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A Matrix Inequality for the Inversions of the Restrictions of a Positive Definite Hermitian Matrix

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ABSTRACT

We exploit the theory of reproducing kernels to deduce a matrix inequality for the inverse of the restriction of a positive definite Hermitian matrix.

Keywords: Reproducing Kernel; Positive Definite Hermitian Matrix; Quadratic Inequality; Inversion of Positive Definite Hermitian Matrix; Restriction of Positive Definite Hermitian Matrix; Schur Complement; Block Matrix

1. Introduction and Results

By exploiting the general structure of reproducing kernel Hilbert spaces, it is possible to prove very interesting norm inequalities (see, e.g., [1,2]). A typical result is as follows.

Let D be an N-ply connected regular domain whose boundary consists of disjoint analytic Jordan curves. Let $\varphi,\psi \in H_2(D)$ be analytic Hardy functions with index two. Then the following generalised isoperimetric inequality holds,

$$\frac{1}{\pi}\iint_D \left|\varphi(z)\psi(z)\right|^2 \mathrm{d}x\mathrm{d}y \le \frac{1}{2\pi}\int_{\partial D}\left|\varphi(z)\right|^2 |\mathrm{d}z| \frac{1}{2\pi}\int_{\partial D}\left|\psi(z)\right|^2 |\mathrm{d}z|, \quad z = x+iy.$$

Moreover, we can completely describe the cases for which we have the equality instead of the inequality here above. Without the theory of reproducing kernels, such a simple and beautiful inequality could not be derived (see [2,3] for the details).

In this paper we introduce a new inequality. Let $A_n = \{a_{i,j}\}_{i,j=1}^{n}$ be a positive definite Hermitian matrix. Let $0 < m < n$ and let $A_m = \{a_{i,j}\}_{i,j=1}^{m}$ be the restriction of A_m to an m dimensional subspace of $\mathbb{C}^n$. Without loss of generality, assume that A_m is the $m\times m$ leading principal minor of A_n. Let A_n^{-1} and A_m^{-1} denote the inverse of A_n and of A_m, respectively. Then we have the following results.

Theorem 1.1 *If* $x \in (x_1,\cdots,x_n) \in \mathbb{C}^n$ *and* $x_{(m)}$ *is the vector of* $\mathbb{C}^m$ *defined by* $x_{(m)} \equiv (x_1,\cdots,x_m)$, *then*

$$x_{(m)}^* A_m^{-1} x_{(m)} \le x^* A_n^{-1} x. \tag{1}$$

Here * denotes conjugate transpose. As an immediate consequence, one also obtains the following corollary.

Corollary 1.2 *If* $\left(A_n^{-1}\right)_m$ *is the restriction of the matrix* A_n^{-1} *to* $\mathbb{C}^m$, *then*

$$A_m^{-1} \le \left(A_n^{-1}\right)_m. \tag{2}$$

Here $\le$ denotes the positive definite order, *i.e.*, if M and N are square matrices, we say that $M \le N$ if $N - M$ is a positive semi-definite matrix.

We observe that for $n = 2$, such results can be checked directly. However, for $n \ge 3$, the result of Theorem 1.1 is not intuitive and appears mysterious, at least at first glance.

2. Proof of the Results

The proof of Theorem 1.1 is based on the theory of reproducing kernels. Therefore, we begin by introducing some notions and results which are used in the sequel.

2.1. Reproducing Kernels

Let E be an arbitrary abstract (non-void) set. Let $\mathscr{F}(E)$ denote the set of all complex-valued functions on E. A reproducing kernel Hilbert spaces (RKHS for short) on the set E is a Hilbert space $\mathscr{H} \subset \mathscr{F}(E)$ endowed with a function $K : E \times E \to \mathscr{H}$, which is called the reproducing kernel and which satisfies **the reproducing property**. Namely we have

$$K_p \equiv K(\cdot, p) \in \mathscr{H} \text{ for all } p \in E \tag{3}$$

and

$$f(p) = \langle f, K_p \rangle_{\mathscr{H}} \tag{4}$$

for all $p \in E$ and for all $f \in \mathscr{H}$. We denote by $H_K(E)$ (or H_K) the reproducing kernel Hilbert space $\mathscr{H}$ whose corresponding reproducing function is K.

A complex-valued function $K : E \times E \to \mathbb{C}$ is called a **positive definite quadratic form function** on the set E, or shortly, **positive definite function**, if, for an arbitrary function $X : E \to \mathbb{C}$ and for any finite subset F of E, one has

$$\sum_{p,q \in F} \overline{X(p)} X(q) K(p,q) \ge 0. \tag{5}$$

By a fundamental theorem, we know that, for any positive definite quadratic form function K on E, there exists a unique reproducing kernel Hilbert space on E with reproducing kernel K. So, in a sense, the correspondence between the reproducing kernel K and the reproducing kernel Hilbert space $H_K(E)$ is one to one.

