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More Results on Singular Value Inequalities for Compact Operators

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ABSTRACT

The well-known arithmetic-geometric mean inequality for singular values, according to Bhatia and Kittaneh, says that if A and B are compact operators on a complex separable Hilbert space, then $2s_j\left(AB^*\right)\le s_j\left(A^*A+B^*B\right)$ for $j=1,2,\cdots$ Hirzallah has proved that if $A_1,A_2,A_3,\text{and }A_4$ are compact operators, then $\sqrt{2}s_j\left(\left|A_1A_2^*+A_3A_4^*\right|^{\frac{1}{2}}\right)\le s_j\left(\begin{bmatrix}A_1 & A_3\\ A_2 & A_4\end{bmatrix}\right)$ for $j=1,2,\cdots$ We give inequality which is equivalent to and more general than the above inequalities, which states that if $A_{i,},B_i,i=1,2,\cdots,n$ are compact operators, then

$$2s_j\left(A_1B_1^*+A_2B_2^*+\cdots+A_nB_n^*\right)\le s_j\left[\begin{vmatrix}A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{vmatrix}^2+\begin{vmatrix}B_1 & B_2 & \cdots & B_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{vmatrix}^2\right]\quad\text{for } j=1,2,\cdots$$

Keywords: Compact Operator; Inequality; Positive Operator; Self-Adjoint Operator; Singular Value

1. Introduction

Let $B(H)$ denote the space of all bounded linear operators on a complex separable Hilbert space H, and let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. For $T\in K(H)$, the singular values of T, denoted by $s_1(T),s_2(T),\cdots$ are the eigenvalues of the positive operator $|\mathrm{T}|=\left(T^*T\right)^{1/2}$ as $s_1(T)\ge s_2(T)\ge\cdots$ repeated according to multiplicity. Note that $s_j(T)=s_j\left(T^*\right)=s_j(|T|)$ for $j=1,2,\cdots$ It follows Weyl's monotonicity principle (see, e.g., [1, p. 63] or [2, p. 26]) that if $S,T\in K(H)$ are positive and $S\le T$, then $s_j(S)\le s_j(T)$ for $j=1,2,\cdots$ Moreover, for $S,T\in K(H)$, $s_j(S)\le s_j(T)$ if and only if $s_j(S\oplus S)\le s_j(T\oplus T)$ for $j=1,2,\cdots$ The singular values of $S\oplus T$ and $\begin{bmatrix}0 & T\\ S & 0\end{bmatrix}$ are the same, and they consist of those of S together with those of T. Here, we use the direct *sum* notation $S\oplus T$ for the block-diagonal operator $\begin{bmatrix}S & 0\\ 0 & T\end{bmatrix}$ defined on $H\oplus H$.

The well-known arithmetic-geometric mean inequality for singular values, according to Bhatia and Kittaneh [3], says that if $A,B\in K(H)$, then

$$2s_j\left(AB^*\right)\le s_j\left(A^*A+B^*B\right)\tag{1.1}$$

for $j=1,2,\cdots$

Hirzallah has proved in [4] that if $A_1,A_2,A_3,\text{and } A_4\in K(H)$, then

$$\sqrt{2}s_j\left(\left|A_1A_2^*+A_3A_4^*\right|^{\frac{1}{2}}\right)\le s_j\left(\begin{bmatrix}A_1 & A_3\\ A_2 & A_4\end{bmatrix}\right)\tag{1.2}$$

for $j=1,2,\cdots$

In this paper, we will give a new inequality which is equivalent to and more general than the inequalities (1.1) and (1.2):

If $A_{i,},B_i\in K(H),i=1,2,\cdots,n$, then

$$2s_j\left(A_1B_1^*+A_2B_2^*+\cdots+A_nB_n^*\right)\le s_j\left[\left|\begin{matrix}A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{matrix}\right|^2+\left|\begin{matrix}B_1 & B_2 & \cdots & B_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{matrix}\right|^2\right] \quad (1.3)$$

for $j=1,2,\cdots$

Audeh and Kittaneh have proved in [5] that if $A,B\in K(H)$ such that A is self-adjoint, $B\ge 0$, and $\pm A\le B$, then

$$2s_j(A)\le s_j\left((B+A)\oplus(B-A)\right) \quad (1.4)$$

for $j=1,2,\cdots$ On the other hand, Tao has proved in [6] that if $A,B,C\in K(H)$ such that $\begin{bmatrix}A & B\\ B^* & C\end{bmatrix}\ge 0$, then

$$2s_j(B)\le s_j\begin{bmatrix}A & B\\ B^* & C\end{bmatrix} \quad (1.5)$$

for $j=1,2,\cdots$ Moreover, Zhan has proved in [7] that if $A,B\in K(H)$ are positive, then

$$s_j(A-B)\le s_j(A\oplus B) \quad (1.6)$$

for $j=1,2,\cdots$ We will give a new inequality which generalizes (1.5), and is equivalent to the inequalities (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6):

