Scientific
Research

On Least Squares Solutions of Matrix Equation $MZN = S$

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ABSTRACT

Let P be a given Hermitian matrix satisfying $P^2 = I$. Using the eigenvalue decomposition of P , we consider the least squares solutions to the matrix equation $MZN = S$ with the constraint $PZ = ZP$.

1. Introduction

Throughout we denote the complex $m \times n$ matrix space by $\mathbb{C}^{m \times n}$. The symbols I, M^*, M^{-1} and $\|M\|$ stand for the identity matrix with the appropriate size, the conjugate transpose, the inverse, and the Frobenius norm of $M \in \mathbb{C}^{m \times n}$, respectively.

The reflexive matrices have extensive applications in engineering and scientific computation. It is a very active research topic to investigate the reflexive solution to the linear matrix equation

$$MZN = S \qquad (1)$$

where M and N are given matrices. For instance, Cvetković-Ilić [1] and Peng *et al*. [2] have given the necessary and sufficient conditions for the existence and the expressions of the reflexive solutions for the matrix Equation (1) by using the structure properties of matrices in required subset of $\mathbb{C}^{m \times n}$ and the generalized singular value decomposition (GSVD); Different from [1,2], Ref. [3] has considered generalized reflexive solutions of the matrix Equation (1); in addition, Herrero and Thome [4] have found the reflexive (with respect to a generalized $\{k+1\}$ —reflection matrix P) solutions of the matrix Equation (1) by the (GSVD) and the lifting technique combined with the Kronecker product.

2. The Reflexive Least Squares Solutions to Matrix Equation (1)

We begin this section with the following lemma, which can be deduced from [5].

   Lemma 1. (Theorem 3.1 in [5]) Let the canonical cor-

relation decomposition of matrix pair $[A, C]$ and B^*, D^* with $A \in \mathbb{C}^{m \times n}, C \in \mathbb{C}^{m \times k}, B \in \mathbb{C}^{p \times q}, D \in \mathbb{C}^{l \times q}$. rank $A = g$, rank $C = h$, rank $B = g'$, rank $D = h'$ be given as

$$A = U\left(\Sigma_A 0\right)E_A^{-1}, C = U\left(\Sigma_C 0\right)E_C^{-1},$$

$$B^* = V\left(\Sigma_B 0\right)E_B^{-1}, D^* = V\left(\Sigma_D 0\right)E_D^{-1},$$

where

$$\Sigma_A = \begin{pmatrix} I_i & & & \\ & \Lambda_j & & \\ \overline{0} & - & \underline{0} \\ & \Delta_j & & \\ & & & I_t \end{pmatrix}, \Sigma_C = \begin{pmatrix} I_h \\ 0 \end{pmatrix}$$

with the same row partitioning, and $g = i + j + t$,

$$\Lambda_j = \mathrm{diag}\left(\lambda_{i+1}, \cdots, \lambda_{i+j}\right), 1 > \lambda_{i+1} \geq \cdots \geq \lambda_{i+j} > 0,$$

$$\Delta_j = \mathrm{diag}\left(\Delta_{i+1}, \cdots, \Delta_{i+j}\right), 1 > \Delta_{i+1} \geq \cdots \geq \Delta_{i+j} > 0, \Lambda_j^2 + \Delta_j^2$$
$$= I_j,$$

$$U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ i & j & h-i-j & m-h-j-t & j & t \end{pmatrix},$$

$$\Sigma_B = \begin{pmatrix} I_{i'} & & & \\ & \overline{\Lambda}_{j'} & & \\ \overline{0} & - & \underline{0} \\ & \overline{\Delta}_{j'} & & \\ & & & I_{t'} \end{pmatrix}, \Sigma_C = \begin{pmatrix} I_{h'} \\ 0 \end{pmatrix}$$

with the same row partitioning, and $g' = i' + j' + t'$,

$$\bar{\Lambda}_{j'} = \text{diag}\left(\bar{\lambda}_{i'+1}, \cdots, \bar{\lambda}_{i'+j'}\right), 1 > \bar{\lambda}_{i'+1} \geq \cdots \geq \bar{\lambda}_{i'+j'} > 0,$$

$$\bar{\Delta}_{j'} = \text{diag}\left(\bar{\Delta}_{i'+1}, \cdots, \bar{\Delta}_{i'+j'}\right), 1 > \bar{\Delta}_{i'+1} \geq \cdots \geq \bar{\Delta}_{i'+j'} > 0, \bar{\Lambda}_{j'}^2 + \bar{\Delta}_{j'}^2 = I_{j'},$$

$$V = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ i' & j' & h'-i'-j' & q-h'-j'-t' & j' & t' \end{pmatrix},$$

