

Proving and Extending Greub-Reinboldt Inequality Using the Two Nonzero Component Lemma

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Abstract

We will use the author's Two Nonzero Component Lemma to give a new proof for the Greub-Reinboldt Inequality. This method has the advantage of showing exactly when the inequality becomes equality. It also provides information about vectors for which the inequality becomes equality. Furthermore, using the Two Nonzero Component Lemma, we will generalize Greub-Reinboldt Inequality to operators on infinite dimensional separable Hilbert spaces.

Keywords

Greub-Reinboldt Inequality, Two Nonzero Component Lemma

1. Introduction

Many authors have established Kantorovich inequality and its generalizations such as Greub-Reinboldt Inequality by variational methods. In a variational approach, one differentiates the functional involved to arrive at an "Euler Equation" and then solves the Euler Equation to obtain the minimizing or maximizing vectors of the functional involved. Solving these Euler Equations is tedious and generally provides little information (see [1], subsection 4.4 for an example of this method). Others have established Kantorovich-type inequalities for positive operators by going through a two-step process which consists of first computing upper bounds for suitable functions on intervals containing the spectrum of suitable matrix and then applying the standard operational calculus to that matrix (see [2]) for an example of this method. This method, which we refer to as "the operational calculus method", has the following two limitations:

First, it does not provide any information about vectors for which the established inequalities become equalities. Second, the operational calculus method is futile in extending Kantorovich-type inequalities to operators on an infinite dimensional Hilbert space. A number of Kantorovich-type inequalities are discussed in [3].

In this paper we use the author's Two Nonzero Component Lemma to prove, improve and extend matrix form of Greub-Reinboldt Inequality.

2. The Two Nonzero Component Lemma

In his investigation on problems of antieigenvalue theory the author has discovered a useful lemma which he calls it the Two Nonzero Component Lemma (see [4]-[6]). Although this Lemma is implicitly used in all of the papers just cited, it was not until 2008 that he stated a formal description of the Lemma in his paper titled, "Antieigenvalue Techniques in Statistics". Below is the statement of the lemma. For the proof of the lemma please see the author's work cited above.

Lemma 1 (The Two Nonzero Component Lemma) Let l_1^+ be the set of all sequences with nonnegative terms in the Banach Space l_1 . That is, let

$$l_1^+ = \{\mathbf{t} = (t_i) \in l_1, t_i \geq 0\}. \quad (1)$$

Let

$$F(x_1, x_2, \dots, x_m) \quad (2)$$

be a function from R^m to R . Assume $g_k(\mathbf{t}) = \sum c_i^k t_i$ for $(c_i^k) \in l_1^+$, $\mathbf{t} \in l_1^+$, and $1 \leq k \leq m$. Then the minimizing vectors for the function

$$F(g_1(\mathbf{t}), g_2(\mathbf{t}), \dots, g_m(\mathbf{t})) \quad (3)$$

on the convex set $C = \{(t_i) \in l_1 : \sum t_i = 1\}$ have at most two nonzero components.

What make the proof of the Lemma possible are the following two facts: First, the convexity of the set

$$C = \{(t_i) \in l_1^+ : \sum t_i = 1\}. \quad (4)$$

Second, a special property that the functions

$$F(g_1(\mathbf{t}), g_2(\mathbf{t}), \dots, g_m(\mathbf{t})) \quad (5)$$

involved possess. If we set

$$D(t_1, t_2, t_3, \dots) = F(g_1(\mathbf{t}), g_2(\mathbf{t}), \dots, g_m(\mathbf{t})) \quad (6)$$

then all restrictions of the form

$$D(t_1, t_2, \dots, t_{i-1}, 0, t_{i+1}, \dots) \quad (7)$$

of

$$D(t_1, t_2, t_3, \dots) \quad (8)$$

have the same algebraic form as $D(t_1, t_2, t_3, \dots)$ itself. For example if

$$D(t_1, t_2, \dots, t_n) = \frac{\beta_1 t_1 + \beta_2 t_2 + \dots + \beta_n t_n}{|\lambda_1|^2 t_1 + |\lambda_2|^2 t_2 + \dots + |\lambda_3|^2 t_3}, \quad (9)$$

then we have

$$D(0, t_2, \dots, t_n) = \frac{\beta_2 t_2 + \dots + \beta_n t_n}{|\lambda_2|^2 t_2 + \dots + |\lambda_3|^2 t_3} \quad (10)$$

which has the same algebraic form as

$$D(t_1, t_2, \dots, t_n) = \frac{\beta_1 t_1 + \beta_2 t_2 + \dots + \beta_n t_n}{|\lambda_1|^2 t_1 + |\lambda_2|^2 t_2 + \dots + |\lambda_3|^2 t_3}. \quad (11)$$

