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Matrices That Commute with Their Conjugate and Transpose

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ABSTRACT

It is known that if $A \in M_n$ is normal $\left(AA^* = A^*A\right)$, then $A\bar{A} = \bar{A}A$ if and only if $AA^{\mathrm{T}} = A^{\mathrm{T}}A$. This leads to the question: do both $A\bar{A} = \bar{A}A$ and $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ imply that A is normal? We give an example to show that this is false when $n = 4$, but we show that it is true when $n = 2$ and $n = 3$.

Keywords: Normal Matrix; Matrix Commuting with Its Conjugate and Transpose

Introduction and Results

Let A be an n-by-n normal matrix, *i.e.*, A is a complex square matrix $\left(A \in M_n\right)$, with the property that $AA^* = A^*A$, where $A^* = \bar{A}^{\mathrm{T}}$ is the conjugate-transpose of A. The Fuglede-Putnam Theorem tells us that if $AB = BA$ for some $B \in M_n$, then $A^*B = BA^*$. Suppose that $A\bar{A} = \bar{A}A$, where A is the conjugate of the matrix A (so we take the complex conjugate of every entry of A). Then taking the transpose gives

$$\bar{A}^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}\bar{A}^{\mathrm{T}} \Rightarrow A^*A^{\mathrm{T}} = A^{\mathrm{T}}A^* \Rightarrow AA^{\mathrm{T}} = A^{\mathrm{T}}A,$$

from the the Fuglede-Putnam Theorem. In a similar way, we see that if $AA^{\mathrm{T}} = A^{\mathrm{T}}A$, then $A\bar{A} = \bar{A}A$, so these two statements are equivalent when A is normal. The question arose in [2], whether the conditions

$$\bar{A}A = A\bar{A} \quad \text{and} \quad A^{\mathrm{T}}A = AA^{\mathrm{T}}$$

imply the third condition $AA^* = A^*A$, so that A is normal.

This is false when $n = 4$. In fact, any matrix of the form $A = \begin{bmatrix} I_{ab} & I_{cd} \\ 0 & I_{ab} \end{bmatrix}$, where $I_{ab} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a,b,c,d \in \mathbb{C}$, $c^2 + d^2 = 0$, c and d not both zero, has the property that both $\bar{A}A = A\bar{A}$ and $A^{\mathrm{T}}A = AA^{\mathrm{T}}$, but A is not normal. In this paper, we prove that if $A \in M_n$ where $n = 2$ or $n = 3$, then these conditions do imply that A is normal. This result was first proposed as a problem by the current author in the International Linear Algebra Society journal IMAGE (fall 2011). My solution for $n = 2$ appeared in the spring 2012 issue, but no solution for the case $n = 3$ has ever been given. In this paper, we give the solution for the case $n = 3$, and for completeness, we also give the solution for $n = 2$. Specifically we prove:

Theorem 1 *If* $A \in M_n$, $n = 2$ *or* $n = 3$, then $A\bar{A} = \bar{A}A$ and $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ imply that A is normal.

Proof. We need the following preliminary result, which is a direct consequence of Theorem 2.3.6 in [3] (using the fact that for $A \in M_n$, $A = B + iC$ where B and C are real then $A\bar{A} = \bar{A}A$ if and only if $BC = CB$), and stated explicitly in [1,2].

Theorem 2 *Let* $A \in M_n$, $n \ge 3$, with $A\bar{A} = \bar{A}A$. Then there exists a real orthogonal matrix $Q \in M_n(\mathbb{R})$ such that $Q^{\mathrm{T}}AQ$ is of the form:

$$\Lambda = \begin{bmatrix} A_1 & * & * & \cdots & \cdots & * \\ 0 & A_2 & * & \cdots & \cdots & * \\ 0 & 0 & A_3 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & \cdots & 0 & A_k \end{bmatrix},$$

where each A_i, $1 \le i \le k$ (for some k) is a 1-by-1 matrix or a 2-by-2 matrix.

