_k(X) = \pm \prod_{j=1}^d (X - \lambda_j^{2^k}), \quad k > 0,$$

the zeros of  $P_k$  are the square of the zeros of  $P_{k-1}$  and the  $2^k$ -power of those of  $P$ .

*Proof.* The basic technique to write down the proclaimed sequence of polynomials is done through an iteration referred to as the Dandelin-Graeffe method (see [6]).

Starting with  $P_0(X) = P(X) = F_0(X^2) - XG_0(X^2)$  and  $P_1(X) = P_0(\sqrt{X})P_0(-\sqrt{X})$ , we have

$$P_1(X) = F_0^2(X) - XG_0^2(X)$$

a polynomial of degree  $d$ , which zeros are the squares of the zeros of  $P_0$ .

For the induction process, one performs the following iteration

$$P_k(X^2) = P_{k-1}(\sqrt{X})P_{k-1}(-\sqrt{X}),$$

which transforms  $P_{k-1}(X) = F_{k-1}(X^2) - XG_{k-1}(X^2)$  as  $P_k = F_{k-1}^2(X) - XG_{k-1}^2(X)$  which zeros are the squares of the zeros of  $P_{k-1}(X)$ . As consequence, the zeros of  $P_k$  are the  $2^k$ -power of those of  $P$ .

**Lemma 3.**

Let  $x_m = \gamma_1^m + \gamma_2^m + \dots + \gamma_d^m$  where  $\gamma_1, \gamma_2, \dots, \gamma_d$  are complex numbers such that

$$|\gamma_1| > |\gamma_2| \geq \dots \geq |\gamma_d|, \text{ then } |\gamma_1| = \lim_{k \rightarrow +\infty} |x_{2^k}|^{2^{-k}}.$$

*Proof.* One can refer to [9] (Théorème 1.4).

As a special case of Lemma 3, we can consider sequences of traces of  $G_m(P)$ , the Graeffe matrix of  $P$  of order  $m = 2^k$ . By this way, we get a slight improvement of the bound for the absolute value of the unique largest zero of a polynomial; moreover, we can noticed a fast convergence of such sequence towards the expected value.

**Proposition 3.** Let  $P$  be a polynomial of degree  $d$  with complex coefficients such that

$$P(X) = X^d + a_1X^{d-1} + \dots + a_{d-1}X + a_d = \prod_{j=1}^d (X - \lambda_j), \quad (a_d \neq 0)$$

where  $\lambda_j, 1 \leq j \leq d$  are the zeros of  $P$  such that  $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_d|$ , then

$$\lim_{k \rightarrow +\infty} \left( \left| \text{Trace}(G_{2^k}(P)) \right| \right)^{2^{-k}} = \lim_{k \rightarrow +\infty} \left( \left| \sum_{j=1}^d \lambda_j^{2^k} \right| \right)^{2^{-k}} = |\lambda_1|$$

**Remark:** It is important to note that the existence of an unique dominant zero is essential to the validity of the previous result, as shown in ([9], p. 8). However, there is a more general result which doesn't rely on neither the existence of a single dominant absolute value of the polynomial's zero, nor the obligation to browse only terms whose indices are powers of 2. Such generalization, due to Mignotte and Ştefănescu [7] includes Lemma 3 as a special case; it can be stated as the following lemma:

**Lemma 4.** Let  $x_m = \gamma_1^m + \gamma_2^m + \dots + \gamma_d^m$  where  $\gamma_1, \gamma_2, \dots, \gamma_d \in \mathbb{C}$  such that  $|\gamma_1| \geq |\gamma_2| \geq \dots \geq |\gamma_d|$ , then

$$|\gamma_1| = \lim_{m \rightarrow +\infty} \max(|x_m|, |x_{m+1}|, \dots, |x_{m+d-1}|)^{\frac{1}{m}}$$

**Remark:** Bearing in mind that there is no need to check the unicity of the largest modulus of the zero of a given polynomial, a few iterations of  $d$  consecutive values of the traces  $x_m, x_{m+1}, \dots, x_{m+d-1}$  of the Graeffe matrices  $G_m(P), G_{m+1}(P), \dots, G_{m+d-1}(P)$ , for an initial value of the exponent  $m$  sufficiently large, yields a good approximation of the maximum of moduli of the zeros of  $P$ .

**Proposition 4.**

Let  $P$  be a polynomial of degree  $d$  with complex coefficients such that

$$P(X) = X^d + a_1X^{d-1} + \dots + a_{d-1}X + a_d = \prod_{j=1}^d (X - \lambda_j), \quad (a_d \neq 0)$$

where  $\lambda_j, 1 \leq j \leq d$  are the zeros of  $P$  such that  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$ , then

$$|\lambda_1| = \lim_{m \rightarrow +\infty} \max \left( \left| \text{Trace}(G_m(P)) \right|, \left| \text{Trace}(G_{m+1}(P)) \right|, \dots, \left| \text{Trace}(G_{m+d-1}(P)) \right| \right)^{\frac{1}{m}},$$

$$\text{where } \text{Trace}(G_m(P)) = \sum_{j=1}^d \lambda_j^m = \lambda_1^m + \lambda_2^m + \dots + \lambda_d^m$$

## 4. Numerical Results

1) We consider the polynomial

$$P_1(X) = X^5 - 10X^4 + 30X^3 - 20X^2 - 31X + 30$$

with an unique absolute value of dominant zero, namely 5. The application of the Graeffe's zero-squaring method (Proposition 3) to the Graeffe's polynomial of order  $2^k$  yields the exact bound with comparatively little effort as the bound is attained from  $k = 6$  (Table 2).

