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On the Spectral Characterization of *H*-Shape Trees

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Abstract

A graph *G* is said to be determined by its spectrum if any graph having the same spectrum as *G* is isomorphic to *G*. An *H*-shape is a tree with exactly two of its vertices having maximal degree 3. In this paper, a formula of counting the number of closed 6-walks is given on a graph, and some necessary conditions of a graph *Γ* cospectral to an *H*-shape are given.

Keywords

Spectra of Graphs, Cospectral Graphs, Spectra Radius, *H*-Shape Trees, Determined by Its Spectrum

1. Introduction

Let $G=(V,E)$ be a simple undirected graph with vertex set $V=\{v_1,v_2,\cdots,v_n\}$ and edge set *E*. Let $A(G)$ be the adjacency matrix of *G*. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)$. The sequence of *n* eigenvalues is called the spectrum of *G*, the largest eigenvalue $\lambda_1(G)$ is often called the spectral radius of *G*. The characteristic polynomial of $A(G)$ is called the characteristic polynomial of the graph *G* and is denoted by $\varphi(G,\lambda)$.

Two graphs are cospectral if they share the same spectrum. A graph *G* is said to be determined by its spetrum (DS for short) if for any graph *H*, $\varphi(H,\lambda)=\varphi(G,\lambda)$ implies that *H* is isomorphic to *G*.

Determining what kinds of graphs are DS is an old problem, yet far from resolved, in the theory of graph spectra. Numerous examples of cospectral but non-isomorphic graphs are reported in literature [1]. However, there are few results known about DS graphs. For the background and some recent surveys of the known results about this problem and related topics, we refer the reader to [2]-[6] and references therein.

Because the kind of problems above are generally very hard to deal with, some more modest ones suggested by van Dam and Haemers [2], say, "Which trees are DS?", this problem is also very hard to deal with, because we know a famous result of Schwenk [7], which says that almost all trees have non-isomorphic cospectral mates.

A T-shape $T(l_1,l_2,l_3)$ is a tree with exactly one of its vertices having maximal degree 3 such that $T(l_1,l_2,l_3)-v=p_{l_1}\cup p_{l_2}\cup p_{l_3}$, where p_{l_i} is the path on $l_i\,(i=1,2,3)$ vertices, and v is the vertex of degree 3. More recently, Wang proved that T-shape tree $T(l_1,l_2,l_3)$ is DS; Wang and Xu [6] proved that T-shape tree $T(l_1,l_2,l_3)(l_1\le l_2\le l_3)$ is DS iff $(l_1,l_2,l_3)\ne(l,l,2l-2)$ for any positive integer $l\ge 2$.

An H-shape is a tree with exactly two of its vertices having maximal degree 3. We denote by $H(l_1,l_2,l_3,l_4,l_5)(l_1\ge 0, l_i\ge 1, i=2,3,4,5)$ is an H-shape tree such that

$H(l_1,l_2,l_3,l_4,l_5)-u-v=p_{l_1}\cup p_{l_2}\cup p_{l_3}\cup p_{l_4}\cup p_{l_5}$, and $H(l_1,l_2,l_3,l_4,l_5)-u=T(l_1,l_4,l_5)$,

$H(l_1,l_2,l_3,l_4,l_5)-v=T(l_1,l_2,l_3)\cup p_{l_4}\cup p_{l_5}$, where u and v are the vertices of degree 3.

In this paper, we give a formula of counting the number of closed 6-walks on a graph, and give some necessary conditions of a graph Γ cospectral to an H-shape.

2. Some Lemmas

In the section, we will present some lemmas which are required in the proof of the main result.

Lemma 2.1 [8] The characteristic polynomial of a graph satisfies the following identities:

1) $\varphi(G_1\cup G_2,\lambda)=\varphi(G_1,\lambda)\varphi(G_2,\lambda)$,

2) $\varphi(G,\lambda)=\varphi(G-e,\lambda)-\varphi(G-v_1v_2,\lambda)$ if $e=v_1v_2$ is a cut-edge of G.

where $G-e$ denotes the graph obtained from G by deleting the edge e and $G-v_1v_2$ denotes the graph obtained from G by deleting the vertices v_1, v_2 and the edges incident to it.

