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The Cartan Invariant Matrix for the Finite Group $G_2(13)$ of Type G_2 *

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Abstract

The Cartan invariant matrix $C=\left(c_{\lambda\mu}^{(1)}\right)_{\lambda,\mu\in X_1(T)}$ for the finite group $G_2(13)$ of type G_2 over the finite field $\mathbb{F}_{13}$ with 13 elements is determined in this paper.

Keywords

Cartan Invariant Matrix; Weyl Module; Projective Indecomposable Module; Simple Module

1. Introduction

Computing the Cartan invariant matrix C for a finite group of Lie type is an important research subject in the representation theory. One has made many great efforts for this.

Firstly, let finite group be of type A. C for $SL(2,2^n)$ and $SL(2,p^n)(p>2)$ are treated combinatorially by Alperin [1] [2] and Upadhyaya [3] that almost all Cartan invariants are powers of 2 but some exceptional ones for the form 2^r+2^{n-r}. Benson, Martin [4] and Humphreys [5] exhibit C for $SU(3,4)$ and $SU(3,5)$ $\left(\det C=5^{12}\right)$. C for $SL(3,p)$ is given by Humphreys [6] when $p=3,5$ $\left(\det C=3^4,5^6\right)$ and by Ye [7] when $p=7$ $\left(\det C=7^{12}\right)$. Zaslawsky [8] computes C for $PSL(3,q)$ when $q=2,3,4,5,7,8,9$ and for $PSU(3,q)$ when $q=2,3,4,5,7,8$. C for $SL(4,2)$ $\left(\det C=2^9\right)$ and $SL(4,4)$ is computed by Benson [9] and Du [10]. C for $SL(5,2)$ $\left(\det C=2^{17}\right)$ has been computed independently by Jantzen (unpublished work) and Ye [11].

Secondly, let finite group be of type B and C. Humphreys [6] and Thackray [12] work out independently C for $Spin(5,3)\cong Sp(4,3)$ $\left(\det C=3^{13}\right)$. Similar computations for $PSp(4,5)$ are summarized in [13] but C is not exhibited. Ye [14], Ye and Zhou [15], Y. Cheng (unpublished work), Liu and Ye [16], Hu and Xu [17]

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compute C for $Sp(4,7)$ $(\det C = 7^{14})$, $SO(7,3)$ $(\det C = 3^{41})$, $Sp(6,2)$, $Sp(6,3)$ $(\det C = 3^{51})$, $Sp(4,11)$ $(\det C = 11^{37})$, respectively.

At last, let finite group be of type G_2. C for $G_2(p)$ has been computed by Mertens [18] when $p = 2,3$ and Hu, Ye [19] when $p = 5$ $(\det C = 5^{16})$. In the present note, we shall compute C for the finite group $G_2(13)$ of type G_2 over a field $\mathbb{F}_{13}$ with 13 elements. Some computations involved in this paper were done by using a computer and the MATLAB software. We shall freely use the notations in [15] without further comments.

2. Preliminaries

Let G be a simply-connected semisimple algebraic group of type G_2 over an algebraically closed field K with characteristic $p > 0$. Take a maximal torus T of G such that $X(T)$ is the weight lattice of G with respect to T. Let $R \subset X(T)$ be the root system associated to (G,T) with the two simple roots α_1, α_2, where α_2 is the long simple root, then $R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$ is the set of positive roots, and X_R be root lattice in $X(T)$. Let $X(T)_+ = \left\{\lambda \in X(T) \middle| \langle \lambda, \alpha^\vee \rangle \ge 0, \forall \alpha \in R_+\right\}$ be the set of dominant weights with the corresponding fundamental weights ω_1, ω_2 satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, and denote by $\lambda = (\lambda_1, \lambda_2)$ the weight $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 \in X(T)$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$. Then $X(T)_+ = \left\{(\lambda_1,\lambda_2) \in X(T) \middle| \lambda_1, \lambda_2 \in \mathbb{Z}^{\ge 0}\right\}$.

It is well known that $V(\lambda)$ with $\lambda \in X(T)_+$ is the Weyl module of the highest weight λ with the unique simple quotient module $L(\lambda)$. In this way, $X(T)_+$ parameterizes the finite dimensional simple G-modules.

Let $G_{(n)} \subset G$ denote the scheme-theoretic kernel of the n-th Frobenius morphism F^n of G and $V^{[n]}$ the Frobenius twist for any G-module V. It is well known that $V^{[n]}$ is trivial as a $G_{(n)}$-module. Moreover, any G-module that becomes trivial upon restriction to $G_{(n)}$ is of this form. Let $X_n(T) = \left\{(\lambda_1,\lambda_2) \in X(T)_+ \middle| \lambda_1, \lambda_2 < p^n\right\}$ be the set of restricted dominant weights, then the simple G-modules $L(\lambda)$'s with $\lambda \in X_n(T)$ remain simple regarded as the $G_{(n)}$-modules. On the other hand, any simple $G_{(n)}$-module is isomorphic to exactly one of them. Denote by $\hat{L}_n(\lambda)$ the simple $G_{(n)}T$-modules with $\lambda \in X(T)_+$, whose projective cover is $\hat{Q}_n(\lambda)$. Then $X_n(T)$ is an index set of isomorphic classes of simple $G_{(n)}T$-modules. Let $G_2(p^n)$ be the finite subgroup consisting of all fixed points of F^n in G, which is called the finite group of type G_2. The following facts are well known. For $\lambda \in X_n(T)$, the restriction of the simple G-module $L(\lambda)$ to $G_2(p^n)$, denoted by $L_n(\lambda)$, remains simple. Furthermore, any simple $G_2(p^n)$-module is isomorphic to exactly one $L_n(\lambda)$ with $\lambda \in X_n(T)$. We denote by $U_n(\lambda)$ the projective indecomposable $G_2(p^n)$-module (or $G_2(p^n)$-PIM, for short), which has $L_n(\lambda)$ as its top and bottom composition factors. Moreover, the restriction $\hat{Q}_n(\lambda)\big|_{G_2(p^n)}$ is also a projective $G_2(p^n)$-module and it is decomposed into a direct sum of $G_2(p^n)$-PIM's such that $U_n(\lambda)$ occurs exactly once. $X_n(T)$ is also an index set of isomorphic classes of simple $G_2(p^n)$-modules $L_n(\lambda)$ and of projective indecomposable $G_2(p^n)$-modules $U_n(\lambda)$.

By the definition, the Cartan invariant $c_{\lambda\mu}^{(n)}(\lambda, \mu \in X_n(T))$ of $G_2(p^n)$ is the multiplicity of the simple $G_2(p^n)$-module $L_n(\mu)$ occurring as a $G_2(p^n)$-composition factor of the projective indecomposable $G_2(p^n)$-module $U_n(\lambda)$, *i.e.* $c_{\lambda\mu}^{(n)} = [U_n(\lambda) : L_n(\mu)]_{G_2(p^n)}$. Symmetric matrix $C = \left(c_{\lambda\mu}^{(n)}\right)_{\lambda,\mu \in X_n(T)}$ is called the Cartan invariant matrix, it is of p^{nm} order.

From now on, we assume $n = 1, p = 13$. For any $\lambda \in X(T)_+$, write $\operatorname{ch}(\lambda) = \operatorname{ch}(V(\lambda))$, $\operatorname{ch}_1(\lambda) = \operatorname{ch}(L(\lambda))$, $\Phi_1(\lambda) = \operatorname{ch}(\hat{Q}_1(\lambda))$. And for any $\lambda \in X_1(T)$, write $\operatorname{ch}^1(\lambda) = \operatorname{ch}(L_1(\lambda))$, $\Psi_1(\lambda) = \operatorname{ch}(U_1(\lambda))$.

3. *G*-Composition Factors of $V(\lambda)\left(\lambda \in X(T)_+\right)$

Let W_{13} be the affine Weyl group of G, which is generated by all affine reflections $s_{\beta,13n}$ with $\beta \in R_+, n \in \mathbb{Z}$, where $s_{\beta,13n}$ sends λ to $s_{\beta,13n}(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta + 13n\beta$. Dot action $\cdot$ of W_{13} on $X(T)$ is defined as follows: $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in W_{13}$ and $\lambda \in X(T)$. For this dot action, the origin of $X(T)$ is placed at $-\rho = (-1,-1)$, and $X(T)$ is partitioned into alcoves, whose closures are fundamental domains for the action of W_{13} on the euclidean space $X(T)\otimes\mathbb{R}$. In case of this paper, $X_1(T)$ lie in a parallelogram with lowest point $-\rho$ and highest point $(12,12)$, which is a union of 12 restricted alcoves (see **Figure 1**). Following [20], any W_{13}-translate of this parallelogram is called a "box"; highest point of every box is a special point (intersection of all possible types of affine reflecting hyperplanes). We number some of the dominant alcoves as in **Figure 2** for easy reference. We say that an alcove is of type 1 if it is a $13X_R$-translate of the alcove marked 1, and so on, for the various restricted alcoves marked 1, 2, 3, 4, 5, 6, 7, 8, 11, 13, 15, 16.

$X_1(T)$ consists of highest weight $(12,12)$, weights inside 12 alcoves and weights on 18 walls. Let k be any positive integer. We denote by k the set of all weights $\lambda=(r,s)$ inside the alcove marked k. Then weights inside 12 restricted alcoves can be represented as follows:

$1=\{(r,s)\in X_1(T)|2r+3s<8\}$;

$2=\{(r,s)\in X_1(T)|r+3s<9,2r+3s>8\}$;

$3=\{(r,s)\in X_1(T)|r+2s<10,r+3s>9\}$;

$4=\{(r,s)\in X_1(T)|2r+3s<21,r+2s>10\}$;

$5=\{(r,s)\in X_1(T)|r+3s<22,r+s<11,2r+3s>21\}$;

$6=\{(r,s)\in X_1(T)|r+s<11,r+3s>22\}$;

$7=\{(r,s)\in X_1(T)|r+3s<22,r+s>11\}$;

$8=\{(r,s)\in X_1(T)|2r+3s<34,r+s>11,r+3s>22\}$;

$11=\{(r,s)\in X_1(T)|r+2s<23,2r+3s>34\}$;

$13=\{(r,s)\in X_1(T)|r+3s<35,r+2s>23\}$;

$15=\{(r,s)\in X_1(T)|2r+3s<47,r+3s>35\}$;

$16=\{(r,s)\in X_1(T)|2r+3s>47\}$.

And, we denote by $\bar{k}$, $\bar{\bar{k}}$, $\bar{\bar{\bar{k}}}$ the sets of all weights on the short right-angled edge, the long right-angled

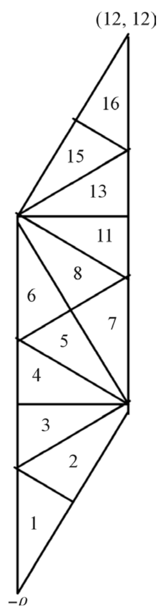

Figure 1. 12 restricted alcoves.

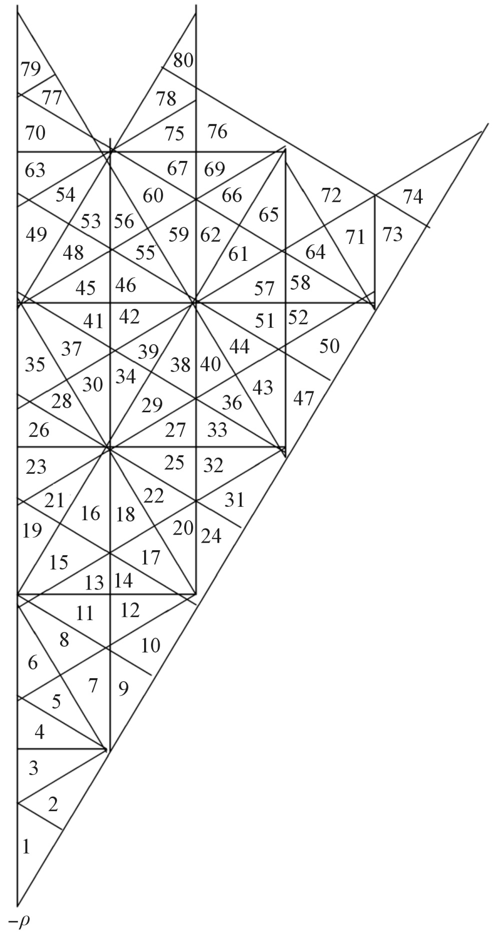

Figure 2. 80 dominant alcoves.

edge, the incline edge of the right triangle which circles the above marked k, respectively, then we can describe weights on 18 walls. For example, $\bar{1}=\{(r,s)\in X_1(T) \mid 2r+3s=8\}$,

$\bar{\bar{6}}=\left\{(r,s)\in X_1(T) \,\middle|\, r+s=11, \frac{11}{2}<s<12\right\}$, $\overline{\overline{\overline{13}}}=\{(r,s)\in X_1(T) \mid r+3s=35\}$, and so on. It is well known that Weyl group $W=N_G(T)/T$ of G is a Coxeter group generated by $w_1=s_{\alpha_1,0}$ and $w_2=s_{\alpha_2,0}$. Write $w_{12\cdots}=w_1w_2\cdots, w_{21\cdots}=w_2w_1\cdots$, then $W=\{1,w_1,w_2,w_{12},w_{21},w_{121},w_{212},w_{1212},w_{2121},w_{12121},w_{21212},w_0\}$, where $w_0=w_{121212}=w_{212121}$. And, the dot action of W_{13} on $X(T)$ induces the dot action of W on $X(T)$.

$\lambda,\mu\in X_1(T)$, saying that λ,μ is linked in $X_1(T)$ means that there exists $w\in W$ such that $\mu\equiv w\cdot\lambda(\mathrm{mod}\,13X(T))$, and write $\mu=\lambda(w)$.

Recall some results of Jantzen [21] or Humphreys [22] on the generic decomposition patterns of Weyl modules, limiting ourselves to weights which lie in the lowest 13^2-alcove (an alcove for the affine Weyl group relative to 13^2). When a dominant weight λ lies inside an alcove sufficiently far from the walls of the dominant Weyl chamber, the pattern of G-composition factors of $V(\lambda)$ depends only on the type of alcove in which λ lies. The corresponding "generic decomposition pattern" consists of the alcoves which contain reflected weights $\mu=w\cdot\lambda(w\in W_{13})$ for which $L(\mu)$ occurs as a G-composition factor of $V(\lambda)$; each such alcove is labelled with the multiplicity of $L(\mu)$ as a G-composition factor. For type G_2 there are 12 type of alcoves, corresponding to the 12 alcoves in the restricted box; hence there are 12 generic patterns. In particular, all patterns involve the same number of alcoves and the same distribution of multiplicities. For type G_2 the total number of composition factors is 119. **Figure 3** shows generic decomposition pattern of $V(\lambda)$ with λ lying inside alcove of type 1. Here the bold alcove marked $\otimes$ means the alcove in which the highest weight of $V(\lambda)$ lies, and $L(\lambda)$ appears once as G-composition factor of $V(\lambda)$, other alcoves marked digits 1 to 4 refer to those alcoves in which the remaining G-composition factors $L(\mu)$'s with the corresponding multiplicities 1 to 4 of $V(\lambda)$ except $L(\lambda)$ lie.