Let $A_1,A_2,\cdots,A_n,D\in K(H)$ such that

$$\begin{bmatrix}A_1 & 0 & \cdots & 0 & D\\ 0 & A_2 & 0 & \cdots & 0\\ \vdots & 0 & \ddots & \cdots & 0\\ 0 & \vdots & \vdots & \ddots & \vdots\\ D^* & 0 & \cdots & 0 & A_n\end{bmatrix}\ge 0, \text{ then}$$

$$2s_j(D)\le s_j\begin{bmatrix}A_1 & 0 & \cdots & 0 & D\\ 0 & A_2 & 0 & \cdots & 0\\ \vdots & 0 & \ddots & \cdots & 0\\ 0 & \vdots & \vdots & \ddots & \vdots\\ D^* & 0 & \cdots & 0 & A_n\end{bmatrix} \quad (1.7)$$

for $j=1,2,\cdots$ Bhatia and Kittaneh have proved in [8] that if $A,B\in K(H)$, such that A is self-adjoint, $B\ge 0$, and $\pm A\le B$, then

$$s_j(A)\le s_j(B\oplus B) \quad (1.8)$$

for $j=1,2,\cdots$ Audeh and Kittaneh have proved in [5] that if $A,B,C\in K(H)$ such that $\begin{bmatrix}A & B\\ B^* & C\end{bmatrix}\ge 0$, then

$$s_j(B)\le s_j(A\oplus C) \quad (1.9)$$

for $j=1,2,\cdots$ We will prove a new inequality which generalizes (1.9), and is equivalent to the inequalities (1.8) and (1.9):

If $A_1,A_2,\cdots,A_n,D\in K(H)$ such that

$$\begin{bmatrix}A_1 & 0 & \cdots & 0 & D\\ 0 & A_2 & 0 & \cdots & 0\\ \vdots & 0 & \ddots & \cdots & 0\\ 0 & \vdots & \vdots & \ddots & \vdots\\ D^* & 0 & \cdots & 0 & A_n\end{bmatrix}\ge 0, \text{ then}$$

$$s_j(D)\le s_j(A_1\oplus A_2\oplus\cdots\oplus A_n) \quad (1.10)$$

for $j=1,2,\cdots$

2. Main Result

Our first singular value inequality is equivalent to and more general than the inequalities (1.1) and (1.2).

Theorem 2.1 Let $A_i,B_i\in K(H), i=1,2,\cdots,n.$ Then

$$2s_j\left(A_1B_1^*+A_2B_2^*+\cdots+A_nB_n^*\right)\le s_j\left[\left|\begin{matrix}A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{matrix}\right|^2+\left|\begin{matrix}B_1 & B_2 & \cdots & B_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{matrix}\right|^2\right]$$

for $j=1,2,\cdots$

Proof. Let $A=\begin{bmatrix}A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$, $B=\begin{bmatrix}B_1 & B_2 & \cdots & B_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0\end{bmatrix}$. Then

$AB^*=A_1B_1^*+A_2B_2^*+\cdots+A_nB_n^*$, and

$$A^*A+B^*B=\begin{bmatrix} A_1^*A_1+B_1^*B_1 & \cdots & A_1^*A_n+B_1^*B_n \\ \vdots & \ddots & \vdots \\ A_n^*A_1+B_n^*B_1 & \cdots & A_n^*A_n+B_n^*n \end{bmatrix}=\left[\begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2\right]+\left[\begin{vmatrix} B_1 & B_2 & \cdots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2\right]$$

Now, using (1.1) we get

$$2s_j\left(A_1B_1^*+A_2B_2^*+\cdots+A_nB_n^*\right)\le s_j\left[\begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2+\begin{vmatrix} B_1 & B_2 & \cdots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2\right]$$

for $j=1,2,\cdots$

Remark 1. As a special case of (1.3), let $Ai=Bi=0$ for $i=2,3,\cdots,n$ .we get (1.1)

Remark 2. As a special case of (1.3), let $Ai=Bi=0$ for $i=3,4,\cdots,n,$ we get (1.2), to see this:

