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BLU Factorization for Block Tridiagonal Matrices and Its Error Analysis

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ABSTRACT

A block representation of the *BLU* factorization for block tridiagonal matrices is presented. Some properties on the factors obtained in the course of the factorization are studied. Simpler expressions for errors incurred at the process of the factorization for block tridiagonal matrices are considered.

Keywords: Block Tridiagonal Matrices; *BLU* Factorization; Error Analysis; BLAS3

1. Introduction

Tridiagonal matrices are connected with different areas of science and engineering, including telecommunication system analysis [1] and finite difference methods for solving partial differential equations [2-4].

The backward error analysis is one of the most powerful tools for studying the accuracy and stability of numerical algorithms. A backward analysis for the *LU* factorization and for the solution of the associated triangular linear systems is presented by Amodio and Mazzia [5]. *BLU* factorization appears to have first been proposed for block tridiagonal matrices, which frequently arise in the discretization of partial differential equations. References relevant to this application include Isaacson and Keller [6], Bank and Rose [7], Mattheij [8], Concus, Golub and Meurant [9], Varah [10], Bank and Rose [11], and Yalamov and Plavlov [12]. For a block dense matrix, Demmel and Higham [13] presented error analysis of *BLU* factorization, and Demmel, Higham and Shreiber [14] also extended earlier analysis.

This paper is organized as follows. We begin, in Section 2 by showing the representation of *BLU* factorization for block tridiagonal matrices. In Section 3 some properties on the factors associated with the factorization are presented. Finally, by the use of BLAS3 based on fast matrix multiplication techniques, an error analysis of the factorization is given in Section 4.

Throughout, we use the "standard model" of floating-point arithmetic in which the evaluation of an expression in floating-point arithmetic is denoted by $fl(\cdot)$, with

$$fl(a \circ b) = (a \circ b)(1+\delta), \quad |\delta| \le u, \quad \circ = +,-,*,/$$

(see Higham [15] for details). Here u is the unit rounding-off associated with the particular machine being used. Unless otherwise stated, in this section an unsubscripted norm denotes $\|A\| = \max_{i,j} |a_{ij}|$.

2. Representation of *BLU* Factorization for Block Tridiagonal Matrices

Consider a nonsingular block tridiagonal matrix

$$A = \begin{bmatrix} A_1 & C_1 & & & \\ B_2 & A_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & C_{s-1} \\ & & & B_s & A_s \end{bmatrix} \in \Re^{n\times n}, \qquad (1)$$

where $s > 1$, $A_i \in \Re^{k_i \times k_i}$ $(i = 1,\cdots,s)$ are nonsingular, $B_i \in \Re^{k_i \times k_{i-1}}$ and $C_i \in \Re^{k_i \times k_{i+1}}$ with $1 \le k_i < n$ and $\sum_{i=1}^{s} k_i = n$ are arbitrary. We present the following factorization of A. The first step is represented as follows:

$$A = \begin{bmatrix} I_1 & & & \\ B_1A_1^{-1} & I_2 & & \\ & & \ddots & \\ & & & I_s \end{bmatrix} \begin{bmatrix} I_1 & \\ & S_1 \end{bmatrix} \begin{bmatrix} A_1 & C_1 & & \\ & I_2 & & \\ & & \ddots & \\ & & & I_s \end{bmatrix} = L_1D_1U_1,$$

where I_i is the identity matrix of order k_i, and

$$S_1 = \begin{bmatrix} A_2 - B_2A_1^{-1}C_1 & C_2 & & & \\ B_3 & A_3 & C_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & C_{s-1} \\ & & & B_s & A_s \end{bmatrix}.$$

The second step of the factorization is applied to D_1 in order to obtain a matrix D_2 with a sub-block S_2, then

$$D_1 = L_2 D_2 U_2 .$$

Applying the method recursively, it follows that

$$D_{i-1} = L_i D_i U_i .$$

After $s-1$ steps the block S_{s-1} is $k_s \times k_s$ and the factorization ends, we obtain

$$A = L_1 \cdots L_{s-1} D_{s-1} U_{s-1} \cdots U_1 = LU ,$$

where $L = \prod_{i=1}^{s-1} L_i$ and $U = D_{s-1} \prod_{i=1}^{s-1} U_{s-i}$. From the process of the representation obtained, we get the results as follows:

1) Taking the second step for example, if $A_2 - B_2 A_1^{-1} C_1$ is nonsingular then we can factor S_1 and D_1 in a similar manner, and this process can be continued recursively to obtain the complete block *LU* factorization;

2) There exists obvious difference between partitioned *LU* factorization (see [15] for further details), GE and block *LU* factorization in this paper.