A simple example of positive definite quadratic form function is a positive definite Hermitian matrix.

Example 2.1 *Let $E = \{p_1, p_2, \cdots, p_n\}$ be a set consisting of n distinct points. Let $A_n \equiv \{a_{ij}\}_{i,j=1}^{n}$ be a strictly positive $n \times n$ Hermitian matrix. Let $A_n^{-1} \equiv \{b_{ij}\}_{i,j=1}^{n}$ denote the inverse of A_n. Then the space $\mathscr{F}(E)$ of the complex valued functions on E, endowed with the inner product*

$$\langle f, g \rangle_{H_{\bar{A}_n}(E)} \equiv \sum_{i,j=1}^{n} f(x_i) b_{ij} \overline{g(p_j)}$$

$$= \overline{\left(g(p_1), g(p_2), \cdots, g(p_n)\right)} A_n^{-1} \begin{pmatrix} f(p_1) \\ f(p_2) \\ \vdots \\ f(p_n) \end{pmatrix},$$

is a reproducing kernel Hilbert (complex Euclidean) space with reproducing kernel K defined by $K(p_i, p_j) = a_{ji}$ for all $i, j = 1, \cdots, n$.

Indeed, the validity of (3) follows by a straightforward calculation. To prove (4) we observe that

$$\langle f, K(\cdot, p_k) \rangle_{H_{K_{\bar{A}_n}}(E)} = \sum_{i,j=1}^{n} f(p_i) b_{ij} \overline{K(p_j, p_k)}$$

$$= \sum_{i,j=1}^{n} f(p_i) b_{ij} a_{jk} = f(p_k)$$

for all $k = 1, \cdots, n$ (note that $\overline{a_{kj}} = a_{jk}$). Thus $H_{\bar{A}_n}(E)$ coincides with the reproducing kernel Hilbert space $H_K(E)$. In particular the norm induced by the product $\langle \cdot, \cdot \rangle_{H_{\bar{A}_n}(E)}$ coincides with the norm of $H_K(E)$.

We can thus combine the two theories of postitive definite Hermitian matrices and of reproducing kernels (see [4-12]).

2.2. Restriction of a Reproducing Kernel

The validity of Theorem 1.1 follows by the properties of the restriction of a reproducing kernel in a general setting. Let E be a non-empty set and let E_0 be a non-empty subset of E. Let $K : E \times E \to \mathbb{C}$ be a positive definite quadratic form function. Then the restriction $K|_{E_0 \times E_0}$ of K to $E_0 \times E_0$ is a positive definite quadratic form function on $E_0 \times E_0$ and the relation between $H_K(E)$ and $H_{K|_{E_0 \times E_0}}(E_0)$ is given by the following statement.

Proposition 2.1 (Restriction of RKHS) *Let E be a non-empty set and let E_0 be a non-empty subset of E. Let $K : E \times E \to \mathbb{C}$ be a positive definite quadratic form function. Then the Hilbert space defined by the positive definite quadratic form function $K|_{E_0 \times E_0}$ is given by*

$$H_{K|_{E_0 \times E_0}}(E_0) = \left\{ f \in \mathscr{F}(E_0) : f = \tilde{f}\big|_{E_0} \text{ for some } \tilde{f} \in H_K(E) \right\}. \tag{6}$$

Furthermore, the norm of $H_{K|_{E_0 \times E_0}}(E_0)$ is expressed in terms of the norm of $H_K(E)$ by the following equality,

$$\|f\|_{H_{K|_{E_0 \times E_0}}(E_0)} = \min\left\{ \|\tilde{f}\|_{H_K(E)} : \tilde{f} \in H_K(E), f = \tilde{f}\big|_{E_0} \right\} \tag{7}$$

which holds for all $f \in H_{K|_{E_0 \times E_0}}(E_0)$.

See [1] for the details.

2.3. Proof of Theorem 1.1

Let $n, m \in \mathbb{N}$ with $0 < m < n$. Let $E \equiv \{p_1, p_2, \cdots, p_n\}$ and $E_0 \equiv \{p_1, p_2, \cdots, p_m\}$. Let K be the positive definite quadratic form function on E defined by $K(p_i, p_j) = a_{ji}$ for all $i, j = 1, \cdots, n.$ Let $x \equiv (x_1, \cdots, x_n) \in \mathbb{C}^n$ and $x_{(m)} \equiv (x_1, \cdots, x_m) \in \mathbb{C}^m$. Let $\tilde{f}$ be the function on E defined by $\tilde{f}(p_i) = x_i$ for all $i = 1, \cdots, n$. Let $f \equiv \tilde{f}\big|_{E_0}$. Then we have $\|\tilde{f}\|^2_{H_K(E)} = \langle f, f \rangle_{H_K(E)} = x^* A_n^{-1} x$ and $\|f\|^2_{H_{K|_{E_0 \times E_0}}(E_0)} = \langle f, f \rangle_{H_{K|_{E_0 \times E_0}}(E_0)} = x^*_{(m)} A_m^{-1} x_{(m)}$. Thus (7) implies that $x^*_{(m)} A_m^{-1} x_{(m)} \le x^* A_n^{-1} x$.