Lemma 2.2 [1] Let C_n, P_n denote the cycle and the path on n vertices respectively. Then

$$\varphi(C_n,\lambda)=\prod_{j=1}^{n}\left(\lambda-2\cos\frac{2\pi j}{n}\right)=2\cos\left(n\arccos\lambda/2\right)-2$$

$$\varphi(P_n,\lambda)=\prod_{j=1}^{n}\left(\lambda-2\cos\frac{\pi j}{n+1}\right)=\frac{\sin\left((n+1)\arccos\lambda/2\right)}{\sin\left(\arccos\lambda/2\right)}$$

Let $\lambda=2\cos\theta$, set $t^{1/2}=\mathrm{e}^{\mathrm{i}\theta}$, we get $\lambda=t^{1/2}+t^{-1/2}$, it is can be write the characteristic polynomial of C_n, P_n in the following form [6]:

$$\varphi\left(C_n,t^{1/2}+t^{-1/2}\right)=t^{n/2}+t^{-n/2}-2=t^{-n/2}\left(t^{n/2}-1\right)^2 \quad (1)$$

$$\varphi\left(P_n,t^{1/2}+t^{-1/2}\right)=t^{-n/2}\left(t^{n+1}-1\right)/(t-1) \quad (2)$$

Lemma 2.3 [4] [9] Let $\varphi(G,x)=\sum_{i=0}^{n}a_i\lambda^{n-i}$ be the characteristic polynomial of graph G with n vertices, then the coefficient of λ^{n-i} is

$$a_i=\sum_{\gamma}(-1)^{comp(\gamma)}2^{cyc(\gamma)} \quad (3)$$

where $a_0=1$ and the sum is over all subgraphs γ of G consisting of disjoint edges and cycles, and having i vertices. If γ is such a subgraph then comp(γ) is the number of components in it and cyc(γ) is the number of cycles.

Lemma 2.4 [2] [10] Let G be a graph. For the adjacency matrix, the following can be obtained from the spectrum.

1) The number of vertices.
2) The number of edges.
3) Whether G is regular.
4) Whether G is regular with any fixed girth.
5) The number of closed walk of any length.
6) Whether G is bipartite.

3. Main Results

The total number of closed k-walks in a graph G, denoted by $|w_k(G)|$.

Lemma 3.1 ([6] p. 657) Let G be a graph with e edges, x_i vertices of degree i, and y 4-cycles. Then

$$|w_4(G)| = 2e + 4\sum_i \binom{i}{2} x_i + 8y \tag{4}$$

Lemma 3.2 Let Γ be a graph with n vertices. If Γ cospectral to an H-shape and $\Gamma \neq W_n$, then

1) Γ have the same degree sequences as the H-shape tree or Γ have the degree sequences $(3,2,2,\cdots,2,1,0)$.

2) Γ contains no 4-cycles.

Proof. Let Γ be a graph with e edges, x_i vertices of degree i, and y 4-cycles. By lemma 2.4 we known that cospectral graphs have the same number of edges and closed 4-walks, respectively. Since Γ is cospectral to an H-shape tree, hence by (4) we have

$$2e + 4\sum_i \binom{i}{2} x_i + 8y = 6n - 2$$

namely

$$\sum_i \binom{i}{2} x_i + 2y = n = \sum_{i\geq 0} x_i \tag{5}$$

Since

$$\sum_{i\geq 2}(i-1)x_i = \sum_{i\geq 2} i x_i - \sum_{i\geq 2} x_i = (2e - x_1) - (n - x_0 - x_1) = 2e - n + x_0 = n - 2 + x_0, \tag{6}$$

from (5), we have

$$\sum_{i\geq 2}\binom{i-1}{2} x_i + 2y = n - \sum_{i\geq 2}(i-1) = 2 - x_0 \tag{7}$$

the (7) imply to $y = 1$ or 0.

Case 1. $y = 1$. by (7) we get $x_0 = 0$ and $x_3 = x_4 = \cdots = 0$, by (5) we get $x_2 = n-2$ and $x_1 = 2$, then $\Gamma = C_4 \cup P_{n-2}$.

We known that "the spectrum of graph W_n is the union of the spectra of the circuit C_4 and the path P_{n-4}" [1], that is

$$\varphi(W_n, \lambda) = \varphi(C_4 \cup C_4 \cup P_{n-4}, \lambda)$$

Case 2. $y = 0$. By (7) we have $x_0 \leq 2$.

If $x_0 = 0$, then $x_3 = 2, x_4 = x_5 = \cdots = 0$, by (5) we get $x_2 = n-6$ and $x_1 = 4$. Thus Γ have the same degree sequences as the H-shape tree.

If $x_0 = 1$, then $x_3 = 1, x_4 = x_5 = \cdots = 0, x_2 = n-3$ and $x_1 = 1$. The degree sequences of Γ is $(3,2,2,\cdots,2,1,0)$.