and let

$$U^* F U = \left(F_{ij}\right), F_{ij} = u_i^* F u_j, i, j, = 1, \cdots, 6.$$

Then general forms of least squares solutions X, Y of matrix equation

$$AXB + CYD = F$$

are as follows:

$$X = E_A \begin{pmatrix} F_{11} - R & F_{15}\bar{\Delta}^{-1} & F_{16} & X_{14} \\ \Delta^{-1}F_{51} & X_{22} & \Lambda F_{26} + \Delta F_{56} & X_{24} \\ F_{61} & F_{62}\bar{\Lambda} + F_{65}\bar{\Delta} & F_{66} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} E_B^*,$$

$$Y = E_C \begin{pmatrix} R & F_{12} - F_{15}\bar{\Delta}^{-1}\bar{\Lambda} & F_{13} & Y_{14} \\ F_{21} - \Lambda\Delta^{-1}F_{51} & Q & F_{23} & Y_{24} \\ F_{31} & F_{32} & F_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix} E_D^*,$$

where

$$X_{22} = \Lambda\left(F_{22} - Q\right)\bar{\Lambda} + \Lambda F_{25}\bar{\Delta} + \Delta F_{52}\bar{\Lambda} + \Delta F_{52}\bar{\Delta}$$

and $Q, R, X_{14}, X_{24}, X_{34}, X_{44}, X_{41}, X_{42}, X_{43}, Y_{14}, Y_{24}, Y_{34}, Y_{44}, Y_{41}, Y_{42}, Y_{43}$

are arbitrary matrices.

Theorem 2. Given $M \in \mathbb{C}^{m \times (n+k)}$, $N \in \mathbb{C}^{(n+k) \times q}$, $S \in \mathbb{C}^{m \times q}$. Then the reflexive least squares solutions to the matrix Equation (1) can be expressed as

$$Z = T\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}T^*, \qquad (2)$$

$$X = E_A \begin{pmatrix} F_{11} - R & F_{15}\bar{\Delta}^{-1} & F_{16} & X_{14} \\ \Delta^{-1}F_{51} & X_{22} & \Lambda F_{26} + \Delta F_{56} & X_{24} \\ F_{61} & F_{62}\bar{\Lambda} + F_{65}\bar{\Delta} & F_{66} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} E_B^*,$$

$$Y = E_C \begin{pmatrix} R & F_{12} - F_{15}\bar{\Delta}^{-1}\bar{\Lambda} & F_{13} & Y_{14} \\ F_{21} - \Lambda\Delta^{-1}F_{51} & Q & F_{23} & Y_{24} \\ F_{31} & F_{32} & F_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix} E_D^*,$$

where

$$X_{22} = \Lambda\left(F_{22} - Q\right)\bar{\Lambda} + \Lambda F_{25}\bar{\Delta} + \Delta F_{52}\bar{\Lambda} + \Delta F_{52}\bar{\Delta}$$

and $Q, R, X_{14}, X_{24}, X_{34}, X_{44}, X_{41}, X_{42}, X_{43}, Y_{14}, Y_{24}, Y_{34}, Y_{44}, Y_{41}, Y_{42}, Y_{43}$

are arbitrary matrices.

Proof. It is required to transform the constrained problem to unconstrained ones. To this end, let

$$P = T\text{diag}\left(I_n, -I_k\right)T^*$$

be the eigenvalue decomposition of the Hermitian matrix P with unitary matrix T. Obviously, $PZ = ZP$ holds if and only if

$$\text{diag}\left(I_n, -I_k\right)\bar{Z} = \bar{Z}\text{diag}\left(I_n, -I_k\right) \qquad (3)$$

where $\bar{Z} = T^*ZT$. Partitioning

$$\bar{Z} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}, Z_{11} \in \mathbb{C}^{n \times n}, Z_{22} \in \mathbb{C}^{k \times k},$$

(3) is equivalent to

$$Z_{12} = -Z_{12}, Z_{21} = -Z_{21}.$$

Therefore,

$$Z = T\text{diag}\left(X, Y\right)T^*, X \in \mathbb{C}^{n \times n}, Y \in \mathbb{C}^{k \times k}. \qquad (4)$$

Partition $T = \left(T_1, T_2\right)$ and denote

$$A = MT_1, C = MT_2, B = T_1^*N, D = T_2^*N, F = S. \qquad (5)$$

According to (4), (5) and the unitary invariance of Frobenius norm

$$\|MZN - S\| = \left\| MT\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}T^*N - S \right\|$$
$$= \|AXB + CYD - F\|.$$

Applying Lemma 2.1, we derive the reflexive least squares solutions to matrix Equation (1) with the constraint $PZ = ZP$ which can be expressed as (2).

3. Acknowledgements

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