Example 1. Note that if $A = Q\Lambda Q^{\mathrm{T}}$, Q real orthogonal, $A\bar{A} = \bar{A}A$ and $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ if and only if $\Lambda\bar{\Lambda} = \bar{\Lambda}\Lambda$ and $\Lambda\Lambda^{\mathrm{T}} = \Lambda^{\mathrm{T}}\Lambda$. Also note that if $A = A^{\mathrm{T}}$ and $A\bar{A} = \bar{A}A$. then A is normal since $A^* = \bar{A}$ in this case.

Lemma 1 *If* $A \in M_2$ with $AA^{\mathrm{T}} = A^{\mathrm{T}}A$, then A is either symmetric or of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, a,b \in \mathbb{C}$.

Proof. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a,b,c,d \in \mathbb{C}$, with $AA^{\mathrm{T}} = A^{\mathrm{T}}A$, then

$$\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+dc & b^2+d^2 \end{bmatrix}$$

Hence $b^2 = c^2$ and $ab+cd = ac+bd$.

Case 1. $b = c$, so that A is symmetric.

Case 2. $b = -c$, then $ab - bd = -ab + bd$ or $ab = bd$. If $b = 0$, then A is symmetric. If $b \neq 0$, $a = d$ and $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Proposition 1 If $A \in M_2$ with $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ and $A\bar{A} = \bar{A}A$, then A is normal.

Proof. From the Lemma 1, we have two cases. If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a,b \in \mathbb{C}$, then A is normal. On the other hand, if A is symmetric with $A\bar{A} = \bar{A}A$, then since $A^* = \bar{A}$ in this case, we must have $AA^* = A^*A$, so A is normal.

Example 2. We now look at the case of $A \in M_3$. We start with a lemma:

Lemma 2 Suppose $A \in M_3$ with $A\bar{A} = \bar{A}A$, $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ and $A = Q\Lambda Q^{\mathrm{T}}$ for some real orthogonal matrix $Q \in M_3(\mathbb{R})$ where Λ is of one of the two forms: see Equation (1).
then A is normal.

Proof. Case 1: $\Lambda = \begin{bmatrix} a & b & x \\ -b & a & y \\ 0 & 0 & \alpha \end{bmatrix}$. Now we require $\Lambda\Lambda^{\mathrm{T}} = \Lambda^{\mathrm{T}}\Lambda$, so that

$$\begin{bmatrix} a & b & x \\ -b & a & y \\ 0 & 0 & \alpha \end{bmatrix}\begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ x & y & \alpha \end{bmatrix} = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ x & y & \alpha \end{bmatrix}\begin{bmatrix} a & b & x \\ -b & a & y \\ 0 & 0 & \alpha \end{bmatrix},$$

or

$$\begin{bmatrix} a^2+b^2+x^2 & xy & \alpha x \\ xy & a^2+b^2+y^2 & \alpha y \\ \alpha x & \alpha y & \alpha^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 & ax-by \\ 0 & a^2+b^2 & bx+ay \\ ax-by & ay+bx & x^2+y^2+\alpha^2 \end{bmatrix}.$$

It follows that $x = y = 0$, and A is normal.

Case 2: If $\Lambda = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}$ then

$$\begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix}\begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix},$$

or

$$\begin{bmatrix} a^2+x^2+y^2 & bx+yz & cy \\ bx+yz & b^2+z^2 & cz \\ cy & cz & c^2 \end{bmatrix} = \begin{bmatrix} a^2 & ax & ay \\ ax & x^2+b^2 & xy+bz \\ ay & xy+bz & c^2+y^2+z^2 \end{bmatrix}.$$

Hence $x^2+y^2 = 0$, $x^2 = z^2$, $y^2+z^2 = 0$, and $x = \pm iy$, $x = \pm z$, $y = \pm iz$ and also $ay = cy$, so $y = 0$ (giving Λ diagonal and A normal) or $a = c$. Suppose $y \neq 0$ so that $a = c$.