If $x_0 = 2$, then $x_3 = x_4 = \cdots = 0$, $x_2 = n$, $|V(\Gamma)| \geq n+2$, a contradiction. □

Lemma 3.3 Let G be a graph with e edges, x_i vertices of degree i, and z 6-cycles. Then

$$|w_6(G)| = 2e + 12\sum_i \binom{i}{2} x_i + 6p_4 + 12k_{1,3} + 12z \tag{8}$$

where p_4 is the number of induced paths of length three and $k_{1,3}$ is the number of induced star $K_{1,3}$.

Proof. A close walk of length 6 can be produced from in the following five classes graphs, they are P_2, P_3, P_4, $K_{1,3}$ and C_6. For an edge and a 6-cycle, it is easy to see that the number of close 6-walks equals 2 and 12, respectively. For a P_3, the number of close 6-walks of a 1-degree vertex is 3 and the number of close 6-walks of the 2-degree vertex is 6, since the number of induced paths of length two is $\sum_i \binom{i}{2} x_i$, hence for all induced paths P_3, the number of close 6-walks is $12\sum_i \binom{i}{2} x_i$. For a P_4, since the number of close 6-walks of a 1-degree vertex is 1 and the number of close 6-walks of a 2-degree vertex is 2, hence for all induced paths P_4, the number of close 6-walks is $6p_4$. Similarly, for a $K_{1,3,}$ the number of close 6-walks of a 1-degree vertex is 2 and the number of close 6-walks of the 3-degree vertex is 6, thus for all induced stars $K_{1,3}$, the number of close 6-walks is $12k_{1,3}$. □

Corollary 3.4 Let $H = H(l_1,l_2,l_3,l_4,l_5)$, then

$$|w_6(H)| = \begin{cases} 20n+28-6k\left(l_1 \geq 1 \text{ and have } k \text{ elements are 1 in } \{l_2,l_3,l_4,l_5\}\right) \\ 20n+34-6k\left(l_1 = 0 \text{ and have } k \text{ elements are 1 in } \{l_2,l_3,l_4,l_5\}\right) \end{cases} \quad (9)$$

where $0 \leq k \leq 4$.

Proof. Case 1. $l_1 \geq 1$.

1) If $k = 0$, that is $l_i \geq 2(i = 2,3,4,5)$, then

$$|w_6(H)| = 2(n-1)+12[(n-6)+3\times 2]+6[(l_2+l_3-2)+(l_4+l_5-2)+(l_1-1)+8]+12\times 2 = 20n+28$$

where $(l_2+l_3-2),(l_4+l_5-2)$ and (l_1-1) are the number of induced paths P_4 in $p_{l_2+l_3+1} \cdot p_{l_4+l_5+1}$ and p_{l_1+2}, respectively. The 8(= 4 + 4) is the number of induced paths of through a 3-degree vertex u (or v). If P_4 is such a induced path, then u is an internal vertex in the P_4 and have at least a vertex in the p_{l_1} (or p_{l_2}).

2) If $k \neq 0$, then

$$|w_6(H)| = 2(n-1)+12[(n-6)+3\times 2]+6[(l_2+l_3-2)+(l_4+l_5-2)+(l_1-1)+(8-k)]+12\times 2 = 20n+28-6k$$

Case 2. $l_1 = 0$.

1) If $k \neq 0$, then

$$|w_6(H)| = 2(n-1)+12[(n-6)+3\times 2]+6[(l_2+l_3-2)+(l_4+l_5-2)+8]+12\times 2 = 20n+34.$$

2) If $k = 0$, similarly, we have $|w_6(H)| = 20n+34-6k$. □

Example 1. Let $H_1 = H(0,1,1,1,1)$, by (9) we have

$$|w_6(H_1)| = 20\times n+34-6k = 20\times 6+34-6\times 4 = 130,$$

if we give to a suitable label for the H_1, by a simple calculation we can get the diagonal matrix of $A^6(H_1)$, that is

$$\text{diag}\left(A^6(H_1)\right) = [11,11,43,43,11,11]$$

clearly, the sum of the elements in the diagonal matrix equals 4 × 11 + 2 × 43 = 130.

Example 2. Let $H_2 = H(2,2,2,2,2)$, by (9) we have $|w_6(H_2)| = 20\times 12+28 = 268$, similarly, if we give to a suitable label for the H_2, then we can get the diagonal matrix of $A^6(H_2)$, that is

$$\text{diag}\left(A^6(H_2)\right) = [6,6,6,6,22,22,22,22,29,29,49,49]$$

clearly, the sum of the elements in the diagonal matrix equals 4 × 6 + 4 × 22 + 2 × 29 + 2 × 49 = 268.

