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Jordan Semi-Triple Multiplicative Maps on the Symmetric Matrices*

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ABSTRACT

In this paper, we show that if an injective map Φ on symmetric matrices $S_n(\mathbf{C})$ satisfies $\Phi(ABA)=\Phi(A)\Phi(B)\Phi(A), \forall A,B\in S_n(\mathbf{C})$, then $\Phi(A)=\pm SA_f S^t$ for all $A\in S_n(\mathbf{C})$, where f is an injective homomorphism on $\mathbf{C}$, S is a complex orthogonal matrix and A_f is the image of A under f applied entrywise.

Keywords: Symmetric Matrices; Orthogonal Matrix; Jordan Homomorphism

1. Introduction

It is an interesting problem to study the interrelation between the multiplicative and the additive structure of a ring or an algebra. Matindale in [1] proved that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Thus, the multiplicative structure determines the ring structure for some rings. This result was utilized by P. Šemrl in [2] to describe the form of the semigroup isomorphisms of standard operator algebras on Banach spaces. Some other results on the additivity of multiplicative maps between operator algebras can be found in [3,4]. Besides ring homomorphisms between rings, sometimes one has to consider Jordan ring homomorphisms. Note that, Jordan operator algebras have important applications in the mathematical foundations of quantum mechanics. So, it is also interesting to ask when the Jordan multiplicative structure determines the Jordan ring structure of Jordan rings or algebras.

Let R,R' be two rings and let $\Phi:R\to R'$ be a map. Recall that Φ is called a Jordan homomorphism if

$$\Phi(A+B)=\Phi(A)+\Phi(B),$$
$$\Phi(AB+BA)=\Phi(A)\Phi(B)+\Phi(B)\Phi(A)$$

for all $A,B\in R$. There are two basic forms of Jordan multiplicative maps, namely,

1) $\Phi(ABA)=\Phi(A)\Phi(B)\Phi(A)$ (Jordan semi-triple multiplicative map) for all $A,B\in R$,

2) $\Phi\left(\frac{1}{2}AB+\frac{1}{2}BA\right)=\frac{1}{2}\Phi(A)\Phi(B)+\frac{1}{2}\Phi(B)\Phi(A)$ (Jordan multiplicative map) for all $A,B\in R$. It is clear that, if Φ is unital and additive, then these two forms of Jordan multiplicative maps are equivalent. But in general, for a unital map, we do not know whether they are still equivalent without the additivity assumption.

The question of when a Jordan multiplicative map is additive was investigated by several authors. Let ϕ be a bijective map on a standard operator algebra. Molnár showed in [5] that if ϕ satisfies

$$\phi(ABA)=\phi(A)\phi(B)\phi(A),$$

then ϕ is additive. Later, Molnár in [5] and then Lu in [6] considered the cases that ϕ preserve the operation $\frac{1}{2}(AB+BA)$ and $AB+BA$, respectively, and proved that such ϕ is also additive. Thus, the Jordan multiplicative structure also determines the Jordan ring structure of the standard operator algebras. Later, in [7] we proved these Jordan multiplicative maps on the space of self-adjoint operators space are Jordan ring isomorphism and thus are equivalent. In this paper, we consider the same question and give affirmative answer for the case of Jordan multiplicative maps on the Jordan algebras of all symmetric matrices. In fact, we study injective Jordan

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semi-triple multiplicative maps on the symmetric matrices $S_n(\mathbf{C})$, and show that such maps must be additive, and hence are Jordan ring homomorphisms.

Let us recall and fix some notations in this paper. Recall that $P \in S_n(\mathbf{C})$ is called an idempotent if $P^2 = P$. We define the order $\leq$ between idempotents as follows: $P \leq Q$ if and only if $PQ = QP = P$ for any idempotents $P, Q \in S_n(\mathbf{C})$. For any $1 \leq j, k \leq n$, let E_{jk} be the matrix with 1 in the position (j,k) and zeros elsewhere, and I_j be the unit of $S_j(\mathbf{C})$.

2. Main Results and Its Proof

In this section, we study injective Jordan semi-triple multiplicative maps on $S_n(\mathbf{C})$, the following is the main result.

