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EP_r Solution to a System of Matrix Equations

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ABSTRACT

A square complex matrix A is called EP if it can be written in the form $U\left(A_1 \oplus 0\right)U$ with U being fixed unitary and A_1 being arbitrary matrix in $\mathbb{C}^{r\times r}$. We give necessary and sufficient conditions for the existence of the EP_r solution to the system of complex matrix equation $AX = B, XC = D$ and present an expression of the EP_r solution to the system when the solvability conditions are satisfied. In addition, the solution to an optimal approximation problem is obtained. Furthermore, the least square EP_r solution with least norm to this system mentioned above is considered. The representation of such solution is also derived.

Keywords: EP Matrix; Matrix Equation; Moore-Penrose Inverse; Approximation Problem; Least Squares Solution

1. Introduction

Throughout we denote the complex $m\times n$ matrix space by $\mathbb{C}^{m\times n}$, the real $m\times n$ matrix space by $\mathbb{R}^{m\times n}$. The symbols $I, A^*, \mathcal{R}(A), \mathcal{N}(A)$, and $\|A\|$ stand for the identity matrix with the appropriate size, the conjugate transpose, the range, the null space, and the Frobenius norm of $A\in\mathbb{C}^{m\times n}$, respectively. The Moore-Penrose inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A^\dagger$ is defined to be the unique matrix $X\in\mathbb{C}^{n\times m}$ of the following matrix equations

$$AXA = A, XAX = X, \left(XA\right)^* = XA, \left(AX\right)^* = AX.$$

Recall that an $n\times n$ complex matrix A is called EP (or range Hermitian) if $AA^\dagger = A^\dagger A$. EP matrices were introduced by Schwerdtfeger in [1], ever since many authors have studied EP matrices with entries from complex number field to semigroups with involution and given various equivalent conditions and many characterizations for matrix to be EP (see, [2-5]).

Investigating the matrix equation

$$AX = B \tag{1}$$

with the unknown matrix X being symmetric, reflexive, Hermitian-generalized Hamiltonian and re-positive definite is a very active research topic (see, [6-9]). As a generalization of (1), the classical system of matrix equations

$$AX = C, XB = D \tag{2}$$

has attracted many people's attention and many results have been obtained about system (2) with various constraints, such as bisymmetric, Hermitian, positive semidefinite, reflexive, and generalized reflexive solutions, and so on (see, [9-12]). It is well-known that EP matrices are a wide class of objects that include many matrices as their special cases, such as Hermitian and skew-Hermitian matrices (*i.e.*, $A^* = \pm A$), normal matrices (*i.e.*, $AA^* = A^*A$), as well as all nonsingular matrices. Therefore investigating the EP solution of the matrix Equation (2) is very meaningful.

Pearl showed in ([2]) that a matrix A is EP if and only if it can be written in the form $U\left(A_1 \oplus 0\right)U^*$ with U unitary and A_1 nonsingular. A square complex matrix A is called EP_r if it can be written in the form $U\left(A_1 \oplus 0\right)U^*$, where U is fixed unitary and A_1 is arbitrary matrix in $\mathbb{C}^{r\times r}$. To our knowledge, so far there has been little investigation of this EP_r solution to (2).

Motivated by the work mentioned above, we investigate EP_r solution to (2). We also consider the optimal approximation problem

$$\left\|\hat{X} - E\right\| = \min_{X\in S_X}\left\|X - E\right\|, \tag{3}$$

where E is a given matrix in $\mathbb{C}^{n\times n}$ and S_X the set of all EP_r solutions to (2). In many case Equation (2) has not an EP_r solution. Hence we need to further study its least squares solution, which can be described as follows: Let $\mathbb{C}_{ep}^{n\times n}(U)$ denote the set of all EP_r matrices with fixed unitary matrix U in $\mathbb{C}^{n\times n}$,

$$S_L = \left\{ X \mid \min_{X\in\mathbb{C}_{ep}^{n\times n}(U)} \|AX - C\|^2 + \|XB - D\|^2 \right\}.$$

Find $\tilde{X} \in \mathbb{C}_{ep}^{n\times n}(U)$ such that

$$\|\tilde{X}\| = \min_{X\in S_L} \|X\|. \tag{4}$$

In Section 2, we present necessary and sufficient conditions for the existence of the EP_r solution to (2), and give an expression of this solution when the solvability conditions are met. In Section 3, we derive an optimal approximation solution to (3). In Section 4, we provide the least squares EP_r solution to (4).