Lemma 3.5 Let Γ be a graph with n vertices, e edges, x_i vertices of degree i, and z 6-cycles. If Γ cospectral to $H(l_1,l_2,l_3,l_4,l_5)$ and $\Gamma \neq W_n$, then

$$2\sum_{i\geq 2}\binom{i-1}{2}x_i + p_4 + 2k_{1,3} + 2z = \begin{cases} n+9-k-2x_0\left(l_1 \geq 1 \text{ and have k elements are 1 in } \{l_2,l_3,l_4,l_5\}\right) \\ n+10-k-2x_0\left(l_1 = 0 \text{ and have k elements are 1 in } \{l_2,l_3,l_4,l_5\}\right) \end{cases} \quad (10)$$

where $k(0 \leq k \leq 4)$ is the number of elements of equals 1 in $\{l_2,l_3,l_4,l_5\}$ and p_4 is the number of induced paths of length three and $k_{1,3}$ is the number of induced star $K_{1,3}$ in Γ.

Proof. If $l_1 \geq 1$, by Lemma 3.3 we have

$$2e+12\sum_i\binom{i}{2}x_i+6p_4+12k_{1,3}+12z = 20n+28-6k, \quad 2\sum_i\binom{i}{2}x_i+p_4+2k_{1,3}+2z = 3n+5-k,$$

$$2\sum_{i\geq 2}\binom{i-1}{2}x_i+p_4+2k_{1,3}+2z = 3n+5-2\sum_{i\geq 2}(i-1)x_i = 3n+5-k-2(n-2+x_0) = n+9-k-2x_0.$$

Similarly, when $l_1 = 0$ the (10) hold. □

Definition 1. Let U be a graph obtained from a cycle C_g (g is even and $6 \le g \le n_1 - 2$) and a path P_{n_1-g}, such that identifying an end vertex in the path and any one vertex in the cycle, and uniting an isolated vertex K_1.

If a graph have the degree sequences $(3,2,2,\cdots,2,1,0)$, then the graph is U uniting some cycle.

Lemma 3.6 Let U' be a graph with degree sequences $(3,2,2,\cdots,2,1,0)$. If U' cospectral to an H-shape, then U' and H satisfying one of the following conditions.

1) There are one 6-cycle in U' and $l_1 \ge 1$, l_2, l_3, l_4, $l_5 \ge 2$.

2) There are one 6-cycle in U' and $l_1 = 0$, have an element is 1 in $\{l_2, l_3, l_4, l_5\}$.

3) No 6-cycle in U' and $l_1 \ge 1$, have two elements are 1 in $\{l_2, l_3, l_4, l_5\}$.

4) No 6-cycle in U' and $l_1 = 0$, have three elements are 1 in $\{l_2, l_3, l_4, l_5\}$.

Proof. Without loss of generality, Let $U' = U \cup C_{n_2}$, where $n_2 (\ge 6)$ is even and $n_1 + n_2 = n$. Let U' have e edges, x_i vertices of degree i, and z 6-cycles.

Case 1. $l_1 \ge 1$. By Lemma 3.5 we have $2\times1+\left[g+(n_1-g-3)+4+n_2\right]+2\times1+2z = n+9-k-2, 2z = 2-k$, get $k = 0$, $z = 1$ or $k = 2$, $z = 0$.

Case 2. $l_1 = 0$, we have $2\times1+\left[g+(n_1-g-3)+4\ +n_2\right]+2\times1+2z = n+10-k-2,\ 2z = 3-k$, get $k = 1$, $z = 1$ or $k = 3$, $z = 0$. □