Theorem 2.1. An injective map

$$\Phi : S_n(\mathbf{C}) \to S_n(\mathbf{C})$$

is a Jordan semi-triple multiplicative map, that is

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A), \forall A, B \in S_n(\mathbf{C}) \quad (2.1)$$

if and only if there is an injective homomorphism f of $\mathbf{C}$ and a complex orthogonal matrix S such that

$$\Phi(A) = \pm S A_f S^t \textbf{ for all } A \in S_n(\mathbf{C}).$$

Firstly, we give some properties of injective Jordan semi-triple multiplicative maps on $S_n(\mathbf{C})$.

Lemma 2.2. Let $\Phi : S_n(\mathbf{C}) \to S_n(\mathbf{C})$ be an injective Jordan semi-triple multiplicative map. Then Φ sends idempotents to tripotents and moreover,

1) $\Phi(I_n)^2$ is an idempotent and

$$\begin{aligned}
&\Phi(A) \\
&= \Phi(I_n)\Phi(A)\Phi(I_n) \\
&= \Phi(I_n)^3 \Phi(A)\Phi(I_n) \\
&= \Phi(I_n)\Phi(A)\Phi(I_n)^3 \\
&= \Phi(I_n)^2 \Phi(A) = \Phi(A)\Phi(I_n)^2
\end{aligned}$$

for all $A \in S_n(\mathbf{C})$, in particular

$$\Phi(I_n)^2 \Phi(A)\Phi(I_n)^2 = \Phi(A);$$

2) $\Phi(I_n)$ commutes with $\Phi(A)$ for every $A \in S_n(\mathbf{C})$;

3) $\Phi(P)^2 = \Phi(I_n)\Phi(P)$ is an idempotent for each idempotent $P \in S_n(\mathbf{C})$;

4) A map $\Psi : S_n(\mathbf{C}) \to S_n(\mathbf{C})$ defined by

$$\Psi(A) = \Phi(I_n)\Phi(A)$$

for all $A \in S_n(\mathbf{C})$, is a Jordan semi-triple multiplicative map, which is injective if and only if Φ is injective.

For Ψ defined in Lemma 2.2, we can see that

$$\Psi(P)^2 = \Psi(P),$$

and $P \leq Q \Rightarrow \Psi(P) \leq \Psi(Q)$ for any idempotents $P, Q \in S_n(\mathbf{C})$. Therefore, we have

Corollary 2.3. Let $n, m \in \mathbf{N}$ and

$$\Phi : S_n(\mathbf{C}) \to S_m(\mathbf{C})$$

be an injective Jordan semi-triple multiplicative map. Then $m \geq n$. In the case $m = n$, for each idempotent $P \in S_n(\mathbf{C})$ the rank of $\Psi(P)$ is equal to the rank of P. In particular,

$$\Psi(0) = \Phi(0) = 0$$

$$\Psi(I) = \Phi(I)^2 = I$$

and

$$\Psi(A^2) = \Psi(A)^2.$$

Now we give proof of Theorem 2.1. The main idea is to use the induction on n, the dimension of the matrix algebra, after proving the result for 2×2 matrices.

Proof of Theorem 2.1. In order to prove Theorem 2.1, it suffices to characterize Ψ. Note if

$$\Psi(A) = S A_f S^t$$

then

$$\Phi(I_n)^2 = \Psi(I_n) = I_n,$$

that is $\Phi(I_n)$ is invertible and

$$\Phi(I_n) = \Phi(I_n)^{-1}.$$

By Lemma 2.1, $\Phi(I_n)$ commutes with $\Phi(A)$ for all $A \in S_n(\mathbf{C})$. It follows that $\Phi(I_n)$ commutes with $\Psi(A)$ for all $A \in S_n(\mathbf{C})$. Therefore, if $\Psi(A) = S A_f S^t$, $\Phi(I_n)$ must be a scalar matrix. As $\Phi(I_n)^2 = I_n$, $\Phi(I_n) = \pm I_n$ and hence Φ has the desired form.

Therefore, we mainly characterize Ψ. The proofs are given in two steps.

Step 1. The proof for $S_2(\mathbf{C})$.