3. Some Properties on the Factors of *BLU* Factorization

The usual property on Schur complements under *BLU* factorization for block diagonal dominance by rows is similar to that of point diagonal dominance, *i.e.*, Schur complements under *BLU* factorization for block diagonal dominance by rows inherit the key property on original matrices. For the factors D_i, U_i and U, we have the following theorem.

Theorem 3.1. Let A in (1) be nonsingular and block diagonally dominant by rows (columns). Then the factors D_i, U and U_i also preserve the similar property.

Proof. By the process of the factorization, it follows that

$$D_i = \operatorname{diag}\left(I_1, \cdots, I_i, S_i\right).$$

Since S_i is block diagonally dominant, by the definition of block diagonal dominance, D_i preserves the same property as the matrix S_i. The proof for U_i and U is as follows. The definition of block diagonal dominance, we have

$$\left\|A_1^{-1}\right\|^{-1} - \left\|C_1\right\| \ge 0, \quad \left\|\left(A_2 - B_2 A_1^{-1} C_1\right)^{-1}\right\| - \left\|C_2\right\| \ge 0.$$

Thus the matrices U_1 and U_2 are also block diagonally dominant. The result follows by induction, that is, U_i also preserves the same property as the matrix S_i. For the matrix *U*, we have

$$U = \operatorname{diag}\left(I_1, \cdots, I_{s-1}, S_{s-1}\right) \begin{bmatrix} A_1 & C_1 & & & \\ & A_2 - B_2 A_1^{-1} C_1 & C_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & C_{s-1} \\ & & & & I_{s-1} \end{bmatrix} = \begin{bmatrix} A_1 & C_1 & & & \\ & A_2 - B_2 A_1^{-1} C_1 & C_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & C_{s-1} \\ & & & & S_{s-1} \end{bmatrix}.$$

By the above proof, it follows that the matrix U is also block diagonally dominant. □

The problem is whether the matrices L_i for all $1 \le i \le s-1$ and L can inherit the same property as the matrix S_i. The result is negative. Take the following block tridiagonal matrix and $\|\bullet\|_2$ for example,

$$A = \begin{bmatrix} 3 & & \lambda & 1 & & \\ & 2\lambda & & \lambda & & \\ 1 & 1 & 3 & & 1/2 & 1 \\ & 1 & & 4 & & 1 \\ & & 1/2 & 1 & 1 & \\ & & & 1 & & 2 \end{bmatrix} = \begin{bmatrix} A_1 & C_1 & \\ B_2 & A_2 & C_2 \\ & B_3 & A_3 \end{bmatrix},$$

where $\lambda = 0.005$ and A_i, B_i and C_i are 2×2 matrices. Since the following inequalities

$$\left\|A_1^{-1}\right\|_2^{-1} = 2\lambda > \lambda = \left\|C_1\right\|_2 ,$$

$$\left\|A_2^{-1}\right\|_2^{-1} = 3 > 1 + 1/2 = \left\|C_2\right\|_2 + \left\|B_2\right\|_2 ,$$

$$\left\|A_3^{-1}\right\|_2^{-1} = 1 = \left\|B_3\right\|_3 ,$$

then the matrix A is block diagonally dominant by rows. Thus the matrix S_i is also block diagonally dominant by rows. However,

$$\left\|B_2 A_1^{-1}\right\|_2 = 1/2\,\lambda^{-1} > 1 = \left\|I_2^{-1}\right\|_2^{-1} ,$$

thus L_1 and L are not block diagonally dominant by rows. □

Only if the matrix A in (1) is block diagonally dominant by columns, the matrices L_i for all $1 \le i \le s-1$ and L can preserve the key property of S_i. The reason is as follows.

Based on the definition of block diagonal dominance by columns and the key property of S_i, we have

$$\left\|B_2 A_1^{-1}\right\| \le \left\|B_2\right\| \left\|A_1^{-1}\right\| \le \left\|B_2\right\| \left\|B_2\right\|^{-1} = 1,$$

$$\left\|B_3 \left(A_2 - B_2 A_1^{-1} C_1\right)^{-1}\right\| \le \left\|B_3\right\| \left\|B_3\right\|^{-1} = 1.$$

Therefore the matrices L_1 and L_2 are also block diagonally dominant by columns. Similarly, L_i for all $s-1 \ge i \ge 3$ block diagonally dominant by columns by induction. Then L can also preserve the key property of S_i.