Replace $Ai=Bi=0$ for $i=3,4,\cdots,n,$ in(1.3), we get

$$2s_j\left(A_1B_1^*+A_2B_2^*\right)\le s_j\left[\begin{vmatrix} A_1 & A_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2+\begin{vmatrix} B_1 & B_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2\right]=s_j\begin{bmatrix} A_1^*A_1+B_1^*B_1 & A_1^*A_2+B_1^*B_2 \\ A_2^*A_1+B_2^*B_1 & A_2^*A_2+B_2^*B_2 \end{bmatrix}$$

$$=s_j\left[\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}^2\right]=s_j^2\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$$

for $j=1,2,\cdots$

Now, we prove that the inequalities (1.1) and (1.3) are equivalent.

Theorem 2.2. The following statements are equivalent:

(i) If $A,B\in K(H)$, then $2s_j\left(AB^*\right)\le s_j\left(A^*A+B^*B\right)$ for $j=1,2,\cdots$

(ii) Let $A_{i,},B_i\in K(H), i=1,2,\cdots,n.$ Then

$$2s_j\left(A_1B_1^*+A_2B_2^*+\cdots+A_nB_n^*\right)\le s_j\left[\begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2+\begin{vmatrix} B_1 & B_2 & \cdots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}^2\right]$$

for $j=1,2,\cdots$

Proof. $(i)\to(ii)$ This implication follows from the proof of Theorem 2.1.

$(ii)\to(i)$ This implication follows from Remark 1.

Remark 3. It can be shown trivially that (1.1) and (1.2) are equivalent. By using this with Theorem 2.2, we conclude that the inequalities (1.2) and (1.3) are equivalent. Chaining this with results in [5], we get that the inequalities (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6) are equivalent.

Our second singular value inequality is equivalent to the inequality (1.4).

***Theorem* 2.3.** Let $A_1,A_2,\cdots,A_n,D\in K(H)$ such that

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix}\ge 0. \text{ Then } 2s_j(D)\le s_j\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \text{ for } j=1,2,\cdots$$

Proof. Since

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \geq 0,$$

it follows that

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & -D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ -D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \geq 0.$$

In fact, if $U = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \cdots & I & \vdots \\ 0 & 0 & \cdots & 0 & -I \end{bmatrix}$, then U is unitary and

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & -D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ -D^* & 0 & \cdots & 0 & A_n \end{bmatrix} = U \begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} U^* \geq 0$$

Thus

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_n \end{bmatrix} \geq \pm \begin{bmatrix} 0 & 0 & \cdots & 0 & D \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and so by applying the inequality (1.4), we get

$$2s_j\left(D \oplus D^*\right) \leq s_j\left(\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \oplus \begin{bmatrix} A_1 & 0 & \cdots & 0 & -D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ -D^* & 0 & \cdots & 0 & A_n \end{bmatrix}\right)$$

for $j = 1, 2, \cdots$ This is equivalent to saying that $2s_j(D) \leq s_j\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix}$ for $j = 1, 2, \cdots$

Remark 4. While the proof of the inequality (1.7), given in Theorem 2.3 is based on the inequality (1.4), it can be obtained by applying the inequality (1.6) to the positive operators

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \text{and} \begin{bmatrix} A_1 & 0 & \cdots & 0 & -D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ -D^* & 0 & \cdots & 0 & A_n \end{bmatrix}.$$

Now, we prove that the inequalities (1.4) and (1.7) are equivalent.

Theorem 2.4. The following statements are equivalent:

(i) Let $A, B \in K(H)$ such that A is self-adjoint, $B \geq 0$, and $\pm A \leq B$. Then

$$2s_j(A) \leq s_j\left((B+A) \oplus (B-A)\right)$$

for $j = 1, 2, \cdots$

(ii) Let $A_1, A_2, \cdots, A_n, D \in K(H)$ such that

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \geq 0.$$

Then $2s_j(D) \leq s_j \begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix}$

for $j = 1, 2, \cdots$

Proof. $(i) \Rightarrow (ii)$ This implication follows from the proof of Theorem 2.3.