Lemma 3.7 Let $\lambda = t^{1/2}+t^{-1/2}$, then

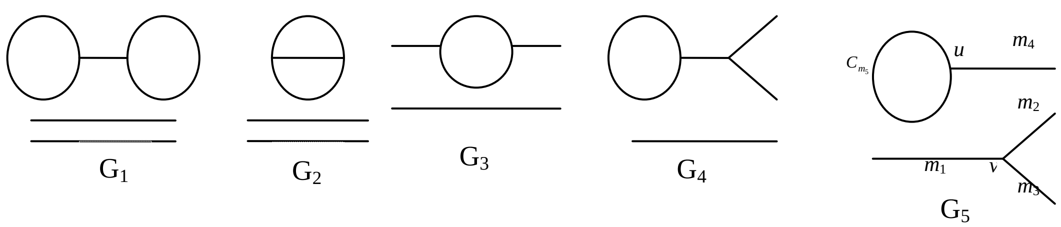
$$\begin{aligned}&\varphi\left(H(l_1,l_2,l_3,l_4,l_5),t^{1/2}+t^{-1/2}\right)\\&=\frac{t^{-n/2}}{(t-1)^5}\Big[(t-1)^2\left(t^{l_1+1}-1\right)\left(t^{l_2+l_3+2}-1\right)\left(t^{l_4+l_5+2}-1\right)-t(t-1)\left(t^{l_1}-1\right)\left(t^{l_2+l_3+2}-1\right)\left(t^{l_4+1}-1\right)\left(t^{l_5+1}-1\right)\\&\quad-t(t-1)\left(t^{l_1}-1\right)\left(t^{l_2+1}-1\right)\left(t^{l_3+1}-1\right)\left(t^{l_4+l_5+2}-1\right)+t^2\left(t^{l_1-1}-1\right)\left(t^{l_2+1}-1\right)\left(t^{l_3+1}-1\right)\left(t^{l_4+1}-1\right)\left(t^{l_5+1}-1\right)\Big]\end{aligned} \tag{11}$$

Proof. By Lemma 2.1 (b) and Lemma 2.2 we have

$$\begin{aligned}&\varphi\left(H(l_1,l_2,l_3,l_4,l_5),\lambda\right)\\&=\varphi\left(P_{l_2+l_3+1},\lambda\right)\varphi\left(T(l_1,l_4,l_5),\lambda\right)-\varphi\left(P_{l_2},\lambda\right)\varphi\left(P_{l_3},\lambda\right)\varphi\left(T(l_1-1,l_2,l_3),\lambda\right)\\&=\varphi\left(P_{l_2+l_3+1},\lambda\right)\varphi\left(P_{l_1},\lambda\right)\varphi\left(P_{l_4+l_5+1},\lambda\right)-\varphi\left(P_{l_2+l_3+1},\lambda\right)\varphi\left(P_{l_1-1},\lambda\right)\varphi\left(P_{l_4},\lambda\right)\varphi\left(P_{l_5},\lambda\right)\\&\quad-\varphi\left(P_{l_1-1},\lambda\right)\varphi\left(P_{l_2},\lambda\right)\varphi\left(P_{l_3},\lambda\right)\varphi\left(P_{l_4+l_5+1},\lambda\right)+\varphi\left(P_{l_1-2},\lambda\right)\varphi\left(P_{l_2},\lambda\right)\varphi\left(P_{l_3},\lambda\right)\varphi\left(P_{l_4},\lambda\right)\varphi\left(P_{l_5},\lambda\right)\end{aligned} \tag{12}$$

$$\begin{aligned}&\varphi\left(H(l_1,l_2,l_3,l_4,l_5),t^{1/2}+t^{-1/2}\right)\\&=\frac{t^{-n/2}}{(t-1)^5}\Big[(t-1)^2\left(t^{l_1+1}-1\right)\left(t^{l_2+l_3+2}-1\right)\left(t^{l_4+l_5+2}-1\right)-t(t-1)\left(t^{l_1}-1\right)\left(t^{l_2+l_3+2}-1\right)\left(t^{l_4+1}-1\right)\left(t^{l_5+1}-1\right)\\&\quad-t(t-1)\left(t^{l_1}-1\right)\left(t^{l_2+1}-1\right)\left(t^{l_3+1}-1\right)\left(t^{l_4+l_5+2}-1\right)+t^2\left(t^{l_1-1}-1\right)\left(t^{l_2+1}-1\right)\left(t^{l_3+1}-1\right)\left(t^{l_4+1}-1\right)\left(t^{l_5+1}-1\right)\Big]\end{aligned}$$ □

If a graph has the same degree sequences as the H-shape, then Γ is one of the following graphs G_1, G_2, G_3, G_4, G_5 in figure or it is an H-shape.



Lemma 3.8 If Γ is cospectral to an H-shape tree, then Γ contains no $P_{n_1} \cup P_{n_2}\ (n_1, n_2 < n)$ as two connected component.