Now we determine the G-composition factors of $V(\lambda)$ with $\lambda \in X(T)_+$. If λ lies inside an alcove, we consider each alcove in the generic pattern which lie outside the dominant chamber. Find the special point at the top of the unique box in which that alcove lies. If that point lies on a reflecting hyperplane through $-\rho$, discard the alcove. Otherwise there is a unique element $w \in W$ taking that point to the special point at the top of a box in the dominant chamber. Find the alcove in this box corresponding to the given alcove, and attach to it the multiplicity in the given alcove, with a sign equal to det w. After carrying out this process for all alcoves, and cancelling multiplicities if necessary, the end result is the pattern of G-composition factors of $V(\lambda)$ with $\lambda \in X(T)_+$. **Figure 4** illustrates this algorithm when λ is inside alcove 25 (marked $\otimes$) which is of type 11. The two long bold lines indicate the walls of the dominant chamber, *i.e.*, w_1-wall and w_2-wall. The bold alcoves marked digits 2, 1, 1, 1, 1, 1, 1 to the right of the w_2-wall correspond to the dominant alcoves marked digits 4, 3, 3, 3, 2, 1, 2, respectively. A single reflection w_2 is involved here, so there are some multiplicity cancellations of alcoves marked digits 4, 3, 3, 3, 2, 1, 2 from the figure. All other alcoves to the right of the w_2-wall disappear, since the special points at the tops of their boxes lie on reflecting hyperplanes through $-\rho$. Similarly, The bold alcoves marked digits 1, 1, 1, 1 to the left of the w_1-wall correspond to the dominant alcoves marked digits 4, 3, 3, 3, respectively. A single reflection w_1 is involved here, so there are some multiplicity cancellations of alcoves marked digits 4, 3, 3, 3 from the figure. All other alcoves to the left of the w_1-wall also disappear, since the special points at the tops of their boxes lie also on reflecting hyperplanes through $-\rho$. So the end result is that $V(\lambda)$ will have 27 G-composition factors, corresponding to λ inside alcove

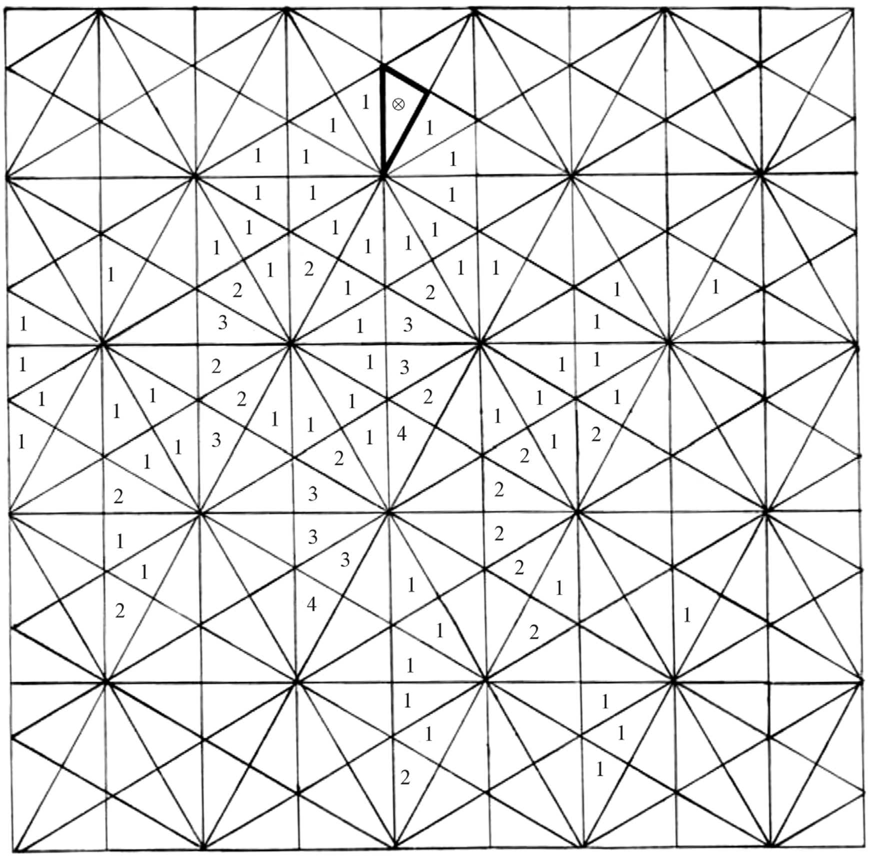

Figure 3. General decomposition pattern for alcove of type 1.

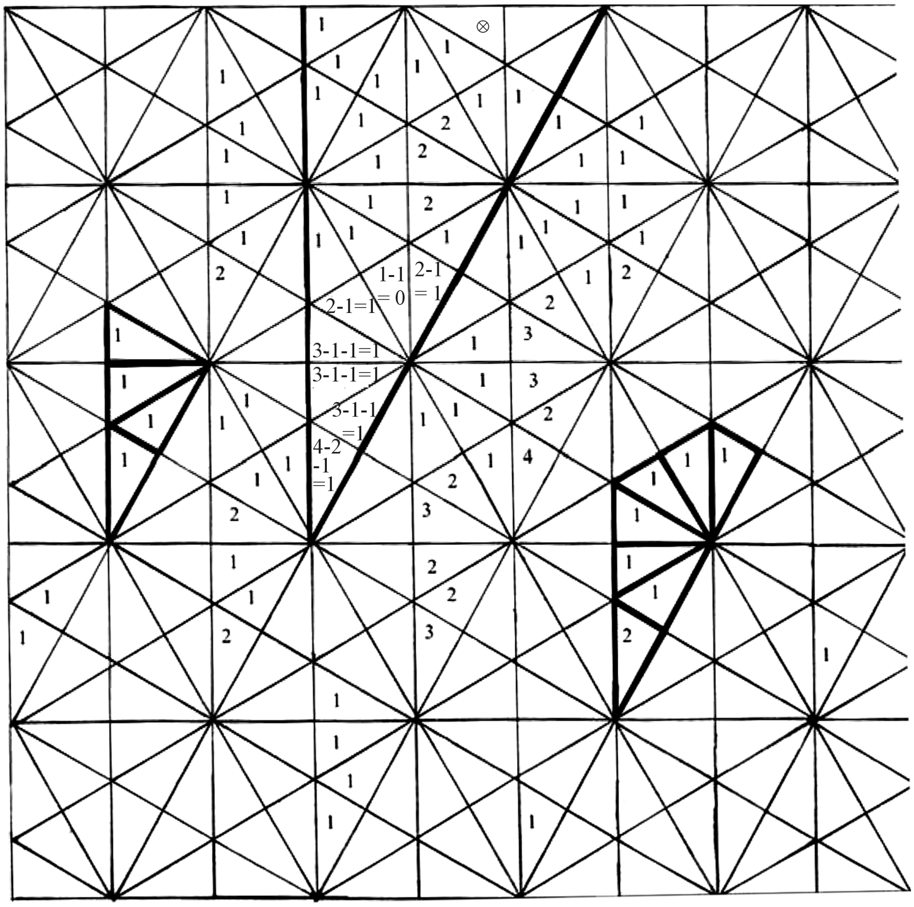

Figure 4. Patterns of G-composition factors of alcove 25.

25 and the reflected weights inside alcoves 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12(twice), 13, 14(twice), 15, 16, 17(twice), 18, 19, 20, 21, 22, 23, 24.

For a weight λ not lying inside an alcove, the G-composition factors of $V(\lambda)$ are obtained by using Jantzen's translation principle (cf. [23]). Find the alcove in whose "upper closure" λ lies and compute the pattern as above for an interior weight of this alcove. Then translate all weights involved to the type of wall in which λ lies; only those in upper closures of alcoves survive to give composition factors of $V(\lambda)$. For example, when $\lambda \in \overline{\overline{25}}$, $V(\lambda)$ will have 13 G-composition factors, corresponding to λ on wall $\overline{\overline{25}}$ and the reflected weights on walls $\overline{\overline{3}}$, $\overline{\overline{5}}$, $\overline{\overline{6}}$, $\overline{\overline{11}}$, $\overline{\overline{12}}$ (twice), $\overline{\overline{15}}$, $\overline{\overline{16}}$, $\overline{\overline{17}}$ (twice), $\overline{\overline{18}}$, $\overline{\overline{23}}$.

In particular, we have the following Proposition.

Proposition 3.1 The decomposition patterns of G-composition factors $L(\mu)$'s of $V(\lambda)$ with $\lambda \in X_1(T)$ are listed as follows (**Table 1**). Here μ's are the reflected weights of λ under W_{13}.

4. Weyl Filtrations of $G_{(1)}T$ − Module $\hat{Q}_1(\lambda)(\lambda \in X_1(T))$

Jantzen [24] shows that when $p \geq 2h-2$, principal indecomposable $G_{(1)}T$-module $\hat{Q}_1(\lambda)(\lambda \in X_1(T))$ admits a G-module filtration with Weyl G-modules as subquotients so called Weyl filtration, and the times of Weyl G-module $V(\mu)$ appearing as a subquotient in Weyl filtration of $\hat{Q}_1(\lambda)$ equals to the multiplicity of the simple G-module $L(\lambda)$ occurring as a G-composition factor of Weyl G-module $V(\mu)$, *i.e.*, $\left[\hat{Q}_1(\lambda):V(\mu)\right]_G = \left[V(\mu):L(\lambda)\right]_G$. In the case of this paper, $h=6, p=13, p>2h-2$. By a lots of concrete computations, we find that when $\lambda \in X_1(T)$, there are at most 80 dominant alcoves corresponding to the highest weights of Weyl G-modules occurring as subquotients in Weyl filtration of $\hat{Q}_1(\lambda)$ with $\lambda \in X_1(T)$, these 80 dominant alcoves are figured in **Figure 2**. So we can get Weyl G-modules $V(\mu)$'s occurring as subquotients in Weyl filtration of $\hat{Q}_1(\lambda)$ with $\lambda \in X_1(T)$ from decomposition patterns described in §3. For λ lying inside restricted dominant alcoves, we have the following Proposition 4.1.

Proposition 4.1 Assume that digit 1 denotes a typical weight inside the alcove 1, digits $2,3,\cdots,80$ denote the reflected weights inside the alcoves $2,3,\cdots,80$ under W_{13}, then we have the following **Table 2**. Here the first column denotes the principal indecomposable $G_{(1)}T$-modules $\hat{Q}_1(\lambda)$, the second column denotes the corresponding Weyl Gmodules $V(\mu)$'s, and n^2 means that n occurs twice.

Similarly, we can obtain Weyl G-modules $V(\mu)$'s occurring as subquotients in Weyl filtration of $\hat{Q}_1(\lambda)$ with λ on walls, *i.e.*, upper closures of restricted dominant alcoves.

For example, $\lambda=(11,11)$ lying inside alcove 16, we can get from the last row of **Table 2** that

$$\Phi_1(11,11)=\mathrm{ch}(11,11)+\mathrm{ch}(13,10)+\mathrm{ch}(8,13)+\mathrm{ch}(16,9)+\mathrm{ch}(7,14)+\mathrm{ch}(17,9)+\mathrm{ch}(7,15)$$
$$+\mathrm{ch}(17,10)+\mathrm{ch}(8,15)+\mathrm{ch}(16,11)+\mathrm{ch}(11,14)+\mathrm{ch}(13,13),$$

and $\lambda=(11,12)\in\overline{\overline{16}}$, so we have

$$\Phi_1(11,12)=\mathrm{ch}(11,12)+\mathrm{ch}(13,11)+\mathrm{ch}(10,13)+\mathrm{ch}(13,11)+\mathrm{ch}(11,13)+\mathrm{ch}(13,12).$$

By Proposition 4.1, we get Weyl G-module $V(\mu)$'s decompositions of $\hat{Q}_1(\lambda)$ with $\lambda\in X_1(T)$, and by the methods given in §3, we obtain G-composition factors of Weyl G-module $V(\mu)$'s, so we can get all G-composition factors of $\hat{Q}_1(\lambda)$ with $\lambda\in X_1(T)$. For example,

Table 1. G-composition factors L(μ)'s of $V(\lambda)(\lambda\in X_1(T))$.

λ	μ's	λ	μ's	λ	μ's	λ	μ's	λ	μ's
8	4 5 6 7 8	$\overline{\overline{8}}$	$\overline{\overline{\overline{4}}}$ $\overline{\overline{\overline{7}}}$ $\overline{\overline{\overline{8}}}$	2	1 2	$\overline{1}$	$\overline{1}$	1	1
11	3 4 5 7 8 9 11	$\overline{\overline{11}}$	$\overline{\overline{3}}$ $\overline{\overline{5}}$ $\overline{\overline{11}}$	3	2 3	$\overline{\overline{\overline{2}}}$	$\overline{\overline{\overline{2}}}$	$\overline{\overline{3}}$	$\overline{\overline{3}}$
$\overline{11}$	$\overline{5}$ $\overline{7}$ $\overline{9}$ $\overline{11}$	$\overline{13}$	$\overline{5}$ $\overline{11}$ $\overline{13}$	4	3 4	$\overline{\overline{6}}$	$\overline{\overline{5}}$ $\overline{\overline{6}}$	$\overline{\overline{\overline{4}}}$	$\overline{\overline{\overline{4}}}$
13	23 4 5 11 13	$\overline{\overline{\overline{13}}}$	$\overline{\overline{\overline{2}}}$ $\overline{\overline{\overline{4}}}$ $\overline{\overline{\overline{13}}}$	5	4 5	$\overline{7}$	$\overline{5}$ $\overline{7}$	$\overline{5}$	$\overline{5}$
15	12 3 4 13 15	$\overline{15}$	$\overline{1}$ $\overline{13}$ $\overline{15}$	6	5 6	$\overline{\overline{15}}$	$\overline{\overline{3}}$ $\overline{\overline{15}}$	$\overline{\overline{5}}$	$\overline{\overline{5}}$
16	12 13 14 15 16	$\overline{\overline{\overline{16}}}$	$\overline{\overline{\overline{2}}}$ $\overline{\overline{\overline{13}}}$ $\overline{\overline{\overline{14}}}$ $\overline{\overline{\overline{16}}}$	7	5 7	$\overline{\overline{16}}$	$\overline{\overline{15}}$ $\overline{\overline{16}}$	$\overline{\overline{\overline{7}}}$	$\overline{\overline{\overline{7}}}$

Table 2. V (μ)-decompositions of $\hat{Q}_1(\lambda)$.

1	1, 2, 15, 16, 20, 22, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 48, 49, 53, 54, 55, 56, 57, 58, 59, 60, 61, 65, 67, 69, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80
2	2, 3, 13, 15, 16, 18, 20, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 53, 54, 56, 57, 60, 61, 63, 67, 70, 71, 73, 75, 77, 78
3	3, 4, 11, 13, 15, 17, 18, 19, 20, 22, 25, 27, 28, 29, 30, 32^2, 33^2, 34, 35, 36, 37, 38^2, 39^2, 40, 41, 42, 43, 44, 45, 46, 51, 53, 54, 56, 57, 60, 61, 62, 63, 64, 67, 70, 71, 75
4	4, 5, 8, 11, 13, 14, 15, 17, 22, 25, 27, 29, 30, 31, 32, 33^2, 34, 36^2, 37, 38^2, 39^2, 40^2, 41, 42^2, 43, 44^2, 45, 46, 47, 48, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 67
5	5, 6, 7, 8, 11, 12, 13, 14, 25, 27, 29, 32, 33, 34, 36^2, 38, 39, 40^2, 41, 42, 43^2, 44^2, 45, 46, 48, 49, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60
6	6, 8, 12, 14, 25, 27, 36, 38, 39, 40, 42, 43, 44, 46, 48, 49, 51, 55, 57, 59,
7	7, 8, 9, 10, 11, 12, 27, 29, 33, 36, 38, 40, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 55, 56
8	8, 10, 11, 12, 14, 17, 22, 25, 27, 29, 33, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 51, 55
11	11, 12, 13, 14, 17, 18, 20, 22, 25, 27, 29, 32, 33, 34, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46
13	13, 14, 15, 16, 17, 18, 20, 22, 24, 25, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42
15	15, 16, 17, 18, 19, 20, 21, 22, 25, 27, 28, 29, 30, 32, 33, 34, 35, 37, 38, 39
16	16, 18, 21, 22, 23, 25, 26, 27, 28, 29, 30, 34

$$\begin{aligned}\Phi_1(11,11) = {} & 8\mathrm{ch}_1(0,0)+8\mathrm{ch}_1(13,0)+8\mathrm{ch}_1(0,13)+4\mathrm{ch}_1(26,0)+\mathrm{ch}_1(13,13)+10\mathrm{ch}_1(8,0)\\ & +6\mathrm{ch}_1(21,0)+10\mathrm{ch}_1(8,13)+8\mathrm{ch}_1(7,1)+12\mathrm{ch}_1(20,1)+8\mathrm{ch}_1(7,14)+6\mathrm{ch}_1(7,2)\\ & +18\mathrm{ch}_1(20,2)+6\mathrm{ch}_1(7,15)+4\mathrm{ch}_1(8,2)+12\mathrm{ch}_1(21,2)+4\mathrm{ch}_1(8,15)+2\mathrm{ch}_1(0,10)\\ & +10\mathrm{ch}_1(13,10)+2\mathrm{ch}_1(11,1)+6\mathrm{ch}_1(24,1)+2\mathrm{ch}_1(11,14)+4\mathrm{ch}_1(3,9)+8\mathrm{ch}_1(16,9)\\ & +4\mathrm{ch}_1(4,9)+6\mathrm{ch}_1(17,9)+8\mathrm{ch}_1(4,10)+4\mathrm{ch}_1(17,10)+10\mathrm{ch}_1(3,11)+2\mathrm{ch}_1(16,11)\\ & +12\mathrm{ch}_1(11,11)\end{aligned}$$

5. $G_2(13)$-Composition Factors of $\hat{Q}_1(\lambda)(\lambda \in X_1(T))$

Let $\Pi(\lambda)$ be the set of all weights of $V(\lambda)$ for $\lambda \in X(T)_+$, and $m_\lambda(\mu) = \dim V(\lambda)_\mu$ for $\mu \in \Pi(\lambda)$, *i.e.*, the multiplicity of μ in $V(\lambda)$. Some values of $m_\lambda(\mu)$ are determined in **Table 3**, where the first row contains the weights μ each of which labels the corresponding column, and the first column contains the wights λ each of which labels the corresponding row. The number lying in the intersection of the λ-row and μ-column is just $m_\lambda(\mu)$.