4. Error Analysis

The use of BLAS3 based on fast matrix multiplication techniques affects the stability only insofar as it increases the constant terms in the normwise backward error bounds [13]. We make assumption about the underlying level-3 BLAS (matrix-matrix operations).

If $A \in \Re^{m\times n}$ and $B \in \Re^{n\times p}$ then the computed approximation $\hat{C}$ to $C = AB$ satisfies

$$\hat{C} = AB + \Delta C, \|\Delta C\| \le c_1(m,n,p)\|A\|\|B\| + O(u^2), \quad (2)$$

where $c(m,n,p)$ denotes a constant depending on m,n and p. For conventional BLAS3 implementations, (2) holds with $c(m,n,p) = n^2$ [13,15].

The computed solution $\hat{K}$ to the triangular systems $JK = Q$, with $J \in R_{m\times m}$ and $Q \in R_{m\times p}$, satisfies

$$J\hat{K} = Q + \Delta Q, \ \|\Delta Q\| \le c_2(m,p)u\|J\|\|\hat{K}\| + O(u^2),$$

where $c_2(m,p)$ denotes a constant depending on m and p. In this section, we present the backward error analysis for the block *LU* factorization by applying BLAS3 based on fast matrix multiplication techniques.

Theorem 4.1. Let $\hat{L}$ and $\hat{U}$ be the computed *BLU* factors of A in (1). Then

$$\hat{L} = L + \Delta L, \ \hat{U} = U + \Delta U,$$
$$\|\Delta L\| \le c_m u\|B_m\|\|\hat{U}_m^{-1}\| + O(u^2),$$
$$\|\Delta U\| \le u\left(\|A_m\| + (1+c'_m)\|\hat{L}_m\|\|C_m\|\right) + O(u^2),$$

where

$$c_m = \max_{1\le i\le s-1}\{c_1(k_{i+1},k_i,k_i)\}, c'_m = \max_{2\le i\le s}\{c_1(k_i,k_{i-1},k_i)\},$$
$$\|A_m\| = \max_{1\le i\le s}\{\|A_i\|\}, \ \|B_m\| = \max_{2\le i\le s}\{\|B_i\|\},$$
$$\|C_m\| = \max_{1\le i\le s-1}\{\|C_i\|\}, \ \|\hat{L}_m\| = \max_{2\le i\le s}\{\|\hat{L}_{i,i-1}\|\},$$
$$\|U_m^{-1}\| = \max_{1\le i\le s-1}\{\|\hat{U}_{ii}^{-1}\|\}.$$

Proof. Applying the standard analysis of errors, we can obtain the above result. Thus we omit it. □

Let $\hat{L}_j = \prod_{i=1}^{j}\hat{L}_i$ and $\hat{U}_j = \prod_{i=1}^{j}\hat{U}_i$. The multiplications $\prod_{i=1}^{j}\hat{L}_i$ and $\prod_{i=1}^{j}\hat{U}_i$ do not produce errors because of their structures. Thus the errors of $\hat{L}_j$ and $\hat{U}_j$ can be represented as $\|\Delta\hat{L}_j\| = \max_{1\le i\le j}\{\|\Delta L_{i+1,i}\|\}$ and $\|\Delta\hat{U}_j\| = \max_{1\le i\le j}\{\|\Delta U_{i+1,i}\|\}$. Then

$$\|\Delta\hat{L}_j\| \le c'_m u\|B'_m\|\|U'^{-1}_m\|,$$
$$\|\Delta\hat{U}_j\| \le u\left(\|A'_m\| + (1+\tilde{c}_m)\|\hat{L}'_m\|\|C'_m\|\right),$$

where $c'_m, \tilde{c}_m, A'_m, B'_m, C'_m, \hat{L}'_m$ and $\hat{U}'^{-1}_m$ are the maximum values of $c_1(k_{i+1},k_i,k_i), c_1(k_{i-1},k_i,k_i), \|A_i\|, \|B_i\|, \|C_i\|$, $\|\hat{L}_{i+1,i}\|$ and $\|\hat{U}_{ii}^{-1}\|$, respectively, when the value i ranges from 1 to j. Although the above error bounds are similar to those of $\|\Delta L\|$ and $\|\Delta U\|$, i in the bounds for $\|\Delta L\|$ and $\|\Delta U\|$ satisfies $1 \le i \le s-1$. On the other hand, based on the structure L_i, the error bounds for $\|\Delta U_i\|$ and $\|\Delta U\|$ is different from those of Theorem 4.1 and we can also obtain the bound for $\|\Delta D_i\|$.