**Table 2.** Unique dominant bound via Graeffe's zero-squaring method.

$k$	3	4	5	6
$\left( \left  \text{Trace}(G_{2^k}(P_1)) \right  \right)^{2^{-k}}$	5.003067358	5.000280728	5.000000012	5.000000000

The previous result turns out as a considerably better bound, comparing with some classical explicit bounds gathered from Dehmer ([4], p. 1, 2), as shown in **Table 3**.

**Table 3.** Classical explicit bounds for  $P_1(X) = X^5 - 10X^4 + 30X^3 - 20X^2 - 31X + 30$  (see [4]).

Referred after	Cauchy	Joyal	Mohammad	Kojima	Jain
Bound	32	17.827723451	20	20	33.219280948

2) Let  $P_2(X) = X^5 - 19X^4 + 126X^3 - 346X^2 + 385X - 147$  be a polynomial with 7 as the largest zero.

By Proposition 4, few iterations of sequences of the maximum of  $x_{m+i} = \left( \left| \text{Trace}(G_{m+i}(P_2)) \right| \right)^{\frac{1}{m}}$ ,  $1 \leq i \leq 5$ , yield some sharp estimations of the real modulus of this double zero, for small values of  $m$  (**Table 4**).

**Table 4.** Computation of the largest (double) bound of  $P_2(X) = X^5 - 19X^4 + 126X^3 - 346X^2 + 385X - 147$ .

$m$	$x_m$	$x_{m+1}$	$x_{m+2}$	$x_{m+3}$	$x_{m+4}$	Max
$10^2$	7.048688850	7.048205127	7.047730920	7.047265952	7.046809955	<b>7.048688850</b>
$10^3$	7.004853712	7.004848862	7.004844021	7.004839190	7.004834368	<b>7.004853712</b>
$10^4$	7.000485220	7.000485171	7.000485123	7.000485074	7.000485026	<b>7.000485220</b>
$10^5$	7.000048520	7.000048520	7.000048519	7.000048519	7.000048519	<b>7.000048520</b>

Moreover, the method leads to better results when locating explicit bounds for zeros of polynomials as shown in **Table 5**.

**Table 5.** Classical explicit bounds for  $P_2(X) = X^5 - 19X^4 + 126X^3 - 346X^2 + 385X - 147$  (see [4]).

Referred after	Cauchy	Joyal	Mohammad	Kojima	Jain
Bound	386	85.165518364	38	38	363.116633802

3) Let's now consider the polynomial  $P_3(X) = X^5 + 11X^4 - 12X^3 - 432X^2 - 864X + 1296$  with 6 as a triple dominant zero. As done in the previous example, dealing with Graeffe's polynomial of  $P_3$  yields a quick computation of the value of dominant eigenvalue of Graeffe's matrix of  $P_3$  (**Table 6**).

**Table 6.** Estimation of the largest (triple) bound of  $P_3(X) = X^5 + 11X^4 - 12X^3 - 432X^2 - 864X + 1296$ .

$m$	$x_m$	$x_{m+1}$	$x_{m+2}$	$x_{m+3}$	$x_{m+4}$	Max
$10^2$	6.066280152	5.999999999	6.064973530	6.000000000	6.063717428	<b>6.066280152</b>
$10^3$	6.008323534	6.004156167	6.008306909	6.004147877	6.008290350	<b>6.008323534</b>
$10^4$	6.000831834	6.000415861	6.000831668	6.000415778	6.000831502	<b>6.000831834</b>
$10^5$	6.000083178	6.000041589	6.000083177	6.000041588	6.000083175	<b>6.000083178</b>

It's important to notice that the convergence seems to become a bit slower than that in the previous case of the existence of a double dominant zero.

Moreover, the method leads to better results when locating explicit bounds for the zeros of polynomials (see **Table 7**).

**Table 7.** Classical explicit bounds for  $P_3(X) = X^5 + 11X^4 - 12X^3 - 432X^2 - 864X + 1296$  (see [4]).

Referred after	Cauchy	Joyal	Mohammad	Kojima	Jain
Bound	7	3.854101966	12	12	39.863137138

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## Appendix: Maple Codes

```
Graeffegen := proc ( P, X , k ::integer)
local G, Y ;
if k < 2 then print("Error")
else G := sort(resultant (P, Y - Xk, X))
end if
end proc:
```

```
GraeffeMatrix := proc ( P, X , k ::integer)
local C, B ;
B := CompanionMatrix (P, X) ;
C[k] := MatrixPower (B, k) ;
sort(CharacteristicPolynomial (C[k], X))
end proc:
```

- In Graeffegen,  $P$  is a polynomial in  $\mathbb{C}[X]$ . The output is the Graeffe polynomial of order  $k$  of  $P$ .
- In GraeffeMatrix,  $B = \text{CompanionMatrix}(P, X)$  stands for the CompanionMatrix of the polynomial  $P \in \mathbb{C}[X]$  and  $\text{CharacteristicPolynomial}(C[k], X)$  gives the characteristic polynomial of  $B^k$ .