Proof. Assume that Γ contains a P_{n_1} as a connected component, by (11) some li is equal, without loss of generality, let $l_1 = l_2 = l_4 = n_1$, then

$$
\begin{aligned}
&\varphi\left(H\left(l_1,l_2,l_3,l_4,l_5\right),\lambda\right)\\
&=\varphi\left(P_{l_1+l_3+1},\lambda\right)\varphi\left(P_{l_1+l_5+1},\lambda\right)-\varphi\left(P_{l_1+l_3+1},\lambda\right)\varphi\left(P_{l_1-1},\lambda\right)\varphi\left(P_{l_5},\lambda\right)\\
&\quad-\varphi\left(P_{l_1-1},\lambda\right)\varphi\left(P_{l_3},\lambda\right)\varphi\left(P_{l_1+l_5+1},\lambda\right)+\varphi\left(P_{l_1-2},\lambda\right)\varphi\left(P_{l_3},\lambda\right)\varphi\left(P_{l_1},\lambda\right)\varphi\left(P_{l_5},\lambda\right)\varphi\left(P_{l_1},\lambda\right)
\end{aligned}
$$

If Γ contains a P_{n_2} as a connected component, then $l_3 = l_5$ and $l_1 + l_3 + 1 = l_1$, a contradiction. □

Thus, if a graph $\Gamma\left(\Gamma \neq W_n\right)$ cospectral to an H-shape and have the same degree sequences as the H-shape, then Γ is one of the following graphs G_3, G_4, G_5 (Fig.) uniting some even cycle, respectively, or it is an H-shape.

Lemma 3.9 If $H_1 = H\left(m_1,m_2,m_3,m_4,m_5\right)$ and $H = H\left(l_1,l_2,l_3,l_4,l_5\right)$ are cospectral, then $H\left(m_1,m_2,m_3,m_4,m_5\right) \cong H\left(l_1,l_2,l_3,l_4,l_5\right)$

Proof. By (11) we have

$$
\begin{aligned}
&\varphi\left(H\left(l_1,l_2,l_3,l_4,l_5\right),t^{1/2}+t^{-1/2}\right)t^{n/2}\left(t-1\right)^5\\
&=\left(t-1\right)^2\left(t^{l_1+1}-1\right)\left(t^{l_2+l_3+2}-1\right)\left(t^{l_4+l_5+2}-1\right)-t\left(t-1\right)\left(t^{l_1}-1\right)\left(t^{l_2+l_3+2}-1\right)\left(t^{l_4+1}-1\right) \qquad (13)\\
&\quad\left(t^{l_5+1}-1\right)-t\left(t-1\right)\left(t^{l_1}-1\right)\left(t^{l_2+1}-1\right)\left(t^{l_3+1}-1\right)\left(t^{l_4+l_5+2}-1\right)+t^2\left(t^{l_1-1}-1\right)\left(t^{l_2+1}-1\right)\left(t^{l_3+1}-1\right)\left(t^{l_4+1}-1\right)\left(t^{l_5+1}-1\right)
\end{aligned}
$$

$$
\begin{aligned}
&=t^{n+5}-4t^{n+4}+4t^{n+3}+t^{l_1+l_2+l_3+l_4+5}+t^{l_1+l_2+l_3+l_5+5}+t^{l_1+l_2+l_4+l_5+5}+t^{l_1+l_3+l_4+l_5+5}-2t^{l_1+l_2+l_3+l_4+4}\\
&\quad-2t^{l_1+l_2+l_3+l_5+4}-2t^{l_1+l_2+l_4+l_5+4}-2t^{l_1+l_3+l_4+l_5+4}-t^{l_2+l_3+l_4+l_5+4}-t^{l_1+l_2+l_3+5}-t^{l_1+l_4+l_5+5}+2t^{l_1+l_2+l_3+4}\\
&\quad+2t^{l_1+l_4+l_5+4}+t^{l_2+l_3+l_4+4}+t^{l_2+l_3+l_5+4}+t^{l_2+l_4+l_5+4}+t^{l_3+l_4+l_5+4}+t^{l_1+l_2+l_4+3}+t^{l_1+l_2+l_5+3}+t^{l_1+l_3+l_4+3}\\
&\quad+t^{l_1+l_3+l_5+3}-t^{l_2+l_4+4}-t^{l_2+l_5+4}-t^{l_3+l_4+4}-t^{l_3+l_5+4}-2t^{l_2+l_3+3}-2t^{l_4+l_5+3}-t^{l_1+l_2+3}-t^{l_1+l_3+3}-t^{l_1+l_4+3} \qquad (14)\\
&\quad-t^{l_1+l_5+3}+t^{l_2+l_3+2}+t^{l_4+l_5+2}+t^{l_1+3}+2t^{l_2+3}+2t^{l_3+3}+2t^{l_4+3}+2t^{l_5+3}-t^{l_2+2}-t^{l_3+2}-t^{l_4+2}-t^{l_5+2}-4t^2+4t-1\\
&=:\psi_{\mathrm{H}}\left(t\right),
\end{aligned}
$$