The matrix E_{11} is an idempotent of rank one. By Corollary 2.3, $\Psi(E_{11})$ is a rank one idempotent. It is well known that every idempotent matrix in $S_n(\mathbf{C})$ can be diagonalizable by complex orthogonal matrix. Thus, there exists a 2×2 orthogonal matrix S such that

$$\Psi(E_{11}) = S E_{11} S^t.$$

Without loss of generality, we may assume that

$$\Psi(E_{11}) = E_{11}.$$

By Corollary 2.3 and from the following fact

$$\Psi\left((E_{12} + E_{21})^2\right) = I_2 = \Psi(E_{12} + E_{21})^2$$

and

$$\begin{aligned}
&E_{11}\Psi(E_{12} + E_{21})E_{11} \\
&= \Psi(E_{11}(E_{12} + E_{21})E_{11}) = \Psi(0) = 0
\end{aligned}$$

we conclude that

$$\Psi(E_{12}+E_{21})=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

or

$$\Psi(E_{12}+E_{21})=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}.$$

Let $V=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, by replacing Ψ with $V^t\Psi(\cdot)V$ if necessary, we may assume that

$$\Psi(E_{12}+E_{21})=E_{12}+E_{21}.$$

For E_{22}, since $\Psi(E_{22})$ is a rank one idempotent and satisfying $\Psi(E_{11})\Psi(E_{22})\Psi(E_{11})=0$ and

$$\Psi(E_{12}+E_{21})\Psi(E_{22})\Psi(E_{12}+E_{21})=\Psi(E_{11})=E_{11},$$

we have $\Psi(E_{22})=E_{22}$. Now for any

$$A=(a_{ij})\in S_2(\mathbf{C}),$$

let $B=(b_{ij})=\Psi(A)$. Then

$$\begin{aligned}b_{ii}E_{ii}&=E_{ii}BE_{ii}=\Psi(E_{ii})\Psi(A)\Psi(E_{ii})\\&=\Psi(E_{ii}AE_{ii})=\Psi(a_{ii}E_{ii}).\end{aligned}$$

Thus, the (i,i)th entry of $\Psi(A)$ depends on the (i,i)th entry of A only. Therefore, there exist injective functionals $f,g,:\mathbf{C}\to\mathbf{C}$ such that f,g satisfy respectively $f(a^2b)=f(a)^2f(b)$ and

$$g(a^2b)=g(a)^2g(b),$$

and

$$\Psi\begin{pmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{pmatrix}=\begin{pmatrix}f(a_{11})&b_{12}\\b_{12}&g(a_{22})\end{pmatrix}.$$

From $f(1)=g(1)=1$, it is easy to verify that f,g is multiplicative. Next we prove that $f=g$. Let $A=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}$, since $(E_{12}+E_{21})A(E_{12}+E_{21})=A$

A and $A^2=A$, we have

$$\begin{aligned}&(E_{12}+E_{21})\Psi(A)(E_{12}+E_{21})\\&=\Psi((E_{12}+E_{21})A(E_{12}+E_{21}))=\Psi(A)\end{aligned}$$

and $\Psi(A)^2=\Psi(A^2)=\Psi(A)$, hence $\Psi(A)=A$ or

$$\Psi(A)=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}\text{ with } b^2=\frac{1}{4}.$$

Thus, $f\left(\frac{1}{2}\right)=g\left(\frac{1}{2}\right)=\frac{1}{2}$ and $f(2)=g(2)=2$ since f,g is multiplicative. Let $J=\begin{pmatrix}1&1\\1&1\end{pmatrix}$, then $\Psi(J)=\begin{pmatrix}1&a\\a&1\end{pmatrix}$. Note that $J(E_{12}+E_{21})J=2J$, and

$$\Phi(J)(E_{12}+E_{21})\Phi(J)=\Phi(J(E_{12}+E_{21})J)=\begin{pmatrix}2&b\\b&2\end{pmatrix},$$

that is

$$\begin{pmatrix}1&a\\a&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&a\\a&1\end{pmatrix}=\begin{pmatrix}2&b\\b&2\end{pmatrix}.$$

This implies $a=1$ and $\Phi(J)=J$. Now by the fact $AJA=J$ and $\Phi(A)J\Phi(A)=J$, we get $\Psi(A)=A$. For any $a\in\mathbf{C}$, since

$$\begin{aligned}&f(a)J\\&=J(f(a)E_{11})J=\Psi(J)\Psi(aE_{11})\Psi(J)\\&=\Psi(aJE_{11}J)=\Psi(aJ)=\begin{pmatrix}f(a)&b\\b&g(a)\end{pmatrix},\end{aligned}$$

thus $f(a)=g(a)$.