$(ii) \Rightarrow (i)$ Let $A, B \in K(H)$ such that A is self-adjoint, $B \geq 0$, and $\pm A \leq B$. Then the matrix

$$\begin{bmatrix} B & 0 & \cdots & 0 & A \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ A & 0 & \cdots & 0 & B \end{bmatrix} \geq 0.$$

In fact, if $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 & \cdots & 0 & -I \\ 0 & \sqrt{2}I & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \cdots & \sqrt{2}I & \vdots \\ I & 0 & \cdots & 0 & I \end{bmatrix}$, then U is unitary and

$$\begin{bmatrix} B & 0 & \cdots & 0 & A \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ A & 0 & \cdots & 0 & B \end{bmatrix} = U^* \begin{bmatrix} B-A & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & B+A \end{bmatrix} U \geq 0.$$

Thus, by applying (ii) we get

$$2s_j(A) \leq s_j \begin{bmatrix} B & 0 & \cdots & 0 & A \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ A & 0 & \cdots & 0 & B \end{bmatrix} = s_j \begin{bmatrix} B+A & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & B-A \end{bmatrix} = s_j\left((B+A) \oplus (B-A)\right)$$

for $j = 1, 2, \cdots$

Remark 5. From equivalence of inequalities (1.4) and (1.7) in Theorem 2.4, and equivalence of the inequalities (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6) in Remark 3, we get that the inequalities (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) are equivalent.

Our third singular value inequality is equivalent to the inequalities (1.8) and (1.9).

Theorem 2.5. Let $A_1, A_2, \cdots, A_n, D \in K(H)$ such that

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \geq 0.$$ Then

$$s_j(D) \leq s_j(A_1 \oplus A_2 \oplus \cdots \oplus A_n)$$

for $j = 1, 2, \cdots$

Proof. As in the proof of Theorem 2.3., we have

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_n \end{bmatrix} \geq \pm \begin{bmatrix} 0 & 0 & \cdots & 0 & D \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and so by applying the inequality (1.8), we get

$$s_j(D \oplus D^*) \leq s_j\left((A_1 \oplus A_2 \oplus \cdots \oplus A_n) \oplus (A_1 \oplus A_2 \oplus \cdots \oplus A_n)\right)$$

for $j=1,2,\cdots$ This is equivalent to saying that $s_j(D)\le s_j(A_1\oplus A_2\oplus\cdots\oplus A_n)$ for $j=1,2,\cdots$

Remark 6. While the proof of the inequality (1.10), given in Theorem 2.5 is based on the inequality (1.8), it can be obtained by employing the inequality (1.7) as follows:

If $\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix}\ge 0.$ Then

$\begin{bmatrix} A_1 & 0 & \cdots & 0 & -D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ -D^* & 0 & \cdots & 0 & A_n \end{bmatrix}\ge 0,$ and so

$$\begin{aligned} \begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} &\le \begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \\ &+ \begin{bmatrix} A_1 & 0 & \cdots & 0 & -D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ -D^* & 0 & \cdots & 0 & A_n \end{bmatrix} \\ &= 2\begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_n \end{bmatrix}. \end{aligned}$$

Following Weyl's monotonicity principle, we have

$$\begin{aligned} s_j\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix} &\le 2s_j\begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_n \end{bmatrix} \\ &= 2s_j(A_1\oplus A_2\oplus\cdots\oplus A_n) \end{aligned}$$

for $j=1,2,\cdots$ Chaining this with the inequality (1.7), yields the inequality (1.10).

Now, we prove that the inequalities (1.8) and (1.10) are equivalent.

Theorem 2.6. The following statements are equivalent:

(i) Let $A,B\in K(H)$, such that A is self-adjoint, $B\ge 0$, and $\pm A\le B$, then

$$s_j(A)\le s_j(B\oplus B)$$

for $j=1,2,\cdots$

(ii) Let $A_1,A_2,\cdots,A_n,D\in K(H)$ such that

$\begin{bmatrix} A_1 & 0 & \cdots & 0 & D \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ D^* & 0 & \cdots & 0 & A_n \end{bmatrix}\ge 0.$ Then

$$s_j(D)\le s_j(A_1\oplus A_2\oplus\cdots\oplus A_n)$$

for $j=1,2,\cdots$

Proof. $(i)\Rightarrow(ii)$ This implication follows the proof of Theorem 2.5.

$(i)\Rightarrow(ii)$ As in the proof of Theorem 2.4, if A is self-adjoint, $B\ge 0,$ and $\pm A\le B.$ Then

$$\begin{bmatrix} B & 0 & \cdots & 0 & A \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ A & 0 & \cdots & 0 & B \end{bmatrix}\ge 0.$$

Thus, by (ii) we have $s_j(A)\le s_j(B\oplus B)$ for $j=1,2,\cdots$

Remark 7. From equivalence of inequalities (1.8) and (1.10) in Theorem 2.6, and equivalence of inequalities (1.8) and (1.9) in [5], we get that the inequalities (1.8), (1.9), and (1.10) are equivalent.

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