3. An Alternative Proof Based on Schur Complement

We provide in this section a direct proof of Theorem 1.1 based on the properties of the Schur complement (cf., e.g., [13]). Let $n, m \in \mathbb{N}$ with $0 < m < n$. Let A_n be a positive definite Hermitian $n \times n$ matrix and assume that

$$A_n \equiv \begin{pmatrix} A_m & B \\ B^* & A_{n-m} \end{pmatrix}$$

where A_m is an $m \times m$ matrix, A_{n-m} is an $(n-m) \times (n-m)$ matrix, and B is an $(n-m) \times m$ matrix. Observe that A_m is positive definite and henceforth invertible. Then the inverse A_n can be written in the form

$$A_n^{-1} \equiv \begin{pmatrix} A_m^{-1} + A_m^{-1} B S_{n-m}^{-1} B^* A_m^{-1} & -A_m^{-1} B S_{n-m}^{-1} \\ -S_{n-m}^{-1} B^* A_m^{-1} & S_{n-m}^{-1} \end{pmatrix}$$

where $S_{n-m} \equiv A_{n-m} - B^* A_m^{-1} B$ is the Schur complement with respect to A_m. Since $A_n > 0$ we also have $A_n^{-1} > 0$ which implies that $S_{n-m}^{-1} > 0$. We now observe that the validity of Theorem 1.1 is equivalent to say that the matrix M defined by

$$M \equiv A_n^{-1} - \begin{pmatrix} A_m^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_m^{-1} B S_{n-m}^{-1} B^* A_m^{-1} & -A_m^{-1} B S_{n-m}^{-1} \\ -S_{n-m}^{-1} B^* A_m^{-1} & S_{n-m}^{-1} \end{pmatrix}$$

is positive semi-definite. Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^{n-m}$. Then we calculate

$$\begin{aligned}
&(x, y)^* M (x, y) \\
&= x^* A_m^{-1} B S_{n-m}^{-1} B^* A_m^{-1} x - y^* S_{n-m}^{-1} B^* A_m^{-1} x \\
&\quad - x^* A_m^{-1} B S_{n-m}^{-1} y + y^* S_{n-m}^{-1} y \\
&= \left(B^* A_m^{-1} x\right)^* S_{n-m}^{-1} \left(B^* A_m^{-1} x\right) - y^* S_{n-m}^{-1} \left(B^* A_m^{-1} x\right) \\
&\quad - \left(B^* A_m^{-1} x\right)^* S_{n-m}^{-1} y + y^* S_{n-m}^{-1} y \\
&= \left(B^* A_m^{-1} x, y\right)^* \begin{pmatrix} S_{n-m}^{-1} & -S_{n-m}^{-1} \\ -S_{n-m}^{-1} & S_{n-m}^{-1} \end{pmatrix} \left(B^* A_m^{-1} x, y\right)
\end{aligned}$$

(here we understand that (x, y) and $\left(B^* A_m^{-1} x, y\right)$ are column vectors). Now we observe that

$$\begin{pmatrix} S_{n-m}^{-1} & -S_{n-m}^{-1} \\ -S_{n-m}^{-1} & S_{n-m}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes S_{n-m}^{-1}$$

where $\otimes$ denotes the Kronecker product of matrices. It is known that the Kronecker product of positive semi-definite matrices is positive semi-definite. Now

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ge 0$$

and $S_{n-m}^{-1} > 0$, hence

$$\begin{pmatrix} S_{n-m}^{-1} & -S_{n-m}^{-1} \\ -S_{n-m}^{-1} & S_{n-m}^{-1} \end{pmatrix}$$

is positive semi-definite and accordingly $(x, y)^* M (x, y) \ge 0$. Our proof is completed.

4. Remark

The results in this paper were given implicitly in the extensive paper [14]. However, such results were not explicitly stated in the corresponding Theorem (**Ultimate realization of reproducing kernel Hilbert spaces**). For this reason, we wrote this paper where we clearly present our Theorem 1.1. We note that such ideas have arisen to our attention while analysing the structure of the theorem from the viewpoint of the support vector machine for the practical calculation.

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