Next we prove that f is additive. Since $\Psi\left(\begin{pmatrix}a&0\\0&b\end{pmatrix}^2\right)=\Psi\begin{pmatrix}a&0\\0&b\end{pmatrix}^2$, and thus we have

$$\Psi\begin{pmatrix}a&0\\0&b\end{pmatrix}=\begin{pmatrix}f(a)&0\\0&f(b)\end{pmatrix}$$

for any $a,b\in\mathbf{C}$. Moreover by the fact $JAJ=(a+b)J$, one can get that

$$\begin{aligned}&(f(a)+f(b))J\\&=J\Psi(A)J=\Psi(JAJ)\\&=\Psi((a+b)J)=\begin{pmatrix}f(a+b)&c\\\overline{c}&f(a+b)\end{pmatrix}\end{aligned}$$

and $f(a+b)=f(a)+f(b)$.

Finally, we prove

$$\Psi\begin{pmatrix}a&b\\b&c\end{pmatrix}=\begin{pmatrix}f(a)&f(b)\\f(b)&f(c)\end{pmatrix}$$

for any $a,b,c\in\mathbf{C}$. Let

$$\Psi\begin{pmatrix}a&b\\b&c\end{pmatrix}=\begin{pmatrix}f(a)&d\\d&f(c)\end{pmatrix}.$$

By the fact that $J\begin{pmatrix}a&b\\b&c\end{pmatrix}J=(a+2b+c)J$

and

$$J\Psi\begin{pmatrix} a & b \\ b & c \end{pmatrix}J$$
$$=\Psi(J)\Psi\begin{pmatrix} a & b \\ b & c \end{pmatrix}\Psi(J)=\Psi((a+2b+c)J),$$

we get $d=f(b)$ and $\Psi\begin{pmatrix} a & b \\ b & c \end{pmatrix}=\begin{pmatrix} f(a) & f(b) \\ f(b) & f(c) \end{pmatrix}$ for any $a,b,c\in\mathbf{C}$.

Step 2. The induction.

Let

$$P=I_{n-1}\oplus[0],$$

then P is a rank $n-1$ idempotent, so is $\Psi(P)$ by Corollary 2.3. Therefore, there exists a orthogonal matrix S such that $\Psi(P)=SPS^t$. Replacing Ψ by the map $A| \to S^t\Psi(A)S$, we may assume that $\Psi(P)=P$.

For any $\hat{A}\in S_{n-1}(\mathbf{C})$, let $A=\hat{A}\oplus[0]$. Then $PAP=A$ implies

$$P\Psi(A)P=\Psi(P)\Psi(A)\Psi(P)=\Psi(PAP)=\Psi(A).$$

It follows that $\Psi(\hat{A}\oplus[0])=\Psi(A)=\hat{X}\oplus[0]$ for some matrix $\hat{X}\in S_{n-1}(\mathbf{C})$. Define the map $\hat{\Psi}$ on $S_{n-1}(\mathbf{C})$ by $\hat{\Psi}(\hat{A})=\hat{X}$. It is easy to check that $\hat{\Psi}$ is an injective Jordan semi-triple multiplicative map on $S_{n-1}(\mathbf{C})$. Furthermore, $\Psi(P)=P$ implies that $\hat{\Psi}(I_{n-1})=I_{n-1}$. By the induction hypothesis there is a $(n-1)\times(n-1)$ orthogonal matrix $\hat{S}$ and an injective homomorphism f on $\mathbf{C}$ such that $\hat{\Psi}(\hat{A})=\pm\hat{S}\hat{A}_f\hat{S}^t$.