Indeed, for any j , $1 \leq j < n$; all restrictions of the function

$$D(t_1, t_2, \dots, t_n) = \frac{\beta_1 t_1 + \beta_2 t_2 + \dots + \beta_n t_n}{|\lambda_1|^2 t_1 + |\lambda_2|^2 t_2 + \dots + |\lambda_3|^2 t_3} \quad (12)$$

obtained by setting an arbitrary set of j components of $D(t_1, t_2, \dots, t_n)$ equal to zeros have the same algebraic form as $D(t_1, t_2, \dots, t_n)$. Obviously, not all functions have this property. For instance, for the function $G(t_1, t_2) = 2t_1 + t_1 t_2$, $G(t_1, 0) = 2t_1$, which does not have the same algebraic form as $G(t_1, t_2)$.

3. Greub-Reinboldt Inequality

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be two real n -tuples. Suppose that m_1, M_1, m_2, M_2 are constants such that $m_1 \leq x_i \leq M_1$ and $m_2 \leq y_i \leq M_2$. Then, for $w_i > 0$ we have

$$\sum_{i=1}^n w_i x_i^2 \sum_{i=1}^n w_i y_i^2 \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2} \left(\sum_{i=1}^n w_i x_i y_i \right)^2 \quad (13)$$

A slightly different form of the above inequality was proved by J. W. S. Cassels in 1951 (see Appendix 1 of [7]). In the following section we provide a proof for the matrix form of Greub-Reinboldt Inequality based on the Two Nonzero Component Lemma. The proof is completely different than the proofs given by others, including Greub and Reinboldt themselves (see [8]). This proof has the advantage of providing information about when the inequality becomes equality and gives information about vectors which make the inequality equality. Furthermore, as we will discuss in the Section 5, our method will indeed extend the Greub-Reinboldt Inequality to operators on an infinite dimensional Hilbert space.

4. The Matrix Form of Greub-Reinboldt Inequality

Theorem 2 Let S and T be two commuting positive operators with eigenvalues $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$ respectively. Assume

$$m_1 \leq \alpha_1 \leq \dots \leq \alpha_n \leq M_1 \quad (14)$$

and

$$m_2 \leq \beta_1 \leq \dots \leq \beta_n \leq M_2.$$

Also assume S and T are diagonalized with diagonal elements $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$ respectively, then

$$(Tx, Tx)(Sx, Sx) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2} (Tx, Sx)^2 \quad (15)$$

for every vector x . In this, case if x is any unit vector which makes the inequality (15) an equality then we have

$$\|x_1\|^2 = \frac{\alpha_n \beta_n}{\alpha_1 \beta_1 + \alpha_n \beta_n} \quad (16)$$

and

$$\|x_n\|^2 = \frac{\alpha \beta_1}{\alpha_1 \beta_1 + \alpha_n \beta_n} \quad (17)$$

and

$$x_i = 0 \text{ if } 1 < i < n \quad (18)$$

where x_i is the projection of x on the eigenspace corresponding to eigenvalue α_i .

Proof. Without loss of generality we can assume $\|x\| = 1$. Consider the functional

$$\frac{(Tx, Tx)(Sx, Sx)}{(Tx, Sx)^2} \quad (19)$$

(19) can be written as

$$\frac{\|Tx\|^2 \|Sx\|^2}{(Tx, Sx)^2}. \tag{20}$$

The reciprocal of (20) is

$$\frac{(Tx, Sx)^2}{\|Tx\|^2 \|Sx\|^2}. \tag{21}$$

The square root of (21) is

$$\frac{(Tx, Sx)}{\|Tx\| \|Sx\|}. \tag{22}$$

To prove (15) we first find

$$\inf_{\|x\|=1} \frac{(STx, x)}{\|Tx\| \|Sx\|}. \tag{23}$$

Since S is invertible, by a change of variable we have

$$\inf_{\|x\|=1} \frac{(STx, x)}{\|Tx\| \|Sx\|} = \inf_{\|x\|=1} \frac{(TS^{-1}x, x)}{\|TS^{-1}x\| \|x\|}. \tag{24}$$