where $l_1+l_2+l_3+l_4+l_5+2=n$. By (14) we have

$$
\begin{aligned}
&\varphi\left(H\left(m_1,m_2,m_3,m_4,m_5\right),t^{1/2}+t^{-1/2}\right)t^{n/2}\left(t-1\right)^5\\
&=t^{n+5}-4t^{n+4}+4t^{n+3}+t^{m_1+m_2+m_3+m_4+5}+t^{m_1+m_2+m_3+m_5+5}+t^{m_1+m_2+m_4+m_5+5}+t^{m_1+m_3+m_4+m_5+5}\\
&\quad-2t^{m_1+m_2+m_3+m_4+4}-2t^{m_1+m_2+m_3+m_5+4}-2t^{m_1+m_2+m_4+m_5+4}-2t^{m_1+m_3+m_4+m_5+4}-t^{m_2+m_3+m_4+m_5+4}\\
&\quad-t^{m_1+m_2+m_3+5}-t^{m_1+m_4+m_5+5}+2t^{m_1+m_2+m_3+4}+2t^{m_1+m_4+m_5+4}+t^{m_2+m_3+m_4+4}+t^{m_2+m_3+m_5+4}\\
&\quad+t^{m_2+m_4+m_5+4}+t^{m_3+m_4+m_5+4}+t^{m_1+m_2+m_4+3}+t^{m_1+m_2+m_5+3}+t^{m_1+m_3+m_4+3}+t^{m_1+m_3+m_5+3}-t^{m_2+m_4+4}\\
&\quad-t^{m_2+m_5+4}-t^{m_3+m_4+4}-t^{m_3+m_5+4}-2t^{m_2+m_3+3}-2t^{m_4+m_5+3}-t^{m_1+m_2+3}-t^{m_1+m_3+3}-t^{m_1+m_4+3}\\
&\quad-t^{m_1+m_5+3}+t^{m_2+m_3+2}+t^{m_4+m_5+2}+t^{m_1+3}+2t^{m_2+3}+2t^{m_3+3}+2t^{m_4+3}+2t^{m_5+3}-t^{m_2+2}-t^{m_3+2}\\
&\quad-t^{m_4+2}-t^{m_5+2}-4t^2+4t-1\\
&=:\psi_{H1}\left(t\right)
\end{aligned}
$$

Let $H_1\left(t\right)=H\left(t\right)$, without loss of generality, we assume that $l_2 \geq l_3 \geq l_4 \geq l_5$ and $m_2 \geq m_3 \geq m_4 \geq m_5$. Comparing the 4th lowest term of $H\left(t\right)$ and $H_1\left(t\right)$, we get $m_5 = l_5$. Similarly, we comparing the 5th, 6th and 7th lowest term of $H\left(t\right)$ and $H_1\left(t\right)$, respectively, we get $m_4 = l_4$, $m_3 = l_3$ and $m_2 = l_2$. By $m_1+m_2+m_3+m_4+m_5+2=l_1+l_2+l_3+l_4+l_5+2=n$, we get $m_1 = l_1$, thus $H\left(m_1,m_2,m_3,m_4,m_5\right) \cong H\left(l_1,l_2,l_3,l_4,l_5\right)$. □

Lemma 3.10 Let G_5 be a graph in Figure, then G_5 and H-shape are not cospectral.

Proof. Let $G_5-u-v=P_{m_1}\cup P_{m_2}\cup P_{m_3}\cup P_{m_4}\cup P_{m_5-1}=1 \quad \left(m_i \geq 1, i=1,2,3,4. m_5 \geq 4\right)$, that is $m_1+m_2+m_3+m_4+m_5+1=n$. Denote the first component by $G_{5,1}$ and the second component by $G_{5,2}$. By Lemma 2.1 and Lemma 2.3 we have

$$\varphi(G_{5,1},\lambda)=\varphi(C_{m_5},\lambda)\varphi(P_{m_4},\lambda)-\varphi(P_{m_5-1},\lambda)\varphi(P_{m_4-1},\lambda)$$

$$\begin{aligned}&\varphi\left(G_{5,1},t^{1/2}+t^{-1/2}\right)\\&=t^{-m_5/2}\left(t^{m_5/2}-1\right)^2\frac{t^{-m_4/2}}{t-1}\left(t^{m_4+1}-1\right)-\frac{t^{-(m_4-1)/2}}{t-1}\left(t^{m_4}-1\right)\frac{t^{-(m_5-1)/2}}{t-1}\left(t^{m_5}-1\right)\\&=\frac{t^{-(m_4+m_5)/2}}{(t-1)^2}\left[(t-1)\left(t^{m_5/2}-1\right)^2\left(t^{m_4+1}-1\right)-t\left(t^{m_4}-1\right)\left(t^{m_5}-1\right)\right],\end{aligned}$$