Let S be the matrix $\hat{S}\oplus[1]$. Without loss of generality, we assume that $\hat{\Psi}(\hat{A})=\hat{A}_f$ for all $\hat{A}\in S_{n-1}(\mathbf{C})$. This is equivalent to $\Psi(\hat{A}\oplus[0])=\hat{A}_f\oplus[0]$. For any $A=\begin{pmatrix} A_{11} & x \\ x^t & a_{22} \end{pmatrix}\in S_n(\mathbf{C})$ with $A_{11}\in S_{n-1}(\mathbf{C}), x\in C^{n-1}$ and $a_{22}\in\mathbf{C}$, we have $PAP=A_{11}\oplus[0]$.

Thus,

$$\begin{aligned}&P\Psi(A)P\\&=\Psi(P)\Psi(A)\Psi(P)=\Psi(0)=(A_{11})_f\oplus[0]\end{aligned}\qquad(*)$$

Let us define matrices R_i for each $i\in\{1,2,\cdots,n-1\}$ by

$$R_i=I_n-E_{ii}-E_{nn}+E_{in}+E_{ni}.$$

For an arbitrary i, From (*) we have

$$P\Psi(R_i)P=(I_{n-1}-E_{ii})\oplus[0].$$

Then there exists $x\in\mathbf{C}^{n-1}$ and $y\in\mathbf{C}$ such that

$$\Psi(R_i)=\begin{pmatrix} I_{n-1}-E_{ii} & x \\ x^t & y \end{pmatrix}.$$

From the equality $\Psi(R_i)^2=\Psi(R_i^2)=\Psi(I_n)=I_n$, we get that $I_{n-1}-E_{ii}+xx^t=I_{n-1}$ and $x^tx+y^2=1$. These equality implies that $xx^t=E_{ii}$ and

$$y^2=1-x^tx=1-tr(xx^t)=1-tr(E_{ii})=0.$$

Hence only the ith entries a_i of x are nonzero and $a_i^2=1$. It follows that

$$\Psi(R_i)=I_n-E_{ii}-E_{nn}+a_iE_{in}+a_iE_{ni}.$$

Next, take any two distinct $i,j\in\{1,2,\cdots,n-1\}$. From

$$R_iR_jR_i=I_n-E_{ii}-E_{jj}+E_{ij}+E_{ji}$$

and using (*), we get

$$\begin{aligned}&\Psi(I_n-E_{ii}-E_{jj}+E_{ij}+E_{ji})\\&=\Psi(R_i)\Psi(R_j)\Psi(R_i)\\&=I_n-E_{ii}-E_{jj}+a_ia_jE_{ij}+a_ia_jE_{ji},\end{aligned}$$

which implies that $a_ia_j=1$. Let $D=I_{n-1}\oplus[-1]$, then $D\Psi(R_1)D^t=R_1$, so we may assume that $\Psi(R_1)=R_1$. Furthermore by the equality

$$R_1R_iR_1=I_n-E_{11}-E_{ii}+E_{1i}+E_{i1}$$

and $R_1\Psi(R_i)R_1=\Psi(R_1R_iR_1)$, we obtain $\Psi(R_i)=R_i$.

Next we prove that $\Psi(A)=A_f$ for any $A\in S_n(\mathbf{C})$.

Let us fix some $i\in\{1,2,\cdots,n-1\}$. As $n>2$, there is another $j\in\{1,2,\cdots,n-1\}$ such that

$$aE_{ni}+aE_{in}=R_j(aE_{ij}+aE_{ji})R_j.$$

Then for any $a,b\in\mathbf{C}$,

$$\begin{aligned}&\Psi(aE_{nn})\\&=\Psi(R_1)\Psi(aE_{11})\Psi(R_1)\\&=R_1f(a)E_{11}R_1=f(a)E_{nn}\end{aligned}$$

and

$$\begin{aligned}&\Psi(bE_{ni}+bE_{in})\\&=\Psi(R_j(bE_{ij}+bE_{ji})R_j)\\&=\Psi(R_j)\Psi(bE_{ij}+\bar{b}E_{ji})\Psi(R_j).\\&=R_j(f(b)E_{ij}+f(b)E_{ij})R_j\\&=f(b)E_{ni}+f(b)E_{in}\end{aligned}$$