For $\lambda', \lambda'' \in X_1(T)$., let $\mathrm{ch}(\lambda') = \sum_{\nu\in\Pi(\lambda')} m_{\lambda'}(\nu)e(\nu)$, then we have $\mathrm{ch}(\lambda')\mathrm{ch}(\lambda'') = \mathrm{ch}(V(\lambda')\otimes V(\lambda'')) = \sum_{\nu\in\Pi(\lambda')} m_{\lambda'}(\nu)\mathrm{ch}(\nu+\lambda'')$, where $\mathrm{ch}(\nu+\lambda'')=0$ if $\nu+\lambda''+\rho$ lie on wall of Weyl chamber or $\mathrm{ch}(\nu+\lambda'') = (\det w)\mathrm{ch}(w\cdot(\nu+\lambda''))$ if $\nu+\lambda''+\rho$ do not lie on wall of Weyl chamber and $w(\nu+\lambda''+\rho) \in X(T)_+$. By using **Table 3**, we have $\mathrm{ch}(\lambda')\mathrm{ch}(\lambda'') = \sum_{\nu\in X(T)_+} a_\nu \mathrm{ch}(\nu), a_\nu \in \mathbb{Z}^{>0}$, for all $\lambda', \lambda'' \in X_1(T)$.

For $\lambda \in X_1(T)$, by §3 and Proposition 4.1, we have $\Phi_1(\lambda) = \sum_{\mu\in X(T)_+} b_\mu \mathrm{ch}_1(\mu), b_\mu \in \mathbb{Z}^{>0}$, where $\mu = \mu_0 + 13\mu_1$, $\mu_0 \in X_1(T)$ is the weight linked to λ; $\mu_1 \in X_1(T)$ and $\mu_1 \preceq (2,1)$. As $G_2(13)$-module, we

Table 3. $m_\lambda(\mu)$ of type G_2.

$m_\lambda(\mu)$	(0,0)	(1,0)	(0,1)	(2,0)	(1,1)	(3,0)	(0,2)	(2,1)	(4,0)	(1,2)	(3,1)	(5,0)	(0,3)	(2,2)
(0,0)	1													
(1,0)	1	1												
(0,1)	2	1	1											
(2,0)	3	2	1	1										
(1,1)	4	4	2	2	1									
(3,0)	5	4	3	2	1	1								
(0,2)	5	3	3	2	1	1	1							
(2,1)	9	8	6	5	3	2	1	1						
(4,0)	8	7	5	5	3	2	1	1	1					
(1,2)	10	10	7	7	5	3	2	2	1	1				
(3,1)	16	14	12	10	7	6	4	3	2	1	1			
(5,0)	12	11	9	8	6	5	3	3	2	1	1	1		
(0,3)	9	7	7	5	4	4	3	2	1	1	1	0	1	
(2,2)	21	19	16	15	11	9	7	6	4	3	2	1	1	1

have $L(\mu)=L(\mu_0)\otimes L(\mu_1)^{[1]}\cong L(\mu_0)\otimes L(\mu_1)$, so can determine $\mathrm{ch}_1(\mu_0)\mathrm{ch}_1(\mu_1)=\mathrm{ch}(L(\mu_0)\otimes L(\mu_1))$ as follows. In case of this paper, $\mathrm{ch}(\mu_1)=\mathrm{ch}_1(\mu_1)$. If $ch(\mu_0)=\mathrm{ch}_1(\mu_0)$, then $\mathrm{ch}_1(\mu_0)\mathrm{ch}_1(\mu_1)=\mathrm{ch}(\mu_0)\mathrm{ch}(\mu_1)$, *i.e.*, $L(\mu_0)\otimes L(\mu_1)=V(\mu_0)\otimes V(\mu_1)$, we get all G-composition factors of $L(\mu_0)\otimes L(\mu_1)$; if $\mathrm{ch}(\mu_0)=\mathrm{ch}_1(\mu_0)+\mathrm{ch}_1(\mu_0')+\cdots$, which μ_0' is maximal one in weights less than μ_0 linked to μ_0 in $X_1(T)$, then we have $\mathrm{ch}_1(\mu_0)\mathrm{ch}_1(\mu_1)=\mathrm{ch}(\mu_0)\mathrm{ch}(\mu_1)-\mathrm{ch}(\mu_0')\mathrm{ch}(\mu_1)-\cdots$. But weights less than μ_0 linked to μ_0 in $X_1(T)$ have at most 12, and the Weyl module corresponding to the smallest one is simple. Repeatedly, we have $\mathrm{ch}_1(\mu)=\mathrm{ch}_1(\mu_0)\mathrm{ch}_1(\mu_1)=\mathrm{ch}(L(\mu_0)\otimes L(\mu_1))=\sum_{\nu\in X(T)_+}c_\nu\mathrm{ch}_1(\nu)$, where $c_\nu\in\mathbb{Z}^{>0}$.

If $\nu\in X_1(T)$, then $L(\nu)=L_1(\nu)$; if $\nu\notin X_1(T)$, then $\nu=\nu_0+13\nu_1$, $\nu_0\in X_1(T),\nu_1=(1,0)$ or $(0,1)$, so $L(\nu)=L(\nu_0)\otimes L(\nu_1)^{[1]}\overset{G_2(13)}{\cong}L(\nu_0)\otimes L(\nu_1)$, return to the above case. In the end, we have $\mathrm{ch}_1(\mu_0)\mathrm{ch}_1(\mu_1)=\mathrm{ch}(L(\mu_0)\otimes L(\mu_1))=\sum_{\nu\in X_1(T)}d_\nu\mathrm{ch}^1(\nu)$, where $d_\nu\in\mathbb{Z}^{>0}$. so determine all $G_2(13)$-composition factors $L_1(\mu)$'s $(\mu\in X_1(T))$ of $\hat{Q}_1(\lambda)(\lambda\in X_1(T))$.

For example, we have the following expression:

$$\begin{aligned}\Phi_1(11,11)={}&26\mathrm{ch}^1(0,0)+50\mathrm{ch}^1(8,0)+64\mathrm{ch}^1(7,1)+66\mathrm{ch}^1(7,2)+44\mathrm{ch}^1(8,2)+10\mathrm{ch}^1(0,10)+22\mathrm{ch}^1(11,1)\\&+20\mathrm{ch}^1(3,9)+18\mathrm{ch}^1(4,9)+16\mathrm{ch}^1(4,10)+14\mathrm{ch}^1(3,11)+12\mathrm{ch}^1(11,11)+10\mathrm{ch}^1(0,1)+10\mathrm{ch}^1(5,1)\\&+8\mathrm{ch}^1(4,2)+6\mathrm{ch}^1(4,4)+4\mathrm{ch}^1(5,4)+2\mathrm{ch}^1(11,2)+28\mathrm{ch}^1(1,0)+26\mathrm{ch}^1(7,0)+28\mathrm{ch}^1(5,2)\\&+30\mathrm{ch}^1(5,3)+20\mathrm{ch}^1(7,3)+10\mathrm{ch}^1(1,9)+10\mathrm{ch}^1(10,2)+8\mathrm{ch}^1(4,8)+6\mathrm{ch}^1(6,8)+4\mathrm{ch}^1(6,9)\\&+2\mathrm{ch}^1(4,11)+\mathrm{ch}^1(1,1)+6\mathrm{ch}^1(2,0)+16\mathrm{ch}^1(6,1)+24\mathrm{ch}^1(6,3)+8\mathrm{ch}^1(12,1)+8\mathrm{ch}^1(5,8)\\&+4\mathrm{ch}^1(5,10)+16\mathrm{ch}^1(9,0)+24\mathrm{ch}^1(9,1)+8\mathrm{ch}^1(12,0)+8\mathrm{ch}^1(2,10)+4\mathrm{ch}^1(2,11)+8\mathrm{ch}^1(4,3)\\&+4\mathrm{ch}^1(8,3)+20\mathrm{ch}^1(6,2)+16\mathrm{ch}^1(9,2)+10\mathrm{ch}^1(2,9)+6\mathrm{ch}^1(5,9)+2\mathrm{ch}^1(2,12)+28\mathrm{ch}^1(8,1)\\&+20\mathrm{ch}^1(10,1)+10\mathrm{ch}^1(1,10)+6\mathrm{ch}^1(3,10)+2\mathrm{ch}^1(1,12)+8\mathrm{ch}^1(10,0)+4\mathrm{ch}^1(11,0).\end{aligned}$$

6. $U_1(\mu)'s$-Decomposition of $\hat{Q}_1(\lambda)(\lambda\in X_1(T))$

For any $\lambda\in X_1(T)$, restriction of $G_{(1)}T$-module $\hat{Q}_1(\lambda)$ on $G_2(13)$ is projective $G_2(13)$-module, and can be decomposed into the direct sum of the principal indecomposable $G_2(13)$-module $U_1(\mu)$, Chastkofsky [25] and Jantzen [26] show the decomposition formula, *i.e.*,

$$\left[\hat{Q}_1(\lambda):U_1(\mu)\right]_{G_2(13)}=\sum_{\nu\in X(T)_+}\left[L(\mu)\otimes L(\nu):L(13\nu+\lambda)\right]_G$$

Using this formula and the knowledge of decomposition of tensor products of two simple G-modules in the above section, and by a series of complicated computations, we have the following Proposition 6.1.

Proposition 6.1 The character $\Phi_1(\lambda)$ of $\hat{Q}_1(\lambda)$ for $\lambda\in X_1(T)$ can be expressed as a sum of character $\Psi_1(\mu)$ of $U_1(\mu)$ with $\mu\in X_1(T)$ as in the following **Table 4**.

By the Proposition 6.1 and the above §5, we can get all the Cartan invariants of $G_2(13)$ $c_{\lambda\mu}^{(1)}=\left[U_1(\lambda):L_1(\mu)\right]_{G_2(13)}$. For example, since $\Phi_1(11,11)=\Psi_1(11,11)$, from the above 4.3, we have $c_{(11,11)(0,0)}^{(1)}=26, c_{(11,11)(8,0)}^{(1)}=50, c_{(11,11)(7,1)}^{(1)}=64$, and so on.

7. Main Results

Let us arrange the row indices from left to right and the column indices from top to bottom in the following order: $\lambda,\lambda(w_{12121}),\lambda(w_{1212}),\lambda(w_{121}),\lambda(w_{12}),\lambda(w_{21212}),\lambda(w_1),\lambda(w_{2121}),\lambda(w_{212}),\lambda(w_{21}),\lambda(w_2),\lambda(w_0)$ for λ taking $(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(2,1),(3,0)$, successively;
$\lambda,\lambda(w_{12}),\lambda(w_1),\lambda(w_{212}),\lambda(w_{21}),\lambda(w_2)$ for λ taking $(4,0),(1,2)$, successively;

Table 4. $\Psi_1(\mu)$-decompositions of $\Phi_1(\lambda)(\lambda\in X_1(T))$.

$\Phi_1(0,0)$	$\Psi_1(0,0)$ $\Psi_1(12,0)$ $\Psi_1(11,1)$ $\Psi_1(12,1)$	$\Phi_1(1,0)$	$\Psi_1(1,0)$ $\Psi_1(12,1)$ $\Psi_1(5,9)$ $\Psi_1(7,8)$
	$\Psi_1(3,10)$ $\Psi_1(0,12)$ $\Psi_1(5,9)$ $\Psi_1(3,11)$		$\Psi_1(1,12)$ $\Psi_1(2,12)$ $\Psi_1(3,12)$ $\Psi_1(4,12)$
	$\Psi_1(2,12)$ $\Psi_1(3,12)$ $\Psi_1(10,12)$ $\Psi_1(12,12)$		$\Psi_1(4,11)$
$\Phi_1(2,0)$	$\Psi_1(2,0)$ $\Psi_1(7,8)$ $\Psi_1(9,7)$ $\Psi_1(5,11)$	$\Phi_1(6,0)$	$\Psi_1(6,0)$ $\Psi_1(9,11)$ $\Psi_1(6,12)$ $\Psi_1(7,12)$
	$\Psi_1(2,12)$ $\Psi_1(3,12)$ $\Psi_1(4,12)$ $\Psi_1(5,12)$		$\Psi_1(8,12)$ $\Psi_1(9,12)$
$\Phi_1(7,0)$	$\Psi_1(7,0)$ $\Psi_1(10,11)$ $\Psi_1(7,12)$ $\Psi_1(8,12)$	$\Phi_1(8,0)$	$\Psi_1(8,0)$ $\Psi_1(11,11)$ $\Psi_1(8,12)$ $\Psi_1(9,12)$
	$\Psi_1(9,12)$ $\Psi_1(10,12)$		$2\Psi_1(10,12)$ $\Psi_1(11,12)$
$\Phi_1(9,0)$	$\Psi_1(9,0)$ $\Psi_1(9,12)$ $\Psi_1(10,12)$ $\Psi_1(11,12)$	$\Phi_1(4,0)$	$\Psi_1(4,0)$ $\Psi_1(11,6)$ $\Psi_1(4,12)$ $\Psi_1(7,11)$
	$2\Psi_1(12,11)$ $5\Psi_1(12,12)$		$\Psi_1(5,12)$
$\Phi_1(5,0)$	$\Psi_1(5,0)$ $\Psi_1(5,12)$ $\Psi_1(8,11)$ $\Psi_1(7,12)$	$\Phi_1(10,0)$	$\Psi_1(10,0)$ $\Psi_1(10,12)$ $4\Psi_1(11,12)$
	$\Psi_1(8,12)$		$10\Psi_1(12,12)$
$\Phi_1(0,1)$	$\Psi_1(0,1)$ $\Psi_1(12,1)$ $\Psi_1(11,2)$ $\Psi_1(12,2)$	$\Phi_1(0,11)$	$\Psi_1(0,11)$ $\Psi_1(12,11)$ $2\Psi_1(11,12)$
	$\Psi_1(3,12)$		$5\Psi_1(12,12)$
$\Phi_1(9,1)$	$\Psi_1(9,1)$ $\Psi_1(10,12)$ $\Psi_1(11,12)$ $3\Psi_1(12,12)$	$\Phi_1(1,5)$	$\Psi_1(1,5)$ $\Psi_1(12,6)$ $\Psi_1(6,12)$ $\Psi_1(7,12)$
$\Phi_1(0,s)$	$\Psi_1(0,s)$ $\Psi_1(12,s)$ $\Psi_1(11,s+1)$ $\Psi_1(12,s+1)$	$\Phi_1(3,0)$	$\Psi_1(3,0)$ $\Psi_1(9,7)$ $\Psi_1(11,6)$ $\Psi_1(3,12)$
	$s=2,4,6,8,9,10$		$\Psi_1(4,12)$ $\Psi_1(6,11)$ $2\Psi_1(5,12)$
$\Phi_1(0,5)$	$\Psi_1(0,5)$ $\Psi_1(12,5)$ $\Psi_1(11,6)$	$\Phi_1(0,7)$	$\Psi_1(0,7)$ $\Psi_1(12,7)$ $2\Psi_1(11,8)$
$\Phi_1(1,1)$	$\Psi_1(1,1)$ $\Psi_1(12,2)$ $\Psi_1(4,12)$	$\Phi_1(3,4)$	$\Psi_1(3,4)$ $\Psi_1(7,12)$ $\Psi_1(8,12)$
$\Phi_1(5,3)$	$\Psi_1(5,3)$ $\Psi_1(8,12)$ $\Psi_1(9,12)$	$\Phi_1(7,2)$	$\Psi_1(7,2)$ $\Psi_1(9,12)$ $\Psi_1(10,12)$
$\Phi_1(12,0)$	$\Psi_1(12,0)$ $\Psi_1(12,12)$	$\Phi_1(0,3)$	$\Psi_1(0,3)$ $\Psi_1(12,3)$
$\Phi_1(0,12)$	$\Psi_1(0,12)$ $\Psi_1(12,12)$	$\Phi_1(1,11)$	$\Psi_1(1,11)$ $3\Psi_1(12,12)$
$\Phi_1(r,1)$	$\Psi_1(r,1)$ $\Psi_1(r+3,12)$ $r=2,3,4,5,6,7,8$	$\Phi_1(1,s)$	$\Psi_1(1,s)$ $\Psi_1(12,s+1)$ $s=2,3,4,6,8,9,10$
$\Phi_1(11,0)$	$\Psi_1(11,0)$ $2\Psi_1(11,12)$ $8\Psi_1(12,12)$	$\Phi_1(r,s)$	$\Psi_1(r,s)$ r,s are not listed above

$\lambda,\lambda(w_{21}),\lambda(w_{2121}),\lambda(w_1),\lambda(w_{121}),\lambda(w_2)$ for λ taking $(0,3),(3,2),(6,1),(9,0)$, successively; $\lambda,\lambda(w_{21}),\lambda(w_2),\lambda(w_1),\lambda(w_{121}),\lambda(w_{12})$ for λ taking $(0,5),(2,4),(4,3),(6,2),(8,1),(10,0)$, successively; (12,12). Then, we write the Cartan invariant matrix C for the finite group $G_2(13)$ of type G_2 as $C=\begin{pmatrix} A & \\ & 1\end{pmatrix}$. For convenience, we block A into $A=(A_{ij})_{7\times 7}$, where A_{ij} with $1\le i\le j\le 6$ is square matrix of 25 order, and $A_{ji}=A_{ij}^{\mathrm{T}}$, A_{i7} with $1\le i\le 6$ are 25×18 matrices, and $A_{7i}=A_{i7}^{\mathrm{T}}$, A_{77} are square matrices of 18 order. We list all matrices $A_{ij}\ (1\le i\le j\le 7)$ in **Table 5**, where the elements below diagonal are omitted for $A_{11},A_{22},A_{33},A_{77}$, the elements above diagonal are omitted for A_{44},A_{55},A_{66}.