Since the factors L_i arising in the factorization in this paper are triangular matrices, from (2) we have

$$\hat{L}_i\hat{U}'_i = D_{i-1} + \Delta D_{i-1},$$
$$\|\Delta D_{i-1}\| \le c_2(n,n)u\|\hat{L}_i\|\|\hat{U}'_i\| + O(u^2),$$

where $\hat{D}_i\hat{U}_i = \hat{U}'_i$. Note that the multiplication $\hat{D}_i\hat{U}_i$ do not produce error because of the structure of D_i and U_i. Then

$$\|\Delta U_i\| = \|\Delta D_{i-1}\| \le c_2(n,n)u\|\hat{L}_i\|\|\hat{U}'_i\| + O(u^2).$$

Thus

$$\|\Delta U\| \le c_2(n,n)u\|\hat{L}_{\max}\|\|\hat{U}'_{\max}\| + O(u^2),$$

where $\|\hat{L}_{\max}\| = \max_i\{\|\hat{L}_i\|\}$ and $\|\hat{U}'_{\max}\| = \max_i\{\|\hat{U}'_i\|\}$.

Compared to the proof of standard analysis of errors, there is a great different in form and the simpler proof of the latter embodies whose superiority. For the former, the error bound does not include $\|\hat{U}'_i\|$, which makes the computation easier.

Applying the result of Theorem 4.1, we have the following theorem.

Theorem 4.2. Let $\hat{L}$ and $\hat{U}$ be the computed *BLU* factors of A in (1). Then

$$A + \Delta A = (L + \Delta L)(U + \Delta U),$$
$$\|\Delta A\| \le u\left(\alpha(i,j)\|A_m\| + \|B_m\|\|\hat{U}_m^{-1}\|\left(\alpha(i,j)\|C_m\| + \beta(i,j)\right)\right) + O(u^2),$$

where

$$\|L_m\| = \max_{2\le i\le s}\left\{\|L_{i,i-1}\|\right\},\ \|U_m\| = \max_{1\le i\le s}\left\{\|U_{ii}\|\right\},$$

$$\alpha(i,j) = \begin{cases} 0, & i=j-1, \\ 1, & i=j, \\ \|L_m\|, & i=j+1, \end{cases} \quad c = \begin{cases} 0, & i=j-1, \\ 1+c'_m+c_m, & \text{others}, \end{cases}$$

$$\beta(i,j) = \begin{cases} \|U_m\|, & i=j+1, \\ 0, & \text{others}. \end{cases}$$

Proof. To save clutter we will omit "$+O(u^2)$" from each bound. For the expression $\hat{L}_{i+1,i}\hat{U}_{ii}$ arising in $\hat{L}\hat{U}$, if nu is sufficiently small, the term $\Delta L_{i+1,i}\Delta U_{ii}$ is small with respect to the other error matrices, in first order approximation, we obtain

$$\begin{aligned}\hat{L}_{i+1,i}\hat{U}_{ii} &= L_{i+1,i}U_{ii} + \Delta L_{i+1,i}U_{ii} + L_{i+1,i}\Delta U_{ii} \\ &= B_{i+1} + \Delta B_{i+1},\end{aligned}$$

where

$$\begin{aligned}\|\Delta B_{i+1}\| \le u\Big(\big(\|A_i\| + (1+c_1(k_i,k_{i-1},k_i))\|\hat{L}_{i,i-1}\|\|C_{i-1}\|\big)\|L_{i+1,i}\| \\ + c_1(k_{i+1},k_i,k_i)\|B_{i+1}\|\|\hat{U}_{ii}^{-1}\|\|U_{ii}\|\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\Delta A_i\| \le u\Big(\|A_i\| + (1+c_1(k_i,k_{i-1},k_i))\|\hat{L}_{i,i-1}\|\|C_{i-1}\| \\ + c_1(k_i,k_{i-1},k_{i-1})\|B_i\|\|\hat{U}_{i-1,i-1}^{-1}\|\|C_{i-1}\|\Big),\end{aligned}$$

$$\|\Delta C_i\| = 0.$$

Therefore the result holds. □

From Theorems 4.1 and 4.2, the bounds for $\|\Delta L\|, \|\Delta U\|$ and $\|\Delta A\|$ just depend on one of factors $\hat{L}_{i,i-1}$ and $\hat{U}_{ii}$ of elements A_{ij}, which make the computation of error bounds simpler.

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