2. EP_r Solution to (2)

In this section, we establish the solvability conditions and the general expression for the EP_r solution to (2).

Throughout we denotes $\mathbb{C}_{ep}^{n\times n}(U)$ the set of all EP_r matrices with fixed unitary matrix U in $\mathbb{C}^{n\times n}$, *i.e.*,

$$\mathbb{C}_{ep}^{n\times n} = \left\{ A \mid A = U(A_1 \oplus 0)U^* \right\},$$

where U is fixed unitary and A_1 is arbitrary matrix in $\mathbb{C}^{r\times r}$.

Lemma 2.1. ([3]) Let $A, C \in \mathbb{C}^{m\times n}$, $B, D \in \mathbb{C}^{n\times p}$. Then the system of matrix equations $AX = C, XB = D$ is consistent if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A), \mathcal{N}(B) \subset \mathcal{N}(D), AD = CB.$$

In that case, the general solution of this system is

$$X = A^\dagger C + DB^\dagger - A^\dagger ADB^\dagger + (I - A^\dagger A)Y(I - BB^\dagger),$$

where $Y \in \mathbb{C}^{n\times n}$ is arbitrary.

Now we consider the EP_r solution to (1). By the definition of EP_r matrix, the solution has the following factorization:

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Let

$$AU = [A_1, A_2], \quad CU = [C_1, C_2],$$

$$U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1, C_1 \in \mathbb{C}^{m\times r}$, $A_2, C_2 \in \mathbb{C}^{m\times(n-r)}$, $B_1, D_1 \in \mathbb{C}^{r\times p}$, $B_2, D_2 \in \mathbb{C}^{(n-r)\times p}$, then (2) has EP_r solution if and only if the system of matrix equations

$$A_1X_1 = C_1, X_1B_1 = D_1, 0 = C_2, 0 = D_2,$$

is consistent. By Lemma 2.1, we have the following theorem.

Theorem 2.2. Let $A, C \in \mathbb{C}^{m\times n}, B, D \in \mathbb{C}^{n\times p}$ and

$$AU = [A_1, A_2], \quad U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $\begin{array}{l} A_1, C_1 \in \mathbb{C}^{m\times r}, A_2, C_2 \in \mathbb{C}^{m\times(n-r)}, \\ B_1, D_1 \in \mathbb{C}^{r\times p}, B_2, D_2 \in \mathbb{C}^{(n-r)\times p}. \end{array}$

Then the matrix Equation (2) has a EP_r solution in $\mathbb{C}^{n\times n}$ if and only if

$$\begin{aligned} &\mathcal{R}(C_1) \subset \mathcal{R}(A_1), \mathcal{N}(B_1) \subset \mathcal{N}(D_1), \\ &A_1D_1 = C_1B_1, C_2 = D_2 = 0. \end{aligned} \tag{5}$$

In that case, the general EP_r solution of (1) is

$$X = U \begin{bmatrix} A_1^\dagger C_1 + D_1B_1^\dagger - A_1^\dagger A_1D_1B_1^\dagger + (I - A_1^\dagger A_1)Y_1(I - B_1B_1^\dagger) & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{6}$$

where $Y_1 \in \mathbb{C}^{r\times r}$ is arbitrary.

3. The Solution of Optimal Approximation Problem (3)

When the set S_X of all EP_r solution to (2) is nonempty, it is easy to verify S_X is a closed set. Therefore the optimal approximation problem (3) has a unique solution by [13]. We first verify the following lemma.

Lemma 3.1. Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}, C \in \mathbb{C}^{n\times n}$. Then the procrustes problem

$$\min_{X\in\mathbb{C}^{n\times n}} \left\| (I - A^\dagger A)X(I - BB^\dagger) - C \right\|$$

has a solution which can be expressed as

$$X = C + A^\dagger G_1 + G_2B^\dagger,$$

where $G_1 \in \mathbb{C}^{m\times n}, G_2 \in \mathbb{C}^{n\times p}$ are arbitrary matrices.