Case 2(a). If $x = -z\ (\neq 0)$, then $ax = bx + yz \Rightarrow y = b - a$ and $x = \pm i(b-a) = -z$ so that $\Lambda = \begin{bmatrix} a & \pm i(b-a) & b-a \\ 0 & b & \pm i(a-b) \\ 0 & 0 & a \end{bmatrix}$. However, this matrix also has the property that $\Lambda\bar{\Lambda} = \bar{\Lambda}\Lambda$, which gives in Equation (2). It follows from equating the entries in the (1, 2) position

$|a|^2 + |b|^2 - 2a\bar{b} = -|a|^2 - |b|^2 + 2\bar{a}b$, or $|a-b|^2 = 0$,

$$\Lambda = \begin{bmatrix} a & b & x \\ -b & a & y \\ 0 & 0 & \alpha \end{bmatrix}, \text{ or } \Lambda = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}, \ a,b,c,x,y,z,\alpha \in \mathbb{C}, \quad (1)$$

$$\begin{bmatrix} |a|^2 & \pm i\left(|a|^2+|b|^2-2a\bar{b}\right) & 2a\bar{b}+2\bar{a}b-3|a|^2-|b|^2 \\ 0 & |b|^2 & \pm i\left(|a|^2+|b|^2-2a\bar{b}\right) \\ 0 & 0 & |a|^2 \end{bmatrix} = \begin{bmatrix} |a|^2 & \pm i\left(-|a|^2-|b|^2+2\bar{a}b\right) & 2a\bar{b}+2\bar{a}b-3|a|^2-|b|^2 \\ 0 & |b|^2 & \pm i\left(-|a|^2-|b|^2+2\bar{a}b\right) \\ 0 & 0 & |a|^2 \end{bmatrix} \quad (2)$$

so $a=b$, and hence Λ is diagonal and A is normal.

Case 2(b). This is where $x=z\neq 0$, and since $ax=bx+xy$ we have $y=a-b$, so that

$$\Lambda=\begin{bmatrix} a & \pm i\left(a-b\right) & a-b \\ 0 & b & \pm i\left(a-b\right) \\ 0 & 0 & a \end{bmatrix},$$

and this is treated in a similar way.

Proposition 2 If $A\in M_3$ with $AA^{\mathrm{T}}=A^{\mathrm{T}}A$ and $A\bar{A}=\bar{A}A$, then A is normal.

Proof. We show that every case reduces to the case of the Lemma 2. From Theorem 1, every matrix Λ with $A=Q\Lambda Q^{\mathrm{T}}$ (Q real orthogonal) can be chosen to be one of the following three forms:

(I) $\Lambda=\begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}$, **(II)** $\Lambda=\begin{bmatrix} a & b & x \\ c & d & y \\ 0 & 0 & e \end{bmatrix}$, or **(III)** $\Lambda=\begin{bmatrix} a & x & y \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$.

We have dealt with Case (I) in Lemma 2, so consider Case (II): $\Lambda\Lambda^{\mathrm{T}}=\Lambda^{\mathrm{T}}\Lambda$ gives in Equation (3).

It follows that $b^2+x^2=c^2$, $c^2+y^2=b^2$, and $x^2+y^2=0$, so that $x=\pm iy$.

Case 1. $x=iy\neq 0$, then since $bx+dy=ey$, $biy+dy=ey$, so $b=i\left(d-e\right)$.