By the spectral mapping theorem the inf on the right hand side of (24) can be represented as

$$\inf \frac{\sum_{i=1}^n \lambda_i \|x_i\|^2}{\sqrt{\sum_{i=1}^n \lambda_i^2 \|x_i\|^2}} \tag{25}$$

over the set

$$\sum_{i=1}^n \|x_i\|^2 = 1, \tag{26}$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is the set of eigenvalues of TS^{-1} . The fact that S and T commute implies that

$$\lambda_i = \alpha_i^{-1} \beta_i \tag{27}$$

for $1 \leq i \leq n$ and

$$m = M_1^{-1} m_2 \text{ and } M = m_1^{-1} M_2 \tag{28}$$

where

$$m = \min \{ \lambda_i \}_{i=1}^n, \tag{29}$$

and

$$M = \max \{ \lambda_i \}_{i=1}^n. \tag{30}$$

If we set $\|x_i\|^2 = t_i$ the problem is reduced to finding

$$\inf \frac{\sum_{i=1}^n \lambda_i t_i}{\sqrt{\sum_{i=1}^n \lambda_i^2 t_i}} \tag{31}$$

over

$$\sum_{i=1}^n t_i = 1. \quad (32)$$

By the Two Nonzero Component Lemma we need to look at

$$\inf \frac{\lambda_i t_i + \lambda_j t_j}{\sqrt{\lambda_i^2 t_i + \lambda_j^2 t_j}} \quad (33)$$

over the convex set

$$t_i + t_j = 1 \quad (34)$$

for pairs of i and j . Notice that since the expression in (33) is positive, for simplicity, we can first compute the infimum of the square of that expression on the convex set (34) and then take square root of the result. Therefore, the problem is now reduced to finding

$$\inf \frac{(\lambda_i t_i + \lambda_j t_j)^2}{\lambda_i^2 t_i + \lambda_j^2 t_j} \quad (35)$$

on (34). By substituting $t_j = 1 - t_i$ in (35) the problem is now reduced to finding

$$\inf \frac{(\lambda_i t_i + \lambda_j (1 - t_i))^2}{\lambda_i^2 t_i + \lambda_j^2 (1 - t_i)} \quad (36)$$

for $0 \leq t_i \leq 1$. To find (36), simply differentiate the expression in (36) and set its derivative with respect to t_i equal to zero (we omit the straight forward computations). The expression in (36) is minimized when

$$t_i = \frac{\lambda_j}{\lambda_i + \lambda_j}.$$

Substituting this value of t_i in (34) and the expression in (36) gives us

$$t_j = \frac{\lambda_i}{\lambda_i + \lambda_j} \quad (37)$$

and

$$\frac{(\lambda_i t_i + \lambda_j (1 - t_i))^2}{\lambda_i^2 t_i + \lambda_j^2 (1 - t_i)} = \frac{4\lambda_i \lambda_j}{(\lambda_i + \lambda_j)^2}. \quad (38)$$

Hence

$$\inf \frac{\lambda_i t_i + \lambda_j t_j}{\sqrt{\lambda_i^2 t_i + \lambda_j^2 t_j}} = \frac{2\sqrt{\lambda_i \lambda_j}}{\lambda_i + \lambda_j} \quad (39)$$

and the inf in (39) is attained at

$$t_i = \frac{\lambda_j}{\lambda_i + \lambda_j} \quad (40)$$

and

$$t_j = \frac{\lambda_i}{\lambda_i + \lambda_j}. \quad (41)$$

Assume that $i < j$. Now we show that we must have $i = 1$ and $j = n$. To prove this we must show

$$\frac{2\sqrt{\lambda_i \lambda_n}}{\lambda_i + \lambda_n} \leq \frac{2\sqrt{\lambda_i \lambda_j}}{\lambda_i + \lambda_j} \quad (42)$$

for $1 \leq i \leq n$ and $1 \leq i \leq n$. Squaring both sides of (42) gives us

$$\frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2} \leq \frac{4\lambda_i\lambda_j}{(\lambda_i + \lambda_j)^2}. \quad (43)$$