Table 5. Matrices $A_{ij}\left(1 \le i \le j \le 7\right)$.

$A_{44} =$ (lower part) / $= A_{11}$. (upper part)

324			168	247	302	327	224	70	110	98	80	71	50	26	142	149	168	173	114	36	54	46	38	30	26	10	16
332	386			442	566	602	420	130	212	186	157	130	89	50	170	184	194	202	132	38	62	50	40	30	20	10	10
168	192	108			770	806	568	180	290	253	214	170	122	64	200	202	210	218	148	42	70	52	40	28	16	8	6
168	192	96	108			874	614	195	314	278	230	186	125	66	198	208	218	224	148	42	74	54	40	26	16	6	4
152	172	92	92	88			452	150	228	198	166	129	84	44	132	136	148	148	94	26	48	36	26	16	12	4	4
128	152	76	76	72	68			66	75	68	52	40	26	10	44	38	42	42	26	10	14	10	6	4	4	0	2
100	116	58	60	56	52	48			128	106	85	66	44	22	62	64	70	74	48	14	24	18	12	6	4	2	0
68	80	40	40	38	36	32	28			98	78	60	40	20	50	50	52	54	36	10	18	12	8	4	2	0	0
32	40	18	20	18	18	16	14	14			72	54	34	18	38	40	40	40	26	6	12	8	6	2	0	0	0
381	402	201	202	176	150	116	82	41	642			50	32	16	30	30	28	26	16	4	6	4	2	0	0	0	0
241	244	132	124	110	86	65	45	20	431	338			30	14	26	20	16	16	12	4	4	2	0	0	0	0	0
244	240	128	120	104	80	59	38	18	456	371	466			12	10	10	8	6	4	0	2	0	0	0	0	0	0
273	262	136	134	117	91	71	48	26	495	390	470	509			422	428	478	522	354	118	176	156	128	108	80	40	202
202	204	102	102	87	72	56	40	20	388	302	372	396	340			502	558	610	408	128	208	188	160	132	92	52	190
104	102	48	51	42	36	28	20	14	194	148	184	200	170	102			650	704	472	146	242	220	190	152	110	56	200
102	102	51	50	43	36	30	20	10	190	148	186	200	170	85	102			794	534	168	276	252	212	172	116	60	198
91	87	42	43	36	30	24	16	10	164	128	162	178	145	81	81	77			378	120	192	172	148	116	76	40	128
73	72	36	36	30	24	18	12	6	138	110	140	145	120	60	60	56	52			50	60	56	44	34	22	10	42
59	56	28	30	24	18	12	8	4	118	95	109	117	88	44	42	40	38	38			108	92	76	60	40	20	62
40	40	20	20	16	12	8	4	2	78	63	78	76	56	28	28	26	24	24	24			88	72	56	38	20	50
22	20	14	10	10	6	4	2	0	39	39	44	42	28	12	14	12	12	12	12	12			68	52	34	18	38
233	264	132	136	120	104	80	54	27	340	231	253	272	224	112	120	102	84	62	44	22	237			48	32	16	30
128	148	74	74	65	56	44	32	16	224	172	192	206	184	92	92	78	64	48	32	16	154	148			30	14	26
66	74	37	38	33	28	24	16	8	120	92	96	107	92	46	44	38	32	28	16	8	77	74	44			12	10
45	56	28	28	24	20	16	12	6	84	62	62	69	64	32	32	26	20	16	12	6	57	56	28	26			648

$A_{55} =$ (lower part) / $= A_{22}$. (upper part)

19			666	622	696	512	160	252	224	188	148	101	60	74	40	36	32	22	8	12	6	4	2	1	0	530	356
12	12			612	675	488	151	246	221	188	146	104	57	74	36	30	26	16	6	8	4	2	1	0	0	522	353
18	14	376			778	564	179	282	256	211	168	112	60	70	32	26	20	12	4	4	2	1	0	0	0	566	384
6	4	220	158			436	142	216	189	158	120	76	40	48	22	16	12	6	2	2	1	0	0	0	0	408	266
3	2	108	80	48			60	70	64	49	36	22	10	15	8	6	4	2	0	1	0	0	0	0	0	136	92
1	0	78	58	28	26			120	100	80	60	40	20	24	12	8	4	2	1	0	0	0	0	0	0	206	136
0	0	63	40	18	18	18			94	74	56	38	20	16	6	4	2	1	0	0	0	0	0	0	0	172	116
0	0	48	28	12	12	12	12			68	52	34	18	12	4	2	1	0	0	0	0	0	0	0	0	142	92
2	1	168	124	66	46	30	18	126			48	32	16	4	2	1	0	0	0	0	0	0	0	0	0	100	68
0	0	166	116	58	42	31	22	104	114			30	14	2	1	0	0	0	0	0	0	0	0	0	0	64	47
0	0	117	86	42	32	27	18	76	81	72			12	1	0	0	0	0	0	0	0	0	0	0	0	35	27
0	0	102	76	37	27	22	16	64	72	62	58			470	378	470	484	336	148	166	142	118	98	70	32	324	199
0	0	60	50	27	17	12	8	48	46	42	38	30			348	434	442	306	136	150	130	112	96	64	34	216	132
0	0	18	18	9	7	4	2	20	14	14	12	10	8			598	592	432	194	212	184	160	124	88	44	252	146
2	1	120	54	24	16	12	6	62	54	28	21	11	6	158			620	438	196	224	196	160	128	84	42	250	148
0	0	134	58	28	14	8	4	62	60	26	20	9	3	180	240			334	150	168	144	122	88	60	28	168	98
0	0	42	20	12	4	0	0	22	20	6	5	3	1	60	84	36			82	76	70	54	40	28	12	79	44
0	0	35	14	8	2	0	0	16	16	5	4	2	0	54	78	30	30			100	80	60	42	28	14	78	48
0	0	21	4	0	0	0	0	8	9	3	2	0	0	42	54	18	18	18			74	56	40	26	12	64	36
0	0	5	0	0	0	0	0	5	3	1	0	0	0	18	18	6	6	6	6			54	38	24	12	50	28
0	0	6	0	0	0	0	0	4	2	0	0	0	0	58	60	20	15	9	5	150			38	24	12	40	20
0	0	2	0	0	0	0	0	2	0	0	0	0	0	60	66	22	17	9	3	178	242			24	12	31	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	22	6	5	3	1	60	84	36			12	15	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	17	5	4	2	0	54	78	30	30			868	556
0	0	0	0	0	0	0	0	0	0	0	0	0	0	9	9	3	2	0	0	42	54	18	18	18			396

$$A_{66} = \begin{matrix}
6 & & & 492 & 518 & 352 & 158 & 172 & 154 & 136 & 106 & 76 & 36 & 402 & 248 & 302 & 315 & 222 & 114 & 110 & 96 & 79 & 58 & 39 & 20 & 602 & 330 & 216 \\
5 & 120 & & & 576 & 384 & 174 & 196 & 176 & 146 & 116 & 76 & 36 & 434 & 274 & 323 & 342 & 230 & 116 & 118 & 101 & 82 & 65 & 44 & 24 & 645 & 360 & 234 \\
3 & 176 & 294 & & & 278 & 122 & 140 & 122 & 106 & 76 & 52 & 24 & 300 & 182 & 222 & 226 & 156 & 77 & 76 & 66 & 57 & 44 & 32 & 16 & 438 & 240 & 164 \\
1 & 66 & 112 & 54 & & & 66 & 62 & 58 & 46 & 34 & 24 & 12 & 136 & 88 & 106 & 110 & 73 & 36 & 36 & 32 & 28 & 22 & 16 & 10 & 203 & 115 & 78 \\
0 & 56 & 98 & 42 & 40 & & & 84 & 68 & 52 & 36 & 24 & 12 & 152 & 94 & 110 & 116 & 76 & 38 & 38 & 33 & 28 & 24 & 16 & 8 & 216 & 120 & 82 \\
0 & 42 & 66 & 24 & 22 & 20 & & & 64 & 50 & 36 & 24 & 12 & 132 & 84 & 96 & 101 & 66 & 32 & 33 & 28 & 24 & 20 & 14 & 8 & 184 & 106 & 74 \\
0 & 16 & 24 & 8 & 6 & 6 & 6 & & & 50 & 36 & 24 & 12 & 112 & 66 & 79 & 82 & 57 & 28 & 28 & 24 & 20 & 16 & 12 & 6 & 152 & 84 & 63 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 86 & & & 36 & 24 & 12 & 84 & 49 & 58 & 65 & 44 & 22 & 24 & 20 & 16 & 12 & 8 & 4 & 112 & 62 & 47 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 110 & 192 & & & 24 & 12 & 57 & 32 & 39 & 44 & 32 & 16 & 16 & 14 & 12 & 8 & 4 & 2 & 71 & 39 & 33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 53 & 96 & 54 & & & 12 & 28 & 16 & 20 & 24 & 16 & 10 & 8 & 8 & 6 & 4 & 2 & 0 & 35 & 22 & 20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 43 & 76 & 38 & 34 & & & 536 & 360 & 448 & 458 & 326 & 160 & 162 & 138 & 118 & 98 & 68 & 34 & 414 & 229 & 162 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & 40 & 20 & 18 & 14 & & & 290 & 366 & 366 & 258 & 124 & 130 & 110 & 96 & 84 & 58 & 34 & 250 & 146 & 101 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 20 & 9 & 9 & 7 & 7 & & & 526 & 512 & 376 & 184 & 188 & 164 & 144 & 114 & 84 & 44 & 290 & 161 & 114 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 98 & 124 & 64 & 44 & 22 & 13 & 203 & & & 536 & 382 & 192 & 196 & 176 & 146 & 120 & 80 & 42 & 286 & 158 & 115 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & 92 & 46 & 32 & 16 & 8 & 146 & 120 & & & 298 & 148 & 148 & 130 & 112 & 84 & 56 & 28 & 186 & 100 & 78 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & 46 & 24 & 16 & 8 & 5 & 74 & 60 & 36 & & & 88 & 74 & 72 & 56 & 42 & 28 & 12 & 92 & 44 & 34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & 32 & 16 & 10 & 6 & 3 & 60 & 48 & 24 & 24 & & & 88 & 72 & 56 & 42 & 28 & 14 & 90 & 48 & 38 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 16 & 8 & 6 & 2 & 1 & 36 & 24 & 12 & 12 & 12 & & & 70 & 54 & 40 & 26 & 12 & 72 & 36 & 28 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 8 & 5 & 3 & 1 & 0 & 18 & 12 & 6 & 6 & 6 & 6 & & & 52 & 38 & 24 & 12 & 54 & 28 & 22 \\
0 & 14 & 20 & 6 & 4 & 1 & 0 & 38 & 28 & 15 & 7 & 3 & 0 & 128 & 98 & 51 & 37 & 19 & 10 & 198 & & & 38 & 24 & 12 & 40 & 20 & 17 \\
0 & 8 & 10 & 2 & 1 & 0 & 0 & 24 & 20 & 10 & 4 & 0 & 0 & 98 & 72 & 36 & 28 & 16 & 8 & 144 & 120 & & & 24 & 12 & 30 & 15 & 12 \\
0 & 3 & 4 & 1 & 0 & 0 & 0 & 13 & 10 & 4 & 2 & 0 & 0 & 51 & 36 & 18 & 14 & 8 & 5 & 72 & 60 & 36 & & & 12 & 15 & 11 & 8 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 7 & 4 & 2 & 0 & 0 & 0 & 37 & 28 & 14 & 10 & 6 & 3 & 60 & 48 & 24 & 24 & & & 1498 & 859 & 464 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 19 & 16 & 8 & 6 & 2 & 1 & 36 & 24 & 12 & 12 & 12 & & & 525 & 284 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 8 & 5 & 3 & 1 & 0 & 18 & 12 & 6 & 6 & 6 & 6 & & & 186
\end{matrix} = A_{33}.$$

$$A_{77} = \begin{pmatrix}
208 & 148 & 72 & 60 & 36 & 18 & 152 & 112 & 49 & 40 & 20 & 10 & 56 & 40 & 11 & 10 & 4 & 0 \\
 & 122 & 60 & 48 & 24 & 12 & 112 & 80 & 35 & 29 & 16 & 8 & 32 & 28 & 9 & 5 & 0 & 0 \\
 & & 36 & 24 & 12 & 6 & 57 & 39 & 18 & 14 & 8 & 5 & 14 & 13 & 4 & 2 & 0 & 0 \\
 & & & 24 & 12 & 6 & 40 & 29 & 14 & 10 & 6 & 3 & 7 & 5 & 2 & 0 & 0 & 0 \\
 & & & & 12 & 6 & 20 & 16 & 8 & 6 & 2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 6 & 10 & 8 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 256 & 196 & 77 & 73 & 39 & 18 & 176 & 150 & 38 & 56 & 27 & 12 \\
 & & & & & & & 160 & 62 & 58 & 28 & 12 & 146 & 124 & 30 & 49 & 25 & 12 \\
 & & & & & & & & 30 & 22 & 12 & 6 & 52 & 44 & 10 & 18 & 8 & 5 \\
 & & & & & & & & & 26 & 12 & 6 & 50 & 43 & 12 & 14 & 8 & 3 \\
 & & & & & & & & & & 12 & 6 & 25 & 23 & 6 & 8 & 2 & 1 \\
 & & & & & & & & & & & 6 & 12 & 12 & 5 & 3 & 1 & 0 \\
 & & & & & & & & & & & & 222 & 184 & 44 & 90 & 52 & 22 \\
 & & & & & & & & & & & & & 180 & 48 & 82 & 40 & 20 \\
 & & & & & & & & & & & & & & 18 & 18 & 10 & 4 \\
 & & & & & & & & & & & & & & & 46 & 22 & 12 \\
 & & & & & & & & & & & & & & & & 20 & 8 \\
 & & & & & & & & & & & & & & & & & 6
\end{pmatrix}$$