Proof. It follows from the properties of Moore-Penrose generalized inverse and the inner product that

$$\begin{aligned} &\left\| (I - A^\dagger A)X(I - BB^\dagger) - C \right\|^2 = \left\langle (I - A^\dagger A)X(I - BB^\dagger) - C, (I - A^\dagger A)X(I - BB^\dagger) - C \right\rangle \\ &= \left\langle (I - A^\dagger A)(X - C)(I - BB^\dagger), (I - A^\dagger A)(X - C)(I - BB^\dagger) \right\rangle + \left\langle (I - A^\dagger A)C(I - BB^\dagger), (I - A^\dagger A)C(I - BB^\dagger) \right\rangle \\ &= \left\| (I - A^\dagger A)(X - C)(I - BB^\dagger) \right\|^2 + \left\| (I - A^\dagger A)C(I - BB^\dagger) \right\|^2. \end{aligned}$$

Hence,

$$\min_{X\in\mathbb{C}^{n\times n}}\left\|\left(I-A^{\dagger}A\right)X\left(I-BB^{\dagger}\right)-C\right\|$$

if and only if

$$\min_{X\in\mathbb{C}^{n\times n}}\left\|\left(I-A^{\dagger}A\right)\left(X-C\right)\left(I-BB^{\dagger}\right)\right\|.$$

It is clear that $X=C+A^{\dagger}G_1+G_2B^{\dagger}$ with $G_1\in\mathbb{C}^{m\times n}$, $G_2\in\mathbb{C}^{n\times p}$ are arbitrary is the solution of the above procrustes problem.

Theorem 3.2. Let $A,C\in\mathbb{C}^{m\times n}, B,D\in\mathbb{C}^{n\times p}, E\in\mathbb{C}^{n\times n}$, and

$$U^{*}EU=\begin{bmatrix}E_1 & E_2\\ E_3 & E_4\end{bmatrix}, \tag{7}$$

where $E_1\in\mathbb{C}^{r\times r}$. Assume S_X is nonempty, then the optimal approximation problem (3) has a unique solution $\hat{X}$ and

$$\hat{X}=U\begin{bmatrix}A_1^{\dagger}C_1+D_1B_1^{\dagger}-A_1^{\dagger}A_1D_1B_1^{\dagger}+\left(I-A_1^{\dagger}A_1\right)E_1\left(I-B_1B_1^{\dagger}\right) & 0\\ 0 & 0\end{bmatrix}U^{*}, \tag{8}$$

Proof. Since S_X is nonempty, $X\in S_X$ has the form of (6). It follows from (7) and the unitary invariance of Frobenius norm that

$$\begin{aligned}
&\left\|X-E\right\|^2\\
&=\left\|U\begin{bmatrix}A_1^{\dagger}C_1+D_1B_1^{\dagger}-A_1^{\dagger}A_1D_1B_1^{\dagger}+\left(I-A_1^{\dagger}A_1\right)Y_1\left(I-B_1B_1^{\dagger}\right) & 0\\ 0 & 0\end{bmatrix}U^{*}-C\right\|^2\\
&=\left\|\begin{bmatrix}A_1^{\dagger}C_1+D_1B_1^{\dagger}-A_1^{\dagger}A_1D_1B_1^{\dagger}+\left(I-A_1^{\dagger}A_1\right)Y_1\left(I-B_1B_1^{\dagger}\right) & 0\\ 0 & 0\end{bmatrix}-U^{*}CU\right\|^2\\
&=\left\|\begin{bmatrix}A_1^{\dagger}C_1+D_1B_1^{\dagger}-A_1^{\dagger}A_1D_1B_1^{\dagger}+\left(I-A_1^{\dagger}A_1\right)Y_1\left(I-B_1B_1^{\dagger}\right) & 0\\ 0 & 0\end{bmatrix}-\begin{bmatrix}E_1 & E_2\\ E_3 & E_4\end{bmatrix}\right\|^2\\
&=\left\|\left(I-A_1^{\dagger}A_1\right)Y_1\left(I-B_1B_1^{\dagger}\right)-\left(E_1+A_1^{\dagger}A_1D_1B_1^{\dagger}-A_1^{\dagger}C_1-D_1B_1^{\dagger}\right)\right\|^2\\
&\quad+\left\|E_2\right\|^2+\left\|E_3\right\|^2+\left\|E_4\right\|^2.
\end{aligned}$$