Also $xe=ax+cy$ gives $xe=ax-cix$, so that $c=i\left(e-a\right)$, so Λ has the form

$$\Lambda=\begin{bmatrix} a & i\left(d-e\right) & iy \\ i\left(e-a\right) & d & y \\ 0 & 0 & e \end{bmatrix}$$

Now we use the fact that $\Lambda\bar{\Lambda}=\bar{\Lambda}\Lambda$. This gives in Equation (4). On equating entries in the (1, 3) position we have:

$$-a\bar{y}+\bar{y}\left(d-e\right)+y\bar{e}=\bar{a}y-y\left(\bar{d}-\bar{e}\right)-\bar{y}e$$

and simplifying gives $\bar{y}\left(d-a\right)=y\left(\bar{a}-\bar{d}\right)$, so if $a\neq d$ we have $\frac{y}{\bar{y}}=-\frac{a-d}{\bar{a}-\bar{d}}$

Equating entries in the (2, 3) position gives:

$$\bar{y}\left(e-a\right)+d\bar{y}+y\bar{e}=y\left(\bar{e}-\bar{a}\right)+\bar{d}y+\bar{y}e,$$

and this reduces to: $\bar{y}\left(d-a\right)=y\left(\bar{d}-\bar{a}\right)$, so if $a\neq d$, $\frac{y}{\bar{y}}=\frac{a-d}{\bar{a}-\bar{d}}$, contradicting the above. We conclude that $a=d$ and Λ is of the form

$$\Lambda=\begin{bmatrix} a & i\left(a-e\right) & iy \\ -i\left(a-e\right) & a & y \\ 0 & 0 & e \end{bmatrix},$$

and we can apply Lemma 2. The other possibility is that $y=0=x$, so that $b=\pm c$ and Λ is either of the form $\Lambda=\begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & e \end{bmatrix}$, a symmetric matrix (when $c=b$), or of the form $\Lambda=\begin{bmatrix} a & b & iy \\ -b & a & y \\ 0 & 0 & e \end{bmatrix}$ (when $c=-b$, since in this case $a=d$).

Case 2. $x=-iy\neq 0$, then $bx+dy=ey$ gives $b=i\left(e-d\right)$, and $ax+cy=xe$ gives $c=i\left(a-e\right)$, so that Λ has the form

$$\Lambda=\begin{bmatrix} a & i\left(e-d\right) & iy \\ i\left(a-e\right) & d & y \\ 0 & 0 & e \end{bmatrix}.$$

We proceed exactly as in Case 1 to reduce Λ to the

$$\begin{bmatrix} a^2+b^2+x^2 & ac+bd+xy & xe \\ ac+bd+xy & c^2+d^2+y^2 & ey \\ xe & ye & e^2 \end{bmatrix}=\begin{bmatrix} a^2+c^2 & ab+cd & ax+cy \\ ab+cd & b^2+d^2 & bx+dy \\ ax+cy & xb+yd & x^2+y^2+e^2 \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} |a|^2+\left(d-e\right)\left(\bar{e}-\bar{a}\right) & -ia\left(\bar{d}-\bar{e}\right)+i\bar{d}\left(d-e\right) & -ia\bar{y}+i\bar{y}\left(d-e\right)+iy\bar{e} \\ i\bar{a}\left(e-a\right)-id\left(\bar{e}-\bar{a}\right) & \left(e-a\right)\left(\bar{d}-\bar{e}\right)+|d|^2 & \bar{y}\left(e-a\right)+d\bar{y}+y\bar{e} \\ 0 & 0 & |e|^2 \end{bmatrix}$$

$$=\begin{bmatrix} |a|^2+\left(\bar{d}-\bar{e}\right)\left(e-a\right) & i\bar{a}\left(d-e\right)-id\left(\bar{d}-\bar{e}\right) & i\bar{a}y-iy\left(\bar{d}-\bar{e}\right)-i\bar{y}e \\ -ia\left(\bar{e}-\bar{a}\right)+i\bar{d}\left(e-a\right) & \left(\bar{e}-\bar{a}\right)\left(d-e\right)+|d|^2 & y\left(\bar{e}-\bar{a}\right)+\bar{d}y+\bar{y}e \\ 0 & 0 & |e|^2 \end{bmatrix} \quad (4)$$

situation of Lemma 2.

In Case (III), where $\Lambda = \begin{bmatrix} a & x & y \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$, we proceed exactly as in Case (II) to deduce the result.

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