$$A_{12}=\begin{pmatrix}
10 & 6 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 209 & 181 & 236 & 249 & 176 & 72 & 98 & 80 & 66 & 44 & 30 & 18 & 68 & 42 \\
8 & 6 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 312 & 276 & 356 & 368 & 254 & 110 & 132 & 110 & 86 & 64 & 46 & 26 & 74 & 40 \\
6 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 396 & 354 & 470 & 470 & 334 & 148 & 166 & 140 & 114 & 84 & 58 & 28 & 78 & 36 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 416 & 368 & 476 & 488 & 340 & 146 & 174 & 144 & 114 & 88 & 58 & 30 & 72 & 32 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 283 & 252 & 338 & 340 & 246 & 106 & 122 & 100 & 82 & 60 & 40 & 20 & 48 & 22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 86 & 84 & 120 & 116 & 86 & 36 & 42 & 34 & 30 & 22 & 14 & 10 & 15 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 150 & 132 & 170 & 174 & 120 & 52 & 56 & 48 & 40 & 34 & 20 & 10 & 24 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 125 & 110 & 140 & 144 & 100 & 42 & 48 & 40 & 32 & 26 & 16 & 8 & 16 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 102 & 86 & 114 & 114 & 82 & 36 & 40 & 32 & 24 & 18 & 14 & 6 & 12 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 74 & 64 & 84 & 88 & 60 & 26 & 34 & 26 & 18 & 12 & 8 & 4 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & 46 & 58 & 58 & 40 & 16 & 20 & 16 & 14 & 8 & 4 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 28 & 26 & 28 & 30 & 20 & 10 & 10 & 8 & 6 & 4 & 2 & 0 & 1 & 0 \\
188 & 190 & 194 & 128 & 42 & 62 & 50 & 38 & 30 & 26 & 10 & 356 & 281 & 348 & 359 & 250 & 107 & 128 & 106 & 90 & 64 & 40 & 23 & 438 & 297 \\
200 & 198 & 204 & 134 & 38 & 64 & 50 & 40 & 30 & 20 & 10 & 388 & 300 & 352 & 374 & 252 & 110 & 130 & 108 & 86 & 65 & 45 & 26 & 449 & 314 \\
204 & 202 & 212 & 144 & 42 & 68 & 52 & 40 & 28 & 16 & 8 & 444 & 336 & 404 & 424 & 294 & 128 & 148 & 124 & 102 & 75 & 52 & 28 & 500 & 342 \\
206 & 212 & 220 & 146 & 42 & 72 & 54 & 40 & 26 & 16 & 6 & 488 & 366 & 430 & 460 & 308 & 132 & 158 & 133 & 107 & 83 & 56 & 30 & 543 & 372 \\
134 & 144 & 146 & 94 & 26 & 48 & 36 & 26 & 16 & 12 & 4 & 338 & 246 & 294 & 306 & 208 & 88 & 102 & 87 & 74 & 56 & 40 & 20 & 372 & 250 \\
38 & 42 & 42 & 26 & 10 & 14 & 10 & 6 & 4 & 4 & 0 & 107 & 84 & 100 & 102 & 68 & 28 & 33 & 28 & 26 & 20 & 14 & 10 & 121 & 86 \\
64 & 68 & 72 & 48 & 14 & 24 & 18 & 12 & 6 & 4 & 2 & 174 & 126 & 146 & 156 & 102 & 43 & 50 & 43 & 36 & 30 & 20 & 10 & 190 & 128 \\
50 & 52 & 54 & 36 & 10 & 18 & 12 & 8 & 4 & 2 & 0 & 148 & 108 & 124 & 133 & 87 & 36 & 43 & 36 & 30 & 24 & 16 & 8 & 160 & 110 \\
40 & 40 & 40 & 26 & 6 & 12 & 8 & 6 & 2 & 0 & 0 & 126 & 86 & 102 & 107 & 74 & 32 & 36 & 30 & 24 & 18 & 14 & 6 & 134 & 88 \\
30 & 28 & 26 & 16 & 4 & 6 & 4 & 2 & 0 & 0 & 0 & 92 & 65 & 75 & 83 & 56 & 24 & 30 & 24 & 18 & 12 & 8 & 4 & 96 & 66 \\
20 & 16 & 16 & 12 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 60 & 45 & 52 & 56 & 40 & 16 & 20 & 16 & 14 & 8 & 4 & 2 & 62 & 46 \\
10 & 8 & 6 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 33 & 26 & 28 & 30 & 20 & 10 & 10 & 8 & 6 & 4 & 2 & 0 & 34 & 26 \\
614 & 563 & 645 & 480 & 161 & 228 & 202 & 164 & 134 & 94 & 47 & 76 & 46 & 46 & 42 & 28 & 11 & 12 & 8 & 8 & 4 & 2 & 1 & 544 & 352
\end{pmatrix}.$$

$$A_{13}=\begin{pmatrix}
42 & 40 & 28 & 11 & 12 & 8 & 8 & 4 & 2 & 1 & 134 & 81 & 92 & 95 & 66 & 33 & 28 & 24 & 20 & 18 & 9 & 4 & 10 & 4 & 4 \\
36 & 32 & 22 & 8 & 12 & 6 & 4 & 2 & 1 & 0 & 149 & 86 & 100 & 100 & 66 & 34 & 30 & 24 & 18 & 13 & 6 & 0 & 8 & 0 & 0 \\
30 & 26 & 16 & 6 & 8 & 4 & 2 & 1 & 0 & 0 & 168 & 100 & 128 & 118 & 78 & 38 & 38 & 28 & 19 & 12 & 7 & 0 & 6 & 0 & 0 \\
26 & 20 & 12 & 4 & 4 & 2 & 1 & 0 & 0 & 0 & 171 & 100 & 118 & 112 & 74 & 36 & 36 & 25 & 16 & 9 & 4 & 0 & 2 & 0 & 0 \\
16 & 12 & 6 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 114 & 66 & 78 & 74 & 52 & 25 & 24 & 16 & 9 & 4 & 0 & 0 & 0 & 0 & 0 \\
6 & 4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 37 & 24 & 30 & 30 & 21 & 8 & 10 & 6 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\
8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 52 & 30 & 38 & 36 & 24 & 12 & 14 & 9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 42 & 24 & 28 & 25 & 16 & 6 & 9 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 18 & 19 & 16 & 9 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 13 & 12 & 9 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 6 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
334 & 363 & 252 & 109 & 130 & 110 & 94 & 66 & 42 & 24 & 426 & 275 & 348 & 359 & 250 & 126 & 128 & 110 & 92 & 70 & 48 & 21 & 280 & 150 & 106 \\
342 & 372 & 250 & 112 & 128 & 110 & 88 & 66 & 46 & 26 & 428 & 262 & 308 & 334 & 224 & 118 & 112 & 98 & 78 & 59 & 37 & 16 & 232 & 120 & 92 \\
380 & 408 & 280 & 122 & 140 & 120 & 100 & 74 & 52 & 28 & 494 & 300 & 358 & 380 & 268 & 138 & 130 & 112 & 92 & 67 & 42 & 20 & 230 & 120 & 100 \\
408 & 444 & 298 & 128 & 152 & 130 & 106 & 82 & 56 & 30 & 536 & 332 & 388 & 414 & 278 & 140 & 142 & 119 & 95 & 73 & 48 & 24 & 234 & 124 & 100 \\
280 & 298 & 204 & 86 & 100 & 86 & 74 & 56 & 40 & 20 & 364 & 220 & 268 & 274 & 186 & 92 & 94 & 79 & 66 & 48 & 36 & 16 & 154 & 78 & 62 \\
94 & 98 & 66 & 28 & 32 & 28 & 26 & 20 & 14 & 10 & 122 & 82 & 102 & 104 & 68 & 34 & 35 & 30 & 26 & 20 & 16 & 10 & 55 & 32 & 24 \\
140 & 152 & 100 & 42 & 48 & 42 & 36 & 30 & 20 & 10 & 184 & 110 & 130 & 140 & 94 & 47 & 50 & 41 & 32 & 24 & 16 & 8 & 72 & 36 & 30 \\
120 & 130 & 86 & 36 & 42 & 36 & 30 & 24 & 16 & 8 & 160 & 98 & 112 & 119 & 79 & 38 & 41 & 32 & 26 & 20 & 14 & 8 & 56 & 30 & 24 \\
100 & 106 & 74 & 32 & 36 & 30 & 24 & 18 & 14 & 6 & 136 & 78 & 92 & 95 & 66 & 32 & 32 & 26 & 22 & 16 & 12 & 6 & 40 & 20 & 17 \\
74 & 82 & 56 & 24 & 30 & 24 & 18 & 12 & 8 & 4 & 104 & 59 & 67 & 73 & 48 & 24 & 24 & 20 & 16 & 12 & 8 & 4 & 28 & 14 & 11 \\
52 & 56 & 40 & 16 & 20 & 16 & 14 & 8 & 4 & 2 & 72 & 37 & 42 & 48 & 36 & 18 & 16 & 14 & 12 & 8 & 4 & 2 & 15 & 7 & 7 \\
28 & 30 & 20 & 10 & 10 & 8 & 6 & 4 & 2 & 0 & 33 & 16 & 20 & 24 & 16 & 10 & 8 & 8 & 6 & 4 & 2 & 0 & 7 & 0 & 0 \\
386 & 414 & 284 & 126 & 146 & 122 & 102 & 70 & 44 & 25 & 242 & 132 & 138 & 142 & 94 & 47 & 40 & 34 & 28 & 22 & 11 & 5 & 860 & 490 & 273
\end{pmatrix}.$$

$$A_{14}=\begin{pmatrix}
2&0&0&0&0&0&0&0&0&30&13&14&15&8&4&8&4&0&0&0&0&4&0&0&0\\
0&0&0&0&0&0&0&0&0&25&10&10&6&4&0&4&0&0&0&0&0&2&0&0&0\\
0&0&0&0&0&0&0&0&0&22&10&12&8&4&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&19&6&8&6&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&10&4&4&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&8&4&2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
126&102&51&48&38&28&22&11&5&274&168&194&196&136&68&66&54&42&34&27&13&80&30&16&8\\
106&76&42&36&30&20&14&7&0&227&138&146&148&98&44&48&36&28&20&15&10&60&18&8&4\\
104&70&36&34&26&17&11&7&0&246&142&152&152&102&46&50&36&27&19&13&8&58&14&6&2\\
102&70&34&32&22&15&8&4&0&255&144&152&152&100&48&50&36&27&18&12&6&56&10&4&1\\
66&48&23&22&15&9&4&0&0&172&94&102&100&64&31&30&23&17&12&8&4&34&4&2&0\\
26&19&8&9&6&4&2&0&0&64&34&38&42&27&16&13&12&8&6&4&0&14&2&1&0\\
32&22&11&12&8&4&0&0&0&82&46&50&50&30&15&12&10&8&6&4&2&16&2&0&0\\
22&15&6&8&4&2&0&0&0&68&36&36&36&23&12&10&8&6&4&2&0&12&1&0&0\\
15&9&4&4&2&0&0&0&0&54&28&27&27&17&8&8&6&4&2&0&0&8&0&0&0\\
8&4&2&0&0&0&0&0&0&40&20&19&18&12&6&6&4&2&0&0&0&4&0&0&0\\
4&0&0&0&0&0&0&0&0&31&15&13&12&8&4&4&2&0&0&0&0&2&0&0&0\\
0&0&0&0&0&0&0&0&0&15&10&8&6&4&0&2&0&0&0&0&0&1&0&0&0\\
371&384&192&194&166&138&106&77&36&388&220&214&228&168&84&86&68&50&38&29&14&222&98&44&32
\end{pmatrix}.$$

$$A_{15}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&24&26&10&7&1&0&127&167&58&53&35\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&24&24&8&6&2&0&186&248&86&78&47\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&24&20&6&4&1&0&228&302&104&92&57\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&24&16&4&2&0&0&250&326&112&99&65\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&16&10&2&1&0&0&170&220&74&65&44\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&1&0&0&0&0&51&67&22&18&12\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&8&4&0&0&0&0&88&114&38&33&24\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&6&2&0&0&0&0&78&99&33&28&20\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&1&0&0&0&0&62&81&28&24&16\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&47&65&24&20&12\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&31&44&16&14&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&16&24&8&8&4\\
4&2&80&28&12&8&4&2&32&26&10&7&1&0&174&216&74&65&39&13&169&215&74&65&39\\
2&1&76&22&12&4&2&1&26&22&8&6&2&0&196&252&88&80&48&16&196&252&88&80&48\\
1&0&74&16&8&2&1&0&24&18&6&4&1&0&224&282&98&88&56&20&224&282&98&88&56\\
0&0&70&12&4&1&0&0&24&16&4&2&0&0&252&312&108&96&64&24&252&312&108&96&64\\
0&0&48&6&2&0&0&0&18&10&2&1&0&0&170&210&72&64&44&16&170&210&72&64&44\\
0&0&15&2&1&0&0&0&5&1&0&0&0&0&53&63&21&18&12&4&53&63&21&18&12\\
0&0&24&2&0&0&0&0&8&4&0&0&0&0&88&108&36&32&24&8&88&108&36&32&24\\
0&0&16&1&0&0&0&0&6&2&0&0&0&0&80&96&32&28&20&8&80&96&32&28&20\\
0&0&12&0&0&0&0&0&4&1&0&0&0&0&64&80&28&24&16&6&64&80&28&24&16\\
0&0&4&0&0&0&0&0&2&0&0&0&0&0&48&64&24&20&12&4&48&64&24&20&12\\
0&0&2&0&0&0&0&0&1&0&0&0&0&0&32&44&16&14&8&2&32&44&16&14&8\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&16&24&8&8&4&0&16&24&8&8&4\\
24&12&441&272&140&98&67&43&224&205&136&115&69&29&204&238&82&69&39&13&26&26&10&7&1
\end{pmatrix}.$$

$$A_{16}=\begin{pmatrix}
12 & 74 & 110 & 42 & 35 & 18 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 21 & 10 & 4 & 2 & 0 & 0 & 75 & 52 & 26 & 14 & 7 & 3\\
16 & 160 & 244 & 92 & 78 & 45 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 10 & 3 & 1 & 0 & 0 & 106 & 72 & 33 & 19 & 8 & 4\\
20 & 226 & 340 & 124 & 105 & 65 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 14 & 6 & 2 & 0 & 0 & 144 & 98 & 46 & 27 & 13 & 8\\
24 & 242 & 364 & 134 & 113 & 71 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 22 & 10 & 4 & 1 & 0 & 0 & 136 & 90 & 43 & 24 & 11 & 6\\
16 & 170 & 260 & 98 & 81 & 49 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 4 & 2 & 0 & 0 & 0 & 98 & 64 & 31 & 17 & 8 & 4\\
4 & 51 & 80 & 31 & 24 & 14 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 38 & 27 & 16 & 8 & 4 & 0\\
8 & 90 & 132 & 46 & 39 & 28 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 1 & 0 & 0 & 0 & 48 & 30 & 15 & 8 & 4 & 2\\
8 & 78 & 113 & 39 & 32 & 22 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 36 & 23 & 12 & 6 & 2 & 0\\
6 & 61 & 91 & 33 & 26 & 16 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 27 & 17 & 8 & 4 & 0 & 0\\
4 & 45 & 71 & 28 & 22 & 12 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 19 & 12 & 6 & 2 & 0 & 0\\
2 & 32 & 47 & 16 & 14 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 8 & 4 & 0 & 0 & 0\\
0 & 16 & 24 & 8 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 0 & 0 & 0 & 0\\
13 & 26 & 28 & 10 & 7 & 1 & 0 & 37 & 36 & 18 & 8 & 2 & 1 & 117 & 82 & 38 & 26 & 10 & 5 & 150 & 108 & 54 & 30 & 15 & 7\\
16 & 24 & 24 & 8 & 6 & 2 & 0 & 18 & 18 & 10 & 4 & 1 & 0 & 82 & 52 & 23 & 13 & 5 & 0 & 124 & 84 & 38 & 22 & 9 & 4\\
20 & 24 & 20 & 6 & 4 & 1 & 0 & 16 & 12 & 6 & 2 & 0 & 0 & 94 & 56 & 26 & 14 & 6 & 0 & 144 & 98 & 46 & 27 & 13 & 8\\
24 & 24 & 16 & 4 & 2 & 0 & 0 & 18 & 10 & 4 & 1 & 0 & 0 & 92 & 60 & 28 & 14 & 4 & 0 & 146 & 98 & 48 & 27 & 12 & 6\\
16 & 16 & 10 & 2 & 1 & 0 & 0 & 12 & 4 & 2 & 0 & 0 & 0 & 56 & 42 & 20 & 8 & 0 & 0 & 98 & 64 & 31 & 17 & 8 & 4\\
4 & 3 & 1 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 26 & 20 & 8 & 4 & 0 & 0 & 38 & 27 & 16 & 8 & 4 & 0\\
8 & 8 & 4 & 0 & 0 & 0 & 0 & 6 & 2 & 1 & 0 & 0 & 0 & 28 & 20 & 10 & 4 & 0 & 0 & 48 & 30 & 15 & 8 & 4 & 2\\
8 & 6 & 2 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 20 & 14 & 6 & 2 & 0 & 0 & 36 & 23 & 12 & 6 & 2 & 0\\
6 & 4 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 14 & 8 & 4 & 0 & 0 & 0 & 27 & 17 & 8 & 4 & 0 & 0\\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 10 & 4 & 2 & 0 & 0 & 0 & 19 & 12 & 6 & 2 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 13 & 8 & 4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 44 & 22 & 8 & 2 & 1 & 54 & 30 & 14 & 8 & 2 & 1 & 20 & 12 & 6 & 0 & 0 & 0
\end{pmatrix}.$$