Thus instead of proving inequality (42) we can prove inequality (43). Let $u = \frac{\lambda_n}{\lambda_1}$ and $v = \frac{\lambda_j}{\lambda_i}$. It is obvious that

$$1 \leq y \leq x. \quad (44)$$

If we substitute $\lambda_n = u\lambda_1$ and $\lambda_j = v\lambda_i$ in (43) we get

$$\frac{4u\lambda_1^2}{\lambda_1^2(1+u)^2} \leq \frac{4v\lambda_i^2}{\lambda_i^2(1+v)^2} \quad (45)$$

which is equivalent to

$$\frac{4u}{(1+u)^2} \leq \frac{4v}{(1+v)^2}. \quad (46)$$

Hence proving inequality (42) is reduced to proving inequality (46). To prove (46), note that based on (44)

$$\frac{4u}{(1+u)^2} - \frac{4v}{(1+v)^2} = \frac{(u-v)(1-uv)}{(u+1)^2(v+1)^2} \leq 0. \quad (47)$$

Therefore, we must have

$$\inf_{\|x\|=1} \frac{(STx, x)}{\|Tx\|\|Sx\|} = \frac{2\sqrt{\lambda_i\lambda_n}}{\lambda_1 + \lambda_n}. \quad (48)$$

The inequality

$$\frac{2\sqrt{\lambda_i\lambda_n}}{\lambda_1 + \lambda_n} \geq \frac{2\sqrt{mM}}{n+M} \quad (49)$$

can be proved the same way we just proved (42). Hence we have

$$\frac{(STx, x)^2}{\|Tx\|^2\|Sx\|^2} \geq \frac{4M_1^{-1}m_2m_1^{-1}M_2}{(M_1^{-1}m_2 + m_1^{-1}M_2)^2}. \quad (50)$$

The right side of 50 is simplified to

$$\frac{4m_1m_2M_1M_2}{(M_1M_2 + m_1m_2)^2}. \quad (51)$$

Thus (50) becomes

$$\frac{(STx, x)^2}{\|Tx\|^2\|Sx\|^2} \geq \frac{4m_1m_2M_1M_2}{(M_1M_2 + m_1m_2)^2}. \quad (52)$$

Finally (50) is equivalent to

$$(Tx, Tx)(Sx, Sx) \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2} (Tx, Sx)^2. \quad (53)$$

5. Generalizing Greub-Reinboldt Inequality to Operators on a Separable Hilbert Space

There are many proofs for Greub-Reinboldt Inequality in the literature. A significant advantage of proving

Greub-Reinboldt Inequality by The Two Nonzero Component Lemma is that we can generalize this inequality to the case of positive operators S and T on an separable infinite dimensional separable Hilbert space. This is because, as the statement of the Two Nonzero Component Lemma shows, this lemma is also when the functions $g_k(\mathbf{t}) = \sum_i^k c_i^k t_i$ infinite linear combinations of t_1, t_2, t_3, \dots . Thus we can replace finite summations in (25), (26), (31), (32) with infinite sums and the arguments made in this paper remain valid. However, in this case it seems difficult to pinpoint the exact pair of i and j for which the projections x_i and x_j of minimizing unit vectors are nonzero.

Theorem 3 Let S and T be two commuting positive operators on a separable Hilbert space such that $\sigma(S) \subseteq [m_1, M_1]$ and $\sigma(T) \subseteq [m_2, M_2]$ where $\sigma(S)$ and $\sigma(T)$ represent the spectrums of S and T respectively, then

$$(Tx, Tx)(Sx, Sx) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2} (Tx, Sx)^2 \quad (54)$$

for any vector x . In this case if x is any unit vector which makes the inequality (54) an equality then there exist a pair of i and j such that

$$\|x_i\|^2 = \frac{\alpha_j \beta_j}{\alpha_i \beta_i + \alpha_j \beta_j} \quad (55)$$

and

$$\|x_j\|^2 = \frac{\alpha_i \beta_i}{\alpha_i \beta_i + \alpha_j \beta_j} \quad (56)$$

and

$$x_k = 0 \text{ if } k \neq i \text{ and } k \neq j \quad (57)$$

where x_i is the projection of x on the eigenspace corresponding to eigenvalue α_i .

There are other generalizations of Greub-Reinboldt Inequality. For example in ([9]) Gustafson extends this inequality to pair noncommuting positive matrices A and B . However, he replaces the standard norm of the Hilbert space with the norm relative to B .

Conclusion 4 The Two Nonzero Component Lemma provides an effective way of proving the Greub-Reinboldt Inequality and extending it to positive operators on separable infinite dimensional Hilbert spaces. The author has also utilized this lemma to prove other Kantorovich-type inequalities. Please see ([4]-[6] [10] [11]).

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