Therefore, there exists $\hat{X}\in S_X$ such that the matrix nearness problem (3) holds if and only if exist $Y_1\in\mathbb{C}^{r\times r}$ such that

$$\min_{Y_1}\left\|\left(I-A_1^{\dagger}A_1\right)Y_1\left(I-B_1B_1^{\dagger}\right)-\left(E_1+A_1^{\dagger}A_1D_1B_1^{\dagger}-A_1^{\dagger}C_1-D_1B_1^{\dagger}\right)\right\|.$$

According to Lemma 3.1, we have

$$Y_1=E_1+A_1^{\dagger}A_1D_1B_1^{\dagger}-A_1^{\dagger}C_1-D_1B_1^{\dagger}+A_1^{\dagger}G_1+G_2B_1^{\dagger},$$

where $G_1\in\mathbb{C}^{m\times r}, G_2\in\mathbb{C}^{r\times p}$ are arbitrary. Substituting Y_1 into (6), we obtain that the solution of the matrix nearness problem (3) can be expressed as (8).

4. The Least Squares EP_r Solution to (4)

In this section, we give the explicit expression of the least squares EP_r solution to (4).

Lemma 4.1. ([12]) Given $E,F\in\mathbb{C}^{m\times n}$, $\Omega_1=\operatorname{diag}\left(a_1,\cdots,a_m\right)$, $\Omega_2=\operatorname{diag}\left(b_1,\cdots,b_n\right)$, $a_i>0\left(i=1,\cdots,m\right)$, $b_j>0\left(j=1,\cdots,n\right)$. Then there exists a unique matrix $\bar{S}\in\mathbb{C}^{m\times n}$ such that

$$\left\|\Omega_1S-E\right\|^2+\left\|S\Omega_2-F\right\|^2=\min.$$

And $\bar{S}$ can be expressed as

$$\bar{S}=\varnothing*\left(\Omega_1E+F\Omega_2\right),$$

where $\varnothing=\left(\dfrac{1}{a_i^2+b_j^2}\right)\in\mathbb{R}^{m\times n}$.

Theorem 4.2. Let $A,C\in\mathbb{C}^{m\times n}$, $B,D\in\mathbb{C}^{n\times p}$ and

$$AU=\left[A_1,A_2\right],\ CU=\left[C_1,C_2\right],$$

$$U^{*}B=\begin{bmatrix}B_1\\ B_2\end{bmatrix},\ U^{*}D=\begin{bmatrix}D_1\\ D_2\end{bmatrix},$$

where $A_1,C_1\in\mathbb{C}^{m\times r}$, $A_2,C_2\in\mathbb{C}^{m\times(n-r)}$, $B_1,D_1\in\mathbb{C}^{r\times p}$,

$B_2, D_2 \in \mathbb{C}^{(n-r)\times p}$. Assume that the singular value decomposition of A_1, B_1 are as follows

$$A_1 = W\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}V^*, B_1 = P\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}Q^*, \tag{9}$$

where $W = [W_1, W_2] \in \mathbb{C}^{m\times m}$, $V = [V_1, V_2] \in \mathbb{C}^{r\times r}$, $P = [P_1, P_2] \in \mathbb{C}^{r\times r}$ and $Q = [Q_1, Q_2] \in \mathbb{C}^{p\times p}$ are unitary matrices, $M_1 = \operatorname{diag}(\sigma_1, \cdots, \sigma_{r_1})$, $\sigma_i > 0\,(i = 1, \cdots, r_1)$, $r_1 = \operatorname{rank}(M_1)$, $W_1 \in \mathbb{C}^{m\times r_1}$, $V_1 \in \mathbb{C}^{r\times r_1}$, $N_1 = \operatorname{diag}(\rho_1, \cdots, \rho_{r_2})$, $\rho_j > 0\,(j = 1, \cdots, r_2)$, $r_2 = \operatorname{rank}(N_1)$, $P_1 \in \mathbb{C}^{r\times r_2}$, $Q_1 \in \mathbb{C}^{p\times r_2}$. Then $X \in S_L$ can be expressed as

$$X = U\begin{bmatrix} V\begin{bmatrix} \varnothing * \left(M_1 W_1^* C_1 P_1 + V_1^* D_1 Q_1 N_1\right) & M_1^{-1} W_1^* C_1 P_2 \\ V_2^* D_1 Q_1 N_1^{-1} & Y_4 \end{bmatrix} P^* & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{10}$$

where $\varnothing = \left(\dfrac{1}{\sigma_i^2 + \rho_j^2}\right) \in \mathbb{R}^{r_1\times r_2}$, and $Y_4 \in \mathbb{C}^{(r-r_1)\times(r-r_2)}$ is an arbitrary matrix.