$$A_{23}=\begin{pmatrix}
379 & 404 & 270 & 123 & 136 & 116 & 92 & 68 & 47 & 27 & 204 & 104 & 106 & 110 & 72 & 38 & 32 & 26 & 20 & 14 & 7 & 0 & 758 & 414 & 237\\
384 & 407 & 278 & 122 & 138 & 118 & 98 & 73 & 52 & 28 & 198 & 102 & 106 & 105 & 66 & 32 & 32 & 24 & 17 & 11 & 7 & 0 & 702 & 381 & 225\\
413 & 446 & 296 & 128 & 150 & 129 & 105 & 82 & 56 & 30 & 210 & 108 & 105 & 104 & 68 & 32 & 32 & 22 & 15 & 8 & 4 & 0 & 782 & 438 & 257\\
292 & 308 & 208 & 88 & 102 & 87 & 74 & 56 & 40 & 20 & 144 & 72 & 66 & 68 & 48 & 23 & 22 & 15 & 9 & 4 & 0 & 0 & 586 & 328 & 194\\
100 & 104 & 68 & 30 & 33 & 29 & 26 & 20 & 14 & 10 & 51 & 28 & 24 & 26 & 19 & 8 & 9 & 6 & 4 & 2 & 0 & 0 & 209 & 128 & 76\\
146 & 156 & 102 & 43 & 48 & 42 & 36 & 30 & 20 & 10 & 64 & 32 & 32 & 32 & 22 & 11 & 12 & 8 & 4 & 0 & 0 & 0 & 290 & 160 & 92\\
124 & 133 & 87 & 37 & 42 & 36 & 30 & 24 & 16 & 8 & 52 & 26 & 24 & 22 & 15 & 6 & 8 & 4 & 2 & 0 & 0 & 0 & 244 & 140 & 82\\
102 & 107 & 74 & 32 & 36 & 30 & 24 & 18 & 14 & 6 & 40 & 20 & 17 & 15 & 9 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 198 & 108 & 67\\
75 & 82 & 56 & 24 & 30 & 24 & 18 & 12 & 8 & 4 & 28 & 14 & 11 & 8 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 148 & 80 & 49\\
52 & 56 & 40 & 16 & 20 & 16 & 14 & 8 & 4 & 2 & 15 & 7 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 105 & 54 & 29\\
28 & 30 & 20 & 10 & 10 & 8 & 6 & 4 & 2 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 49 & 21 & 12\\
218 & 225 & 152 & 71 & 70 & 60 & 50 & 40 & 31 & 15 & 380 & 247 & 322 & 329 & 232 & 116 & 124 & 104 & 84 & 60 & 43 & 21 & 114 & 56 & 46\\
138 & 142 & 94 & 44 & 46 & 36 & 28 & 20 & 15 & 10 & 283 & 194 & 260 & 270 & 188 & 98 & 100 & 86 & 66 & 48 & 33 & 16 & 64 & 30 & 24\\
152 & 152 & 102 & 46 & 50 & 36 & 27 & 19 & 13 & 8 & 362 & 260 & 364 & 360 & 262 & 134 & 130 & 112 & 91 & 67 & 45 & 20 & 64 & 28 & 20\\
152 & 152 & 100 & 48 & 50 & 36 & 27 & 18 & 12 & 6 & 373 & 270 & 360 & 366 & 258 & 130 & 132 & 112 & 89 & 70 & 46 & 24 & 58 & 26 & 20\\
102 & 100 & 64 & 31 & 30 & 23 & 17 & 12 & 8 & 4 & 260 & 188 & 262 & 258 & 192 & 95 & 94 & 79 & 65 & 48 & 32 & 16 & 34 & 16 & 12\\
46 & 48 & 31 & 16 & 15 & 12 & 8 & 6 & 4 & 0 & 116 & 88 & 126 & 124 & 91 & 44 & 45 & 38 & 32 & 24 & 16 & 10 & 17 & 8 & 6\\
50 & 50 & 30 & 15 & 12 & 10 & 8 & 6 & 4 & 2 & 136 & 100 & 130 & 132 & 94 & 47 & 44 & 38 & 32 & 28 & 16 & 8 & 16 & 8 & 6\\
36 & 36 & 23 & 12 & 10 & 8 & 6 & 4 & 2 & 0 & 114 & 86 & 112 & 112 & 79 & 38 & 38 & 32 & 26 & 22 & 14 & 8 & 12 & 6 & 4\\
27 & 27 & 17 & 8 & 8 & 6 & 4 & 2 & 0 & 0 & 92 & 66 & 91 & 89 & 65 & 32 & 32 & 26 & 20 & 16 & 12 & 6 & 8 & 4 & 2\\
19 & 18 & 12 & 6 & 6 & 4 & 2 & 0 & 0 & 0 & 66 & 48 & 67 & 70 & 48 & 24 & 28 & 22 & 16 & 12 & 8 & 4 & 4 & 2 & 1\\
13 & 12 & 8 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 47 & 33 & 45 & 46 & 32 & 16 & 16 & 14 & 12 & 8 & 4 & 2 & 2 & 1 & 0\\
8 & 6 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 23 & 16 & 20 & 24 & 16 & 10 & 8 & 8 & 6 & 4 & 2 & 0 & 1 & 0 & 0\\
616 & 656 & 446 & 202 & 224 & 192 & 160 & 132 & 98 & 46 & 550 & 341 & 422 & 437 & 304 & 152 & 156 & 132 & 108 & 80 & 53 & 26 & 892 & 493 & 310\\
426 & 456 & 298 & 138 & 148 & 132 & 112 & 96 & 66 & 36 & 350 & 216 & 256 & 276 & 186 & 98 & 96 & 84 & 66 & 49 & 32 & 16 & 545 & 302 & 192
\end{pmatrix}.$$

$$A_{24} = \begin{pmatrix}
328 & 322 & 173 & 160 & 140 & 108 & 80 & 53 & 21 & 298 & 174 & 165 & 166 & 114 & 52 & 58 & 42 & 32 & 22 & 16 & 11 & 168 & 60 & 28 & 16 \\
306 & 298 & 157 & 144 & 123 & 97 & 69 & 43 & 19 & 282 & 157 & 149 & 149 & 102 & 48 & 50 & 38 & 29 & 20 & 13 & 8 & 149 & 48 & 24 & 12 \\
342 & 322 & 164 & 160 & 135 & 105 & 78 & 52 & 26 & 298 & 160 & 149 & 154 & 104 & 52 & 52 & 39 & 28 & 18 & 12 & 6 & 160 & 54 & 25 & 13 \\
250 & 246 & 123 & 122 & 102 & 83 & 60 & 44 & 20 & 214 & 112 & 103 & 104 & 68 & 33 & 32 & 24 & 17 & 12 & 8 & 4 & 122 & 42 & 20 & 8 \\
93 & 93 & 45 & 47 & 39 & 34 & 26 & 20 & 14 & 84 & 42 & 40 & 46 & 29 & 18 & 14 & 13 & 8 & 6 & 4 & 0 & 50 & 20 & 10 & 4 \\
124 & 122 & 61 & 60 & 50 & 40 & 30 & 20 & 10 & 106 & 56 & 50 & 52 & 32 & 16 & 12 & 10 & 8 & 6 & 4 & 2 & 56 & 20 & 12 & 4 \\
107 & 101 & 49 & 50 & 40 & 32 & 24 & 16 & 10 & 84 & 42 & 38 & 39 & 24 & 13 & 10 & 8 & 6 & 4 & 2 & 0 & 44 & 14 & 8 & 2 \\
86 & 83 & 41 & 40 & 32 & 26 & 18 & 12 & 6 & 62 & 32 & 29 & 28 & 17 & 8 & 8 & 6 & 4 & 2 & 0 & 0 & 36 & 8 & 4 & 0 \\
66 & 60 & 30 & 30 & 24 & 18 & 12 & 8 & 4 & 44 & 22 & 20 & 18 & 12 & 6 & 6 & 4 & 2 & 0 & 0 & 0 & 24 & 4 & 0 & 0 \\
44 & 44 & 22 & 20 & 16 & 12 & 8 & 4 & 2 & 33 & 16 & 13 & 12 & 8 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
22 & 20 & 14 & 10 & 10 & 6 & 4 & 2 & 0 & 16 & 11 & 8 & 6 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
50 & 34 & 17 & 16 & 12 & 8 & 4 & 2 & 1 & 170 & 107 & 122 & 123 & 80 & 39 & 36 & 30 & 24 & 20 & 10 & 5 & 36 & 8 & 8 & 0 \\
24 & 16 & 8 & 8 & 6 & 4 & 2 & 1 & 0 & 115 & 72 & 82 & 80 & 52 & 26 & 22 & 18 & 14 & 11 & 5 & 0 & 19 & 4 & 4 & 0 \\
20 & 12 & 6 & 6 & 4 & 2 & 1 & 0 & 0 & 128 & 82 & 108 & 100 & 66 & 32 & 32 & 24 & 16 & 11 & 6 & 0 & 18 & 4 & 2 & 0 \\
16 & 10 & 4 & 4 & 2 & 1 & 0 & 0 & 0 & 127 & 80 & 100 & 96 & 64 & 32 & 32 & 23 & 15 & 9 & 4 & 0 & 17 & 2 & 0 & 0 \\
10 & 4 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 82 & 52 & 66 & 64 & 44 & 22 & 22 & 15 & 8 & 4 & 0 & 0 & 8 & 0 & 0 & 0 \\
4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 39 & 26 & 32 & 32 & 22 & 8 & 11 & 6 & 4 & 2 & 0 & 0 & 4 & 0 & 0 & 0 \\
4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & 22 & 32 & 32 & 22 & 11 & 14 & 9 & 4 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & 18 & 24 & 23 & 15 & 6 & 9 & 4 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 14 & 16 & 15 & 8 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 11 & 11 & 9 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 5 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
413 & 408 & 205 & 204 & 174 & 144 & 106 & 67 & 33 & 602 & 379 & 418 & 439 & 326 & 160 & 164 & 138 & 116 & 86 & 56 & 28 & 304 & 152 & 74 & 46 \\
260 & 242 & 128 & 120 & 106 & 84 & 62 & 39 & 22 & 368 & 232 & 249 & 270 & 192 & 98 & 100 & 84 & 66 & 49 & 32 & 17 & 175 & 82 & 40 & 24
\end{pmatrix}.$$

$$A_{25} = \begin{pmatrix}
12 & 6 & 450 & 270 & 136 & 92 & 68 & 48 & 224 & 230 & 152 & 134 & 78 & 26 & 216 & 268 & 92 & 84 & 48 & 16 & 24 & 24 & 8 & 6 & 2 \\
9 & 6 & 415 & 239 & 118 & 77 & 57 & 41 & 208 & 213 & 134 & 117 & 67 & 21 & 224 & 279 & 96 & 88 & 56 & 20 & 24 & 20 & 6 & 4 & 1 \\
8 & 4 & 465 & 272 & 138 & 90 & 64 & 44 & 244 & 241 & 153 & 132 & 78 & 30 & 256 & 310 & 108 & 96 & 64 & 24 & 24 & 16 & 4 & 2 & 0 \\
4 & 0 & 360 & 212 & 102 & 75 & 56 & 40 & 182 & 178 & 120 & 102 & 60 & 24 & 178 & 212 & 72 & 64 & 44 & 16 & 16 & 10 & 2 & 1 & 0 \\
2 & 0 & 119 & 69 & 32 & 26 & 20 & 14 & 61 & 53 & 35 & 28 & 16 & 8 & 59 & 65 & 21 & 18 & 12 & 4 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 182 & 102 & 48 & 36 & 30 & 20 & 92 & 90 & 60 & 50 & 30 & 10 & 92 & 108 & 36 & 32 & 24 & 8 & 8 & 4 & 0 & 0 & 0 \\
0 & 0 & 154 & 88 & 42 & 30 & 24 & 16 & 84 & 82 & 50 & 42 & 24 & 10 & 84 & 96 & 32 & 28 & 20 & 8 & 6 & 2 & 0 & 0 & 0 \\
0 & 0 & 130 & 75 & 36 & 24 & 18 & 14 & 64 & 68 & 40 & 34 & 18 & 6 & 64 & 80 & 28 & 24 & 16 & 6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 94 & 56 & 30 & 18 & 12 & 8 & 48 & 54 & 30 & 24 & 12 & 4 & 48 & 64 & 24 & 20 & 12 & 4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 61 & 40 & 20 & 14 & 8 & 4 & 38 & 40 & 20 & 16 & 8 & 2 & 32 & 44 & 16 & 14 & 8 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 34 & 20 & 10 & 6 & 4 & 2 & 16 & 22 & 10 & 10 & 4 & 0 & 16 & 24 & 8 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 30 & 4 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 118 & 134 & 42 & 35 & 21 & 5 & 230 & 296 & 100 & 90 & 62 \\
0 & 0 & 14 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & 84 & 24 & 20 & 12 & 0 & 186 & 248 & 86 & 78 & 47 \\
0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 86 & 26 & 20 & 10 & 0 & 228 & 302 & 104 & 92 & 57 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84 & 88 & 28 & 20 & 8 & 0 & 250 & 326 & 112 & 99 & 65 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 54 & 58 & 20 & 14 & 4 & 0 & 170 & 220 & 74 & 65 & 44 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23 & 25 & 9 & 6 & 2 & 0 & 72 & 94 & 31 & 26 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 28 & 12 & 8 & 0 & 0 & 88 & 114 & 38 & 33 & 24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 20 & 8 & 4 & 0 & 0 & 78 & 99 & 33 & 28 & 20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 14 & 4 & 2 & 0 & 0 & 62 & 81 & 28 & 24 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 8 & 0 & 0 & 0 & 0 & 47 & 65 & 24 & 20 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 4 & 0 & 0 & 0 & 0 & 31 & 44 & 16 & 14 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 24 & 8 & 8 & 4 \\
36 & 28 & 376 & 166 & 78 & 50 & 40 & 31 & 132 & 122 & 66 & 55 & 25 & 6 & 288 & 348 & 116 & 102 & 66 & 22 & 122 & 136 & 42 & 35 & 21 \\
18 & 14 & 228 & 98 & 48 & 28 & 20 & 15 & 86 & 78 & 36 & 30 & 14 & 1 & 202 & 254 & 88 & 80 & 48 & 16 & 76 & 84 & 24 & 20 & 12
\end{pmatrix}.$$

$$A_{26}=\begin{pmatrix}
0&0&0&0&0&0&0&16&22&14&4&1&0&34&18&8&4&1&0&12&6&0&0&0&0\\
0&0&0&0&0&0&0&14&16&10&2&0&0&26&14&6&2&0&0&8&4&0&0&0&0\\
0&0&0&0&0&0&0&15&12&6&1&0&0&23&10&4&1&0&0&6&2&0&0&0&0\\
0&0&0&0&0&0&0&8&4&2&0&0&0&14&4&2&0&0&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&6&2&0&0&0&0&6&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&2&1&0&0&0&6&2&1&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&4&1&0&0&0&0&4&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
21&81&101&32&26&17&4&22&12&6&2&0&0&96&68&34&18&8&4&182&130&62&42&18&9\\
16&62&76&22&18&11&0&14&8&3&1&0&0&82&56&25&15&7&4&178&132&66&46&22&10\\
20&76&98&32&24&11&0&20&12&6&2&0&0&120&84&40&24&12&8&266&200&102&74&38&20\\
24&76&96&32&23&9&0&20&10&4&1&0&0&116&80&39&23&11&6&260&192&97&71&37&18\\
16&50&64&22&15&4&0&12&4&2&0&0&0&84&56&28&16&8&4&200&146&72&56&32&16\\
4&21&29&10&6&2&0&6&2&0&0&0&0&40&28&16&8&4&0&102&72&36&28&16&10\\
8&22&32&14&9&0&0&6&2&1&0&0&0&42&28&14&8&4&2&100&72&36&28&16&8\\
8&18&23&9&4&0&0&4&1&0&0&0&0&32&22&12&6&2&0&88&64&32&24&14&8\\
6&14&15&4&2&0&0&2&0&0&0&0&0&24&16&8&4&0&0&74&56&28&20&12&6\\
4&11&9&0&0&0&0&1&0&0&0&0&0&18&12&6&2&0&0&56&44&22&16&8&4\\
2&5&4&0&0&0&0&0&0&0&0&0&0&12&8&4&0&0&0&38&32&16&12&4&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&8&4&0&0&0&0&20&16&10&6&2&0\\
5&6&2&0&0&0&0&98&120&60&34&12&6&182&128&64&34&16&8&133&88&43&26&10&5\\
0&0&0&0&0&0&0&55&62&31&17&6&1&102&70&32&18&8&4&76&48&23&13&5&0
\end{pmatrix}.$$