Thus, for any

$$A=\begin{pmatrix} aE_{ii} & x \\ x^t & a \end{pmatrix}$$

where $x\in\mathbf{C}^{n-1}$ has only one nonzero entry in the ith position, we have $\Psi(A)=A_f$. For any $A\in S_n(\mathbf{C})$, let

$$A=\begin{pmatrix}\hat{A} & x\\ \bar{x}^t & a\end{pmatrix}$$

and

$$\Psi(A)=\begin{pmatrix}\hat{A}_f & y\\ y^t & b\end{pmatrix}.$$

From $E_{nn}AE_{nn}=aE_{nn}$, we have

$$\begin{aligned}&bE_{nn}\\&=E_{nn}\Psi(A)E_{nn}=\Psi(E_{nn})\Psi(A)\Psi(E_{nn})\\&=\Psi(E_{nn}AE_{nn})=f(a)E_{nn}\end{aligned}$$

And $f(a)=b$. For any $i\in\{1,2,\cdots,n-1\}$, since

$$(E_{in}+E_{ni})A(E_{in}+E_{ni})=B=\begin{pmatrix}\hat{B} & z\\ z^t & \alpha\end{pmatrix}$$

where $\hat{B}\in S_{n-1}(\mathbf{C})$ and z have only one nonzero entry a_{nn} and a_{in} in the (i,i)th and ith position respectively, α is equal to the (i,i)th entry of A, thus we have

$$\begin{aligned}&(E_{in}+E_{ni})\Psi(A)(E_{in}+E_{ni})\\&=\Psi(E_{in}+E_{ni})\Psi(A)\Psi(E_{in}+E_{ni})\\&=\Psi((E_{in}+E_{ni})A(E_{in}+E_{ni}))\end{aligned}$$

and so $y=x_f$. The proofs are complete.

By Theorem 2.1, we can characterize another two forms of Jordan multiplicative maps on $S_n(\mathbf{C})$.

Theorem 2.4. An injective map

$$\Phi:S_n(\mathbf{C})\to S_n(\mathbf{C})$$

satisfies

$$\begin{aligned}&\Phi\left(\frac{1}{2}(ABC+CBA)\right)\\&=\frac{1}{2}\Phi(A)\Phi(B)\Phi(C)+\frac{1}{2}\Phi(C)\Phi(B)\Phi(A) \quad (2.2)\\&\forall A,B,C\in S_n(\mathbf{C})\end{aligned}$$

if and only if there is an injective homomorphism f on $\mathbf{C}$ and a complex orthogonal matrix S such that

$$\Phi(A)=\pm SA_fS^t \quad \text{for all} \quad A\in S_n(\mathbf{C}).$$

Proof. Let $C=A$ in Equation (2.2), we get

$$\Phi(ABA)=\Phi(A)\Phi(B)\Phi(A),$$

that is, Φ is a Jordan semi-triple multiplicative map. Consequently, Φ has the desired form by Theorem 2.1.

Since every ring homomorphism on $\mathbf{R}$ is an identity map, thus by Theorem 2.1, Theorem 2.4, we get

Corollary 2.5. Let $\Phi:S_n(\mathbf{R})\to S_n(\mathbf{R})$ be an injective map. Then the following condition are equivalent,

1) $\Phi(ABA)=\Phi(A)\Phi(B)\Phi(A),\forall A,B\in S_n(\mathbf{R})$

2) $$\begin{aligned}&\Phi\left(\frac{1}{2}(ABC+CBA)\right)\\&=\frac{1}{2}\Phi(A)\Phi(B)\Phi(C)+\frac{1}{2}\Phi(C)\Phi(B)\Phi(A)\\&\forall A,B,C\in S_n(\mathbf{R}),\end{aligned}$$

3) there is a real orthogonal matrix S such that

$$\Phi(A)=\pm SAS^t \quad \text{for all} \quad A\in S_n(\mathbf{R}).$$

At the end of this section, we characterize bijective maps on $S_n(\mathbf{C})$ preserving $ABC+CBA$.