$$\varphi(G_{5,2},\lambda)=\varphi(P_{m_1},\lambda)\varphi(P_{m_2+m_3+1},\lambda)-\varphi(P_{m_1-1},\lambda)(P_{m_2},\lambda)(P_{m_3},\lambda)$$

$$\begin{aligned}&\varphi\left(G_{5,2},t^{1/2}+t^{-1/2}\right)\\&=\frac{t^{-m_1/2}}{t-1}\left(t^{m_1+1}-1\right)\frac{t^{-(m_2+m_3+1)/2}}{t-1}\left(t^{m_2+m_3+2}-1\right)-\frac{t^{-(m_1-1)/2}}{t-1}\left(t^{m_1}-1\right)\frac{t^{-m_2/2}}{t-1}\left(t^{m_2+1}-1\right)\frac{t^{-m_3/2}}{t-1}\left(t^{m_3+1}-1\right)\\&=\frac{t^{-(m_1+m_2+m_3+1)/2}}{(t-1)^2}\left[\left(t^{m_1+1}-1\right)\left(t^{m_2+m_3+2}-1\right)-t\left(t^{m_1}-1\right)\left(t^{m_2+1}-1\right)\left(t^{m_3+1}-1\right)\right].\end{aligned}$$

By Lemma 2.1 (a) we have

$$\begin{aligned}&\varphi\left(G_5,t^{1/2}+t^{-1/2}\right)t^{n/2}(t-1)^5\\&=\left[(t-1)\left(t^{m_5/2}-1\right)^2\left(t^{m_4+1}-1\right)-t\left(t^{m_4}-1\right)\left(t^{m_5}-1\right)\right]\times\left[\left(t^{m_1+1}-1\right)\left(t^{m_2+m_3+2}-1\right)-t\left(t^{m_1}-1\right)\left(t^{m_2+1}-1\right)\left(t^{m_3+1}-1\right)\right]\\&=:\psi_{G5}(t)\end{aligned}\tag{15}$$

Comparing (14) and (15), since $\psi_H(0)=-1$ for any $l_i\ (i=1,2,\cdots,5)$ and $\psi_{G5}(0)=1$ for any $m_i\ (i=1,2,\cdots,5)$, hence $\psi_H(t)\neq\psi_{G5}(t)$. G_5 and H-shape are not cospectral.

Remark. If G_5 uniting some C_{n_i}, without loss of generality, let $G_{5,1}=G_5\cup C_{n-n_1}$, where $m_1+m_2+m_3+m_4+m_5+1=n_1$. Since $\varphi\left(C_{n-n_1}t^{1/2}+t^{-1/2}\right)=t^{-(n-n_1)/2}\left(t^{(n-n_1)/2}-1\right)^2$, we have $\psi_{G5,1}(t)=\psi_{G5}(t)\ \left(t^{(n-n_1)/2}-1\right)^2$, $\psi_{G5,1}(0)=\psi_{G5}(0)=1$, $\psi_H(t)\neq\psi_{G5,1}(t)$. Thus, $G_{5,1}$ and H-shape are not cospectral. □

Theorem 3.11 Let $H=H(l_1,l_2,l_3,l_4,l_5)$ $(l_1\geq 0, l_i\geq 1, i=2,3,4,5)$, if a graph Γ ($\Gamma\neq W_n$) cospectral to an H-shape, then either Γ is U (Definition 1) uniting some even cycles C_{n_i} ($n_i\geq 6$), denoted by U', and U', H satisfying one of the following conditions.

1) There are one 6-cycle in U' and $l_1\geq 1$, l_2, l_3, l_4, $l_5\geq 2$.

2) There are one 6-cycle in U' and $l_1=0$, have 1 element is 1 in $\{l_2,l_3,l_4,l_5\}$.

3) No 6-cycle in U' and $l_1\geq 1$, have 2 elements are 1 in $\{l_2,l_3,l_4,l_5\}$.

4) No 6-cycle in U' and $l_1=0$, have 3 elements are 1 in $\{l_2,l_3,l_4,l_5\}$, or Γ is the graph G_3 and G_4 in Figure uniting some even cycles $C_{n_i}\ (n_i\geq 6)$, respectively.

Proof. This result is contained from Lemma 3.2 up to Lemma 3.10. □

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