$$A_{34}=\begin{pmatrix}
280&264&136&130&112&91&67&45&26&420&259&288&306&224&114&112&96&78&58&38&20&194&88&44&24\\
306&278&142&140&120&95&72&48&28&452&283&306&333&232&118&118&102&82&65&44&24&203&90&46&26\\
206&192&95&94&79&65&48&32&16&314&188&208&216&154&76&76&66&56&44&32&16&138&56&28&16\\
100&91&48&45&40&32&24&16&10&148&98&106&112&76&38&38&33&28&22&16&10&68&28&14&8\\
104&94&47&48&40&32&24&16&8&160&98&104&110&76&38&38&33&28&24&16&8&66&28&12&8\\
92&79&40&40&34&26&20&14&8&136&84&90&98&66&33&33&28&24&20&14&8&56&22&10&6\\
75&65&32&32&26&20&16&12&6&116&66&74&80&56&28&28&24&20&16&12&6&46&16&8&4\\
60&48&24&24&20&16&12&8&4&84&49&56&65&44&22&24&20&16&12&8&4&36&12&6&2\\
40&32&16&16&14&12&8&4&2&56&32&38&44&32&16&16&14&12&8&4&2&28&8&4&0\\
24&16&10&8&8&6&4&2&0&28&17&20&24&16&10&8&8&6&4&2&0&14&4&2&0\\
193&170&84&82&68&54&40&30&15&394&256&308&320&234&117&128&106&84&62&44&22&162&80&36&24\\
116&96&44&46&36&28&20&15&11&256&180&232&247&176&94&96&84&62&46&32&16&101&50&22&14\\
127&102&46&50&36&26&19&12&8&332&246&334&339&252&132&126&110&88&66&44&20&118&64&32&16\\
130&100&50&50&37&27&18&12&6&340&258&335&347&250&128&128&110&88&70&46&24&121&64&32&15\\
84&60&30&30&23&16&12&8&4&234&176&244&246&184&92&92&78&64&48&32&16&80&44&22&8\\
42&30&18&15&13&8&6&4&0&117&94&128&126&92&44&46&38&32&24&16&10&40&22&11&4\\
44&30&15&12&10&8&6&4&2&128&96&122&128&92&46&44&38&32&28&16&8&36&22&14&4\\
33&23&13&10&8&6&4&2&0&106&84&108&110&78&38&38&32&26&22&14&8&30&15&9&2\\
25&16&8&8&6&4&2&0&0&84&62&88&88&64&32&32&26&20&16&12&6&24&8&4&0\\
18&12&6&6&4&2&0&0&0&62&46&66&70&48&24&28&22&16&12&8&4&20&4&0&0\\
12&8&4&4&2&0&0&0&0&44&32&44&46&32&16&16&14&12&8&4&2&10&0&0&0\\
6&4&0&2&0&0&0&0&0&22&16&20&24&16&10&8&8&6&4&2&0&5&0&0&0\\
655&722&358&364&316&272&216&152&76&820&513&526&573&450&225&226&190&154&118&82&41&506&296&152&108\\
382&416&199&212&180&160&132&92&52&465&282&286&323&252&136&128&112&86&65&45&20&292&170&88&62\\
226&224&106&112&96&86&68&50&24&272&166&176&196&152&84&76&70&56&41&26&12&162&88&44&28
\end{pmatrix}.$$

$$A_{35}=\begin{pmatrix}
18 & 12 & 240 & 99 & 48 & 27 & 19 & 13 & 91 & 78 & 34 & 26 & 11 & 0 & 228 & 283 & 98 & 88 & 56 & 20 & 80 & 86 & 26 & 20 & 10 \\
18 & 12 & 246 & 98 & 48 & 27 & 18 & 12 & 94 & 80 & 32 & 23 & 8 & 0 & 254 & 312 & 108 & 96 & 64 & 24 & 84 & 88 & 28 & 20 & 8 \\
12 & 8 & 164 & 64 & 30 & 17 & 12 & 8 & 60 & 52 & 22 & 15 & 4 & 0 & 170 & 210 & 72 & 64 & 44 & 16 & 54 & 58 & 20 & 14 & 4 \\
6 & 4 & 77 & 31 & 15 & 8 & 6 & 4 & 26 & 23 & 10 & 7 & 2 & 0 & 74 & 90 & 30 & 26 & 16 & 4 & 23 & 25 & 9 & 6 & 2 \\
6 & 4 & 78 & 30 & 12 & 8 & 6 & 4 & 24 & 26 & 12 & 8 & 0 & 0 & 88 & 108 & 36 & 32 & 24 & 8 & 24 & 28 & 12 & 8 & 0 \\
4 & 2 & 64 & 23 & 10 & 6 & 4 & 2 & 22 & 19 & 8 & 4 & 0 & 0 & 80 & 96 & 32 & 28 & 20 & 8 & 20 & 20 & 8 & 4 & 0 \\
2 & 0 & 50 & 17 & 8 & 4 & 2 & 0 & 20 & 13 & 4 & 2 & 0 & 0 & 64 & 80 & 28 & 24 & 16 & 6 & 16 & 14 & 4 & 2 & 0 \\
0 & 0 & 40 & 12 & 6 & 2 & 0 & 0 & 14 & 8 & 0 & 0 & 0 & 0 & 48 & 64 & 24 & 20 & 12 & 4 & 12 & 8 & 0 & 0 & 0 \\
0 & 0 & 31 & 8 & 4 & 0 & 0 & 0 & 7 & 4 & 0 & 0 & 0 & 0 & 32 & 44 & 16 & 14 & 8 & 2 & 6 & 4 & 0 & 0 & 0 \\
0 & 0 & 15 & 4 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 16 & 24 & 8 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 10 & 110 & 34 & 16 & 8 & 4 & 2 & 32 & 20 & 8 & 4 & 0 & 0 & 186 & 216 & 74 & 60 & 36 & 14 & 169 & 201 & 66 & 56 & 36 \\
11 & 5 & 54 & 16 & 8 & 4 & 2 & 1 & 20 & 8 & 4 & 0 & 0 & 0 & 114 & 118 & 40 & 30 & 18 & 10 & 108 & 118 & 38 & 30 & 18 \\
11 & 6 & 54 & 12 & 6 & 2 & 1 & 0 & 16 & 6 & 2 & 0 & 0 & 0 & 124 & 128 & 42 & 32 & 18 & 8 & 122 & 132 & 44 & 32 & 18 \\
9 & 4 & 54 & 10 & 4 & 1 & 0 & 0 & 10 & 4 & 0 & 0 & 0 & 0 & 124 & 134 & 44 & 34 & 18 & 6 & 124 & 136 & 46 & 34 & 18 \\
4 & 0 & 34 & 4 & 2 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 82 & 88 & 28 & 22 & 12 & 4 & 82 & 90 & 28 & 22 & 12 \\
2 & 0 & 17 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & 44 & 14 & 12 & 6 & 0 & 36 & 44 & 14 & 12 & 6 \\
0 & 0 & 16 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 42 & 44 & 12 & 10 & 6 & 2 & 40 & 46 & 12 & 10 & 6 \\
0 & 0 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & 34 & 10 & 8 & 4 & 0 & 30 & 34 & 10 & 8 & 4 \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 26 & 8 & 6 & 2 & 0 & 24 & 26 & 8 & 6 & 2 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 & 18 & 6 & 4 & 0 & 0 & 18 & 18 & 6 & 4 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 12 & 4 & 2 & 0 & 0 & 14 & 12 & 4 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 6 & 2 & 0 & 0 & 0 & 10 & 6 & 2 & 0 & 0 \\
82 & 54 & 632 & 350 & 176 & 124 & 88 & 59 & 266 & 220 & 138 & 112 & 64 & 22 & 252 & 274 & 90 & 72 & 40 & 15 & 32 & 22 & 8 & 4 & 0 \\
47 & 31 & 352 & 198 & 100 & 72 & 50 & 33 & 156 & 116 & 72 & 56 & 32 & 17 & 148 & 148 & 48 & 36 & 20 & 11 & 18 & 8 & 4 & 0 & 0 \\
21 & 14 & 187 & 95 & 46 & 31 & 22 & 15 & 71 & 52 & 30 & 22 & 10 & 8 & 99 & 107 & 36 & 28 & 17 & 8 & 12 & 6 & 2 & 0 & 0
\end{pmatrix}.$$

$$A_{36}=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 68 & 64 & 32 & 16 & 6 & 0 & 120 & 84 & 40 & 24 & 12 & 8 & 88 & 54 & 26 & 14 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 68 & 68 & 34 & 15 & 4 & 0 & 126 & 88 & 44 & 26 & 12 & 6 & 88 & 58 & 28 & 14 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & 44 & 22 & 8 & 0 & 0 & 84 & 56 & 28 & 16 & 8 & 4 & 54 & 40 & 20 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 22 & 10 & 4 & 0 & 0 & 40 & 28 & 16 & 8 & 4 & 0 & 26 & 20 & 8 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 22 & 11 & 4 & 0 & 0 & 42 & 28 & 14 & 8 & 4 & 2 & 26 & 20 & 10 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 15 & 7 & 2 & 0 & 0 & 32 & 22 & 12 & 6 & 2 & 0 & 20 & 14 & 6 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 8 & 4 & 0 & 0 & 0 & 24 & 16 & 8 & 4 & 0 & 0 & 14 & 8 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 4 & 2 & 0 & 0 & 0 & 18 & 12 & 6 & 2 & 0 & 0 & 10 & 4 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 12 & 8 & 4 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 19 & 17 & 8 & 4 & 0 & 0 & 77 & 80 & 40 & 20 & 8 & 4 & 183 & 132 & 66 & 42 & 18 & 9 & 184 & 134 & 67 & 42 & 18 & 9 \\
10 & 10 & 6 & 4 & 0 & 0 & 0 & 69 & 64 & 30 & 16 & 7 & 4 & 167 & 126 & 66 & 46 & 22 & 10 & 166 & 126 & 66 & 46 & 22 & 10 \\
8 & 10 & 8 & 2 & 0 & 0 & 0 & 102 & 96 & 46 & 26 & 12 & 8 & 258 & 196 & 102 & 74 & 38 & 20 & 256 & 196 & 102 & 74 & 38 & 20 \\
6 & 6 & 6 & 0 & 0 & 0 & 0 & 97 & 90 & 45 & 24 & 11 & 6 & 255 & 190 & 97 & 71 & 37 & 18 & 254 & 190 & 97 & 71 & 37 & 18 \\
4 & 4 & 2 & 0 & 0 & 0 & 0 & 72 & 60 & 30 & 16 & 8 & 4 & 196 & 144 & 72 & 56 & 32 & 16 & 196 & 144 & 72 & 56 & 32 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 34 & 30 & 18 & 8 & 4 & 0 & 102 & 72 & 36 & 28 & 16 & 10 & 102 & 72 & 36 & 28 & 16 & 10 \\
2 & 4 & 0 & 0 & 0 & 0 & 0 & 36 & 30 & 15 & 8 & 4 & 2 & 98 & 72 & 36 & 28 & 16 & 8 & 98 & 72 & 36 & 28 & 16 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 28 & 23 & 13 & 6 & 2 & 0 & 88 & 64 & 32 & 24 & 14 & 8 & 88 & 64 & 32 & 24 & 14 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 22 & 16 & 8 & 4 & 0 & 0 & 74 & 56 & 28 & 20 & 12 & 6 & 74 & 56 & 28 & 20 & 12 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 17 & 12 & 6 & 2 & 0 & 0 & 56 & 44 & 22 & 16 & 8 & 4 & 56 & 44 & 22 & 16 & 8 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 8 & 4 & 0 & 0 & 0 & 38 & 32 & 16 & 12 & 4 & 2 & 38 & 32 & 16 & 12 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 0 & 0 & 0 & 0 & 20 & 16 & 10 & 6 & 2 & 0 & 20 & 16 & 10 & 6 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 137 & 212 & 106 & 60 & 26 & 13 & 178 & 120 & 60 & 34 & 12 & 6 & 57 & 36 & 18 & 8 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & 114 & 53 & 31 & 13 & 9 & 96 & 62 & 31 & 17 & 6 & 1 & 26 & 18 & 10 & 4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & 56 & 26 & 14 & 6 & 8 & 68 & 40 & 20 & 12 & 6 & 0 & 16 & 12 & 6 & 2 & 0 & 0
\end{pmatrix}.$$

$$A_{45}=\left(\begin{array}{ccccccccccccccccccccccccc}
34&24&271&134&67&46&34&24&96&77&43&33&14&6&114&126&42&33&18&6&8&4&0&0&0\\
44&32&300&154&74&57&44&32&104&84&50&38&16&4&96&98&30&23&12&4&6&2&0&0&0\\
22&16&153&77&37&28&22&16&47&43&25&21&8&0&44&50&15&13&6&0&0&0&0&0&0\\
24&16&152&74&36&28&24&16&48&40&24&18&6&2&48&48&12&10&6&2&4&0&0&0&0\\
20&14&134&65&32&24&20&14&38&31&18&14&4&0&36&37&10&8&4&0&0&0&0&0&0\\
16&12&112&57&28&20&16&12&32&22&12&8&2&0&28&27&8&6&2&0&0&0&0&0&0\\
12&8&82&44&24&16&12&8&26&14&6&4&0&0&20&18&6&4&0&0&0&0&0&0&0\\
8&4&55&32&16&12&8&4&18&8&4&2&0&0&15&12&4&2&0&0&0&0&0&0&0\\
4&2&27&16&8&6&4&2&15&6&2&0&0&0&11&6&2&0&0&0&0&0&0&0&0\\
62&44&254&106&48&32&24&12&72&46&24&16&4&1&150&162&48&40&24&6&48&50&16&12&4\\
46&32&146&56&24&16&12&6&32&24&12&10&2&0&80&86&24&20&12&0&24&24&8&6&2\\
47&32&133&46&22&12&9&6&28&19&8&6&1&0&78&85&26&20&10&0&24&20&6&4&1\\
51&34&144&52&25&13&8&4&34&14&4&2&0&0&84&88&28&20&8&0&24&16&4&2&0\\
48&32&104&40&20&8&4&0&22&8&2&1&0&0&56&58&20&14&4&0&16&10&2&1&0\\
24&16&51&20&10&4&2&0&14&4&1&0&0&0&30&28&10&6&2&0&8&4&1&0&0\\
28&16&48&20&12&4&0&0&12&4&0&0&0&0&24&28&12&8&0&0&8&4&0&0&0\\
22&14&40&14&8&2&0&0&10&2&0&0&0&0&20&20&8&4&0&0&6&2&0&0&0\\
16&12&32&8&4&0&0&0&4&1&0&0&0&0&16&14&4&2&0&0&4&1&0&0&0\\
12&8&24&4&0&0&0&0&2&0&0&0&0&0&12&8&0&0&0&0&2&0&0&0&0\\
8&4&12&0&0&0&0&0&1&0&0&0&0&0&6&4&0&0&0&0&1&0&0&0&0\\
4&2&6&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
49&34&170&74&37&23&18&14&42&28&16&12&4&1&52&52&16&12&4&1&6&2&0&0&0\\
42&28&76&28&14&8&6&4&14&4&2&1&0&0&16&10&2&1&0&0&0&0&0&0&0\\
21&14&39&14&6&4&3&2&8&1&0&0&0&0&8&4&0&0&0&0&0&0&0&0&0\\
19&12&23&8&4&2&1&0&4&0&0&0&0&0&4&1&0&0&0&0&0&0&0&0&0
\end{array}\right).$$