Theorem 2.6. A bijective map $\Phi:S_n(\mathbf{C})\to S_n(\mathbf{C})$ satisfies

$$\begin{aligned}&\Phi(ABC+CBA)\\&=\Phi(A)\Phi(B)\Phi(C)+\Phi C\Phi(B)\Phi(A) \quad (2.3)\\&\forall A,B,C\in S_n(\mathbf{C}),\end{aligned}$$

if and only if there is a ring isomorphism f on $\mathbf{C}$ and a complex orthogonal matrix S such that

$$\Phi(A)=\pm SA_fS^t \quad \text{for all} \quad A\in S_n(\mathbf{C}).$$

Proof. It is enough to check the "only if" part. Letting $A=B$ in Equation (2.3), we get

$$\Phi\left(A^2C+CA^2\right)=\Phi(A)^2\Phi(C)+\Phi(C)\Phi(A)^2.$$

Taking $A=\frac{I}{\sqrt{2}}$ and $\Phi(C)=I$, we get

$I=2\Phi\left(\frac{I}{\sqrt{2}}\right)^2$, and thus

$$\Phi\left(\frac{I}{\sqrt{2}}\right)^2=\frac{I}{2} \quad (2.4)$$

Letting $A=C$ in Equation (2.3), we get

$$\Phi(2ABA)=2\Phi(A)\Phi(B)\Phi(A).$$

Taking $A=\frac{I}{\sqrt{2}}$, we get

$$\Phi(B)=2\Phi\left(\frac{I}{\sqrt{2}}\right)\Phi(B)\Phi\left(\frac{I}{\sqrt{2}}\right).$$

Multiplying this equality by $\Phi\left(\frac{I}{\sqrt{2}}\right)$ from the left side, by Equation (2.4) we get

$$\Phi\left(\frac{I}{\sqrt{2}}\right)\Phi(B)=\Phi(B)\Phi\left(\frac{I}{\sqrt{2}}\right)$$

for any $B\in S_n(\mathbf{C})$, and hence $\Phi\left(\frac{I}{\sqrt{2}}\right)=\lambda I$ for some scalar $\lambda \in \mathbf{C}$. By Equation (2.4), we obtain

$$\Phi\left(\frac{I}{\sqrt{2}}\right) = \pm\frac{I}{\sqrt{2}}.$$

If $\Phi\left(\frac{I}{\sqrt{2}}\right) = -\frac{I}{\sqrt{2}}$, let $\Psi(A) = -\Phi(A)$, then Ψ also meets Equation (2.3) and $\Psi\left(\frac{I}{\sqrt{2}}\right) = \frac{I}{\sqrt{2}}$. So without loss of generality, we assume $\Phi\left(\frac{I}{\sqrt{2}}\right) = \frac{I}{\sqrt{2}}$. By letting $C = A$ and $B = \frac{I}{\sqrt{2}}$ in Equation (2.3), we get $\Phi\left(\sqrt{2}A^2\right) = \sqrt{2}\Phi(A)^2$ and $\Phi\left(2A^2\right) = 2\Phi(A)^2$ for all $A \in S_n(\mathbf{C})$. Consequently

$$\Phi(I) = \Phi\left(2\left(\frac{I}{\sqrt{2}}\right)^2\right) = 2\Phi\left(\frac{I}{\sqrt{2}}\right)^2 = I.$$

Now letting $B = C = I$ in Equation (2.3) we get

$$\Phi(2A) = 2\Phi(A).$$

Thus, $2\Phi(ABA) = \Phi(2ABA) = 2\Phi(A)\Phi(B)\Phi(A)$ and $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ by taking $A = C$ in Equation (2.3). Therefore, Φ has desired form by surjectivity of Φ and Theorem 2.1.

In particular, we have

Corollary 2.7. A bijective map $\Phi : S_n(\mathbf{R}) \to S_n(\mathbf{R})$ satisfies

$$\Phi(ABC + CBA)$$
$$= \Phi(A)\Phi(B)\Phi(C) + \Phi(C)\Phi(B)\Phi(A)$$
$$\forall A, B, C \in S_n(\mathbf{R}),$$

if and only if there is a real orthogonal matrix S such that

$$\Phi(A) = \pm SAS^t \quad \text{for all} \quad A \in S_n(\mathbf{R}).$$

Remark 2.8. We do not know whether the surjective assumption in Theorem 2.6 and Corollary 2.7 can be omitted.

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