Proof. It yields from (9) that

$$\begin{aligned}
\|AX - C\|^2 + \|XB - D\|^2 &= \left\| AU\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^* - C \right\|^2 + \left\| U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^* B - D \right\|^2 \\
&= \left\| [A_1, A_2]\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} - [C_1, C_2] \right\|^2 + \left\| \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \right\|^2 \\
&= \|A_1 X_1 - C_1\|^2 + \|X_1 B_1 - D_1\|^2 + \|C_2\|^2 + \|D_2\|^2 \\
&= \left\| W\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}V^* X_1 - C_1 \right\|^2 + \left\| X_1 P\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}Q^* - D_1 \right\|^2 + \|C_2\|^2 + \|D_2\|^2 \\
&= \left\| \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}V^* X_1 P - W^* C_1 P \right\|^2 + \left\| V^* X_1 P\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} - V^* D_1 Q \right\|^2 + \|C_2\|^2 + \|D_2\|^2 .
\end{aligned}$$

Assume that

$$V^* X_1 P = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}, Y_1 \in \mathbb{C}^{r_1\times r_2}, Y_4 \in \mathbb{C}^{(r-r_1)\times(r-r_2)}. \tag{11}$$

Then we have

$$\begin{aligned}
&\|AX - C\|^2 + \|XB - D\|^2 \\
&= \left\| M_1 Y_1 - W_1^* C_1 P_1 \right\|^2 + \left\| Y_1 N_1 - V_1^* D_1 Q_1 \right\|^2 \\
&\quad + \left\| M_1 Y_2 - W_1^* C_1 P_2 \right\|^2 + \left\| Y_3 N_1 - V_2^* D_1 Q_1 \right\|^2 + \left\| W_2^* C_1 P_1 \right\|^2 \\
&\quad + \left\| W_2^* C_1 P_2 \right\|^2 + \left\| V_1^* D_1 Q_2 \right\|^2 + \left\| V_2^* D_1 Q_2 \right\|^2 + \|C_2\|^2 + \|D_2\|^2 .
\end{aligned}$$

Hence

$$\min_{X \in \mathbb{C}_{ep}^{n\times n}(U)} \|AX - C\|^2 + \|XB - D\|^2$$

is solvable if and only if there exist Y_1, Y_2, Y such that

$$\left\| M_1 Y_1 - W_1^* C_1 P_1 \right\|^2 + \left\| Y_1 N_1 - V_1^* D_1 Q_1 \right\|^2 = \min, \tag{12}$$

$$\left\| M_1 Y_2 - W_1^* C_1 P_2 \right\|^2 = \min, \left\| Y_3 N_1 - V_2^* D_1 Q_1 \right\|^2 = \min. \tag{13}$$

It follows from (12) and (13) that

$$Y_1 = \varnothing * \left(M_1 W_1^* C_1 P_1 + V_1^* D_1 Q_1 N_1\right), \tag{14}$$

$$Y_2 = M_1^{-1} W_1^* C_1 P_2, Y_3 = V_2^* D_1 Q_1 N_1^{-1} \tag{15}$$

where $\varnothing = \left(\dfrac{1}{\sigma_i^2 + \rho_j^2}\right) \in \mathbb{R}^{r_1\times r_2}$. Substituting (14) and (15) into (11), we can get the form of elements in S_L is (10).

Theorem 4.3. Assume the notations and conditions are the same as Theorem 4.2. Then

$$\|\tilde{X}\| = \min_{X \in S_L} \|X\|$$

if and only if

$$\tilde{X} = U\begin{bmatrix} V\begin{bmatrix} \varnothing * \left(M_1 W_1^* C_1 P_1 + V_1^* D_1 Q_1 N_1\right) & M_1^{-1} W_1^* C_1 P_2 \\ V_2^* D_1 Q_1 N_1^{-1} & 0 \end{bmatrix} P^* & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{16}$$

where $\varnothing = \left(\dfrac{1}{\sigma_i^2 + \rho_j^2}\right) \in \mathbb{R}^{r_1 \times r_2}$.

Proof. In Theorem 4.2, it implies from (10) that $\min_{X \in S_L} \|X\|$ is equivalent to X has the expression (10) with $Y_4 = 0$. Hence (16) holds.

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