$$A_{46}=\left(\begin{array}{ccccccccccccccccccccccccc}
0&0&0&0&0&0&0&54&86&44&22&8&6&80&52&26&13&4&0&18&10&4&1&0&0\\
0&0&0&0&0&0&0&56&96&48&24&8&4&64&44&22&8&0&0&12&4&2&0&0&0\\
0&0&0&0&0&0&0&26&48&28&12&4&0&32&22&10&4&0&0&6&2&0&0&0&0\\
0&0&0&0&0&0&0&28&48&24&12&4&2&32&22&11&4&0&0&6&2&1&0&0&0\\
0&0&0&0&0&0&0&20&36&20&8&2&0&24&15&7&2&0&0&4&1&0&0&0&0\\
0&0&0&0&0&0&0&14&24&12&4&0&0&16&8&4&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&10&16&8&2&0&0&11&4&2&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&6&8&4&0&0&0&6&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&8&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&134&200&100&60&26&13&203&144&72&44&18&9&99&68&34&18&8&4\\
0&0&0&0&0&0&0&121&182&97&61&29&13&183&134&71&47&22&10&83&56&25&15&7&4\\
0&0&0&0&0&0&0&162&224&118&78&38&18&256&192&100&72&38&20&120&84&40&24&12&8\\
0&0&0&0&0&0&0&167&226&116&78&40&19&256&188&97&70&37&18&116&80&39&23&11&6\\
0&0&0&0&0&0&0&144&200&100&72&40&20&208&148&74&56&32&16&84&56&28&16&8&4\\
0&0&0&0&0&0&0&78&100&48&36&20&14&108&74&38&28&16&10&40&28&16&8&4&0\\
0&0&0&0&0&0&0&72&100&50&36&20&10&104&74&37&28&16&8&42&28&14&8&4&2\\
0&0&0&0&0&0&0&66&86&42&30&16&10&92&65&33&24&14&8&32&22&12&6&2&0\\
0&0&0&0&0&0&0&54&72&36&24&12&6&76&56&28&20&12&6&24&16&8&4&0&0\\
0&0&0&0&0&0&0&40&56&28&18&8&4&57&44&22&16&8&4&18&12&6&2&0&0\\
0&0&0&0&0&0&0&26&40&20&12&4&2&38&32&16&12&4&2&12&8&4&0&0&0\\
0&0&0&0&0&0&0&12&20&14&6&2&0&20&16&10&6&2&0&8&4&0&0&0&0\\
0&0&0&0&0&0&0&81&132&66&42&18&9&98&68&34&18&8&4&23&12&6&2&0&0\\
0&0&0&0&0&0&0&84&144&72&56&32&16&84&56&28&16&8&4&12&4&2&0&0&0\\
0&0&0&0&0&0&0&42&72&36&28&16&8&42&28&14&8&4&2&6&2&1&0&0&0\\
0&0&0&0&0&0&0&28&56&28&20&12&6&24&16&8&4&0&0&2&0&0&0&0&0
\end{array}\right).$$

$$A_{56} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 44 & 22 & 16 & 8 & 4 & 18 & 12 & 6 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 32 & 16 & 12 & 4 & 2 & 12 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 32 & 16 & 8 & 2 & 1 & 24 & 12 & 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 4 & 2 & 0 & 0 & 0 & 0 & 12 & 6 & 0 & 0 & 0 & 0 & 24 & 16 & 8 & 4 & 1 & 0 & 24 & 16 & 8 & 4 & 1 & 0 \\
3 & 2 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 20 & 10 & 4 & 1 & 0 & 0 & 20 & 10 & 4 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 1 & 0 & 0 & 0 & 6 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 50 & 64 & 20 & 15 & 9 & 5 & 6 & 2 & 0 & 0 & 0 & 0 & 24 & 16 & 8 & 4 & 1 & 0 & 70 & 46 & 23 & 13 & 5 & 0 \\
18 & 58 & 70 & 24 & 17 & 9 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 20 & 10 & 4 & 1 & 0 & 0 & 86 & 58 & 28 & 14 & 4 & 0 \\
6 & 18 & 24 & 6 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 1 & 0 & 0 & 0 & 26 & 20 & 10 & 4 & 0 & 0 \\
6 & 15 & 17 & 5 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 20 & 14 & 6 & 2 & 0 & 0 \\
6 & 9 & 9 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 10 & 4 & 2 & 0 & 0 & 0
\end{pmatrix}.$$

$$A_{17} = \begin{pmatrix}
131 & 92 & 46 & 29 & 13 & 4 & 118 & 84 & 33 & 26 & 10 & 5 & 54 & 36 & 9 & 9 & 4 & 0 \\
236 & 172 & 89 & 59 & 27 & 10 & 260 & 192 & 80 & 64 & 30 & 14 & 128 & 94 & 22 & 27 & 12 & 4 \\
328 & 248 & 127 & 87 & 41 & 20 & 364 & 272 & 116 & 94 & 46 & 22 & 172 & 140 & 35 & 40 & 16 & 8 \\
340 & 248 & 125 & 85 & 41 & 18 & 382 & 282 & 117 & 97 & 49 & 24 & 186 & 146 & 36 & 40 & 16 & 6 \\
256 & 182 & 90 & 66 & 36 & 16 & 280 & 206 & 86 & 74 & 40 & 20 & 132 & 100 & 25 & 27 & 12 & 4 \\
96 & 66 & 34 & 26 & 16 & 10 & 94 & 66 & 28 & 26 & 14 & 10 & 38 & 29 & 10 & 6 & 4 & 0 \\
124 & 92 & 46 & 32 & 16 & 8 & 140 & 102 & 42 & 36 & 20 & 10 & 62 & 54 & 14 & 13 & 4 & 2 \\
108 & 78 & 38 & 26 & 14 & 8 & 120 & 86 & 36 & 30 & 16 & 8 & 49 & 39 & 10 & 8 & 2 & 0 \\
90 & 66 & 32 & 22 & 12 & 6 & 100 & 74 & 32 & 24 & 14 & 6 & 37 & 27 & 6 & 6 & 0 & 0 \\
66 & 48 & 24 & 16 & 8 & 4 & 74 & 56 & 24 & 18 & 8 & 4 & 27 & 17 & 4 & 2 & 0 & 0 \\
42 & 36 & 18 & 12 & 4 & 2 & 52 & 40 & 16 & 14 & 4 & 2 & 15 & 12 & 4 & 0 & 0 & 0 \\
20 & 16 & 10 & 6 & 2 & 0 & 28 & 20 & 10 & 6 & 2 & 0 & 8 & 4 & 0 & 0 & 0 & 0 \\
107 & 74 & 37 & 22 & 9 & 4 & 43 & 28 & 11 & 8 & 2 & 1 & 6 & 4 & 2 & 0 & 0 & 0 \\
94 & 62 & 31 & 17 & 6 & 0 & 36 & 22 & 8 & 4 & 1 & 0 & 6 & 2 & 0 & 0 & 0 & 0 \\
108 & 66 & 32 & 17 & 7 & 0 & 30 & 16 & 6 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
106 & 68 & 32 & 15 & 4 & 0 & 26 & 12 & 4 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
66 & 48 & 23 & 9 & 0 & 0 & 16 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 19 & 8 & 4 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 22 & 11 & 4 & 0 & 0 & 8 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 15 & 6 & 2 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 9 & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 4 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

$$A_{27}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
190&140&70&42&17&8&122&82&37&22&11&5&35&29&7&6&0&0\\
190&142&74&50&23&10&106&72&33&19&8&4&20&14&3&1&0&0\\
280&210&108&77&39&20&144&98&46&27&13&8&26&18&6&2&0&0\\
274&200&101&72&37&18&136&90&43&24&11&6&24&14&4&1&0&0\\
210&152&75&57&32&16&98&64&31&17&8&4&14&6&2&0&0&0\\
100&71&36&28&16&10&46&31&16&8&4&0&6&2&0&0&0&0\\
104&74&37&28&16&8&48&30&15&8&4&2&8&2&1&0&0&0\\
92&65&32&24&14&8&36&23&12&6&2&0&4&1&0&0&0&0\\
77&57&28&20&12&6&27&17&8&4&0&0&2&0&0&0&0&0\\
57&44&22&16&8&4&19&12&6&2&0&0&1&0&0&0&0&0\\
39&32&16&12&4&2&13&8&4&0&0&0&0&0&0&0&0&0\\
20&16&10&6&2&0&8&4&0&0&0&0&0&0&0&0&0&0\\
51&34&17&8&2&1&10&4&2&0&0&0&0&0&0&0&0&0\\
24&16&8&4&1&0&6&2&0&0&0&0&0&0&0&0&0&0\\
\end{pmatrix}.$$

$$A_{37}=\begin{pmatrix}
20&12&6&2&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
20&10&4&1&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
12&4&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
6&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
6&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
4&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
108&70&34&18&8&4&27&14&7&2&0&0&3&0&0&0&0&0\\
88&58&25&15&7&4&14&8&3&1&0&0&0&0&0&0&0&0\\
124&86&40&24&12&8&20&12&6&2&0&0&0&0&0&0&0&0\\
118&80&39&23&11&6&20&10&4&1&0&0&0&0&0&0&0&0\\
86&56&28&16&8&4&12&4&2&0&0&0&0&0&0&0&0&0\\
40&28&16&8&4&0&6&2&0&0&0&0&0&0&0&0&0&0\\
42&28&14&8&4&2&6&2&1&0&0&0&0&0&0&0&0&0\\
32&22&12&6&2&0&4&1&0&0&0&0&0&0&0&0&0&0\\
24&16&8&4&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
18&12&6&2&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
12&8&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
8&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
8&4&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
6&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\end{pmatrix}.$$

$$A_{47} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 23 & 12 & 6 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 8 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 12 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 10 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_{57} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 100 & 68 & 30 & 18 & 8 & 4 & 70 & 46 & 18 & 13 & 5 & 0 & 24 & 16 & 3 & 4 & 1 & 0 \\ 126 & 88 & 44 & 26 & 12 & 6 & 86 & 58 & 25 & 14 & 4 & 0 & 20 & 10 & 1 & 1 & 0 & 0 \\ 42 & 28 & 14 & 8 & 4 & 2 & 26 & 20 & 9 & 4 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 \\ 32 & 22 & 12 & 6 & 2 & 0 & 20 & 14 & 6 & 2 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 18 & 12 & 6 & 2 & 0 & 0 & 10 & 4 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_{67}=\begin{pmatrix}
8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
80 & 56 & 25 & 15 & 7 & 4 & 156 & 124 & 48 & 45 & 22 & 10 & 128 & 118 & 31 & 46 & 20 & 10\\
120 & 82 & 39 & 23 & 11 & 6 & 254 & 202 & 78 & 74 & 38 & 18 & 222 & 192 & 48 & 79 & 40 & 18\\
44 & 28 & 14 & 8 & 4 & 2 & 98 & 78 & 30 & 29 & 16 & 8 & 94 & 74 & 17 & 33 & 20 & 8\\
32 & 22 & 12 & 6 & 2 & 0 & 85 & 67 & 26 & 24 & 14 & 8 & 81 & 66 & 16 & 28 & 16 & 8\\
18 & 12 & 6 & 2 & 0 & 0 & 55 & 45 & 16 & 16 & 8 & 4 & 50 & 49 & 12 & 18 & 8 & 4\\
8 & 4 & 0 & 0 & 0 & 0 & 20 & 16 & 4 & 6 & 2 & 0 & 20 & 16 & 4 & 6 & 2 & 0\\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
42 & 26 & 13 & 7 & 3 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
26 & 20 & 10 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
13 & 10 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
7 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
128 & 98 & 51 & 37 & 19 & 10 & 46 & 28 & 13 & 7 & 3 & 0 & 4 & 2 & 0 & 0 & 0 & 0\\
98 & 72 & 36 & 28 & 16 & 8 & 28 & 20 & 10 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
51 & 36 & 18 & 14 & 8 & 5 & 13 & 10 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
37 & 28 & 14 & 10 & 6 & 3 & 7 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
19 & 16 & 8 & 6 & 2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
10 & 8 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

By MATLAB soft, we have $\det C = 13^{32}$, which is also known by a general result in the representation theory of finite groups on the determinant of the Cartan invariant matrix.

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References

[1] Alperlin, J.L. (1975) On Modules for the Linear Fractional Groups. International Symposium on Theory of Finite Groups. RIMS, Kyoto.

[2] Alperlin, J.L. (1979) Projective Modules for $SL(2, 2^n)$. *Journal of Pure and Applied Algebra*, **15**, 219-234. http://dx.doi.org/10.1016/0022-4049(79)90017-3

[3] Upadhyaya, B.S. (1978) Composition Factors of the Principal Indecomposable Modules for the Speciallinear Group $SL(2, q)$. *Journal of London Mathematical Society*, **17**, 437-445.

[4] Benson, D.J. and Martin, S. (1991) Mod 2 Cohomology of the Unitary Groups $U_3(2^n)$. *Communications in Algebra*, **19**, 3125-3144. http://dx.doi.org/10.1080/00927879108824311

[5] Humphreys, J.E. (1976) Ordinary and Modular Representations of Chevalley Groups. Lecture Notes in Mathematics 528, Springer, Berlin.

[6] Humphreys, J.E. (1973) Some Computations of Cartan Invariants for Finite Groups of Lie Type. *Communications on Pure and Applied Mathematics*, **26**, 745-755. http://dx.doi.org/10.1002/cpa.3160260516

[7] Ye, J.C. (1982) The Cartan Invariants of $SL(3, p^n)$. *Journal of Mathematical Research with Applications*, **2**, 9-19. (In Chinese)

[8] Zaslawsky, E. (1974) Computational Methods Applied to Ordinary and Modular Characters of Some Finite Simple

Groups. Ph.D. Thesis, University of California, Santa Cruz.

[9] Benson, D.J. (1983) The Loewy Structure of the Projective Indecomposable Modules for a_8 in Characteristic 2. *Communications in Algebra*, **11**, 1395-1432. http://dx.doi.org/10.1080/00927878308822912

[10] Du, J. (1986) The Cartan Invariants of *SL*(4, 2n). *Journal of East China Normal University* (Natural Sciences Ed.), **4**, 17-25. (In Chinese)

[11] Ye, J.C. (1997) The Cartan Invariant Matrix for the Finite Group *SL*(5, 2). *Algebra Colloquium*, **4**, 203-211.

[12] Thackray, J.G. (1980) Modular Representations of Some Finite Groups. Ph.D. Thesis, Cambridge University, Cambridge.

[13] Humphreys, J.E. (1980) Modular Representations of Finite Groups of Lie Type. Finite Simple Groups II, Academic Press, London, 259-290.

[14] Ye, J.C. (1985) The Cartan Invariants of *Sp*(4, pn). *Journal of East China Normal University* (Natural Sciences Ed.), **2**, 18-25. (In Chinese)

[15] Ye, J.C. and Zhou, Z.G. (2001) Irreducible Characters of Algebraic Groups in Characteristic Three. *Communications in Algebra*, **29**, 201-223. http://dx.doi.org/10.1081/AGB-100000795

[16] Liu, A.L. and Ye, J.C. (2003) Cartan Matrix for the Finite Symplectic Group *Sp*(6, 3). *Algebra Colloquium*, **10**, 33-39. http://dx.doi.org/10.1007/s100110300005

[17] Hu, Y.W. and Xu, L. (2011) The Cartan Invariant Matrix for Finite Symplectic Group *Sp*(4, 11). *Advances in Mathematics* (China), **40**, 483-491.

[18] Mertens, D. (1985) Zur Darstellungstheorie der endlichen Chevalley-Gruppen vom Typ G2. Diplomarbeit, University of Bonn, Bonn.

[19] Hu, Y.W. and Ye, J.C. (2001) Cartan Invariant Matrix for Finite Group of Lie Type G2(5). *Journal of Tongji University* (Natural Sciences Ed.), **29**, 1205-1208. (In Chinese)

[20] Lusztig, G. (1980) Hecke Algebras and Jantzen's Generic Decomposition Patterns. *Advances in Mathematics*, **37**, 121-164. http://dx.doi.org/10.1016/0001-8708(80)90031-6

[21] Jantzen, J.C. (1977) Uber das Dekompositionsverhalten Gewisser Modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren. *Journal of Algebra*, **49**, 441-469. http://dx.doi.org/10.1016/0021-8693(77)90252-6

[22] Humphreys, J.E. (1987) Cohomology of Link Bundles on *G*/*B* for the Ecceptional Group G_2. *Journal of Pure and Applied Algebra*, **44**, 160-168.

[23] Andersen, H.H. (1981) On the Structure of the Conhomology of Line Bundles on *G*/*B*. *Journal of Algebra*, **71**, 245-258. http://dx.doi.org/10.1016/0021-8693(81)90119-8

[24] Jantzen, J.C. (1980) Darstellungen Halbeinfacher Gruppen and Ihrer Frobenius-Kerne. *Journal für die Reine und Angewandte Mathematik*, **317**, 157-199.

[25] Chastkofsky, L. (1981) Projective Characters for Finite Chevalley Groups. *Journal of Algebra*, **69**, 347-357. http://dx.doi.org/10.1016/0021-8693(81)90209-X

[26] Jantzen, J.C. (1981) Zur Reduktion modulo *p* der Character von Deligne und Lusztig. *Journal of Algebra*, **70**, 452-474. http://dx.doi.org/10.1016/0021-8693(81)90229-5