Scientific Research

On Real Matrices to Least-Squares g-Inverse and Minimum Norm g-Inverse of Quaternion Matrices[*]

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ABSTRACT

Through the real representations of quaternion matrices and matrix rank method, we give the expression of the real matrices in least-squares g-inverse and minimum norm g-inverse. From these formulas, we derive the extreme ranks of the real matrices. As applications, we establish necessary and sufficient conditions for some special least-squares g-inverse and minimum norm g-inverse.

1. Introduction

Throughout this paper, $\mathbb{R}$ stands for the real number field, $\mathbb{H}^{m \times n}$ stands for the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{ \, a_0 + a_1 i + a_2 j + a_3 k \, \big| \, i^2 = j^2 = k^2 = ijk = -1,$$
$$a_0, a_1, a_2, a_3 \in \mathbb{R} \, \}.$$

I, A^T, A^* and $A^{\dagger}$ stand for the identity matrix' the transpose' the conjugate transpose and the Moore-Penrose inverse of a quaternion matrix A. In [1], for a quaternion matrix A, $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$. $\dim \mathcal{R}(A)$ is called the rank of a quaternion matrix A and denoted by $r(A)$.

The well known Moore-Penrose inverse $A^{\dagger}$ of $A \in \mathbb{H}^{m \times n}$ is defined to be the unique matrix $X \in \mathbb{H}^{m \times n}$ satisfying the following four Penrose equations

1) $AXA = A$,
2) $XAX = X$,
3) $(AX)^* = AX$,
4) $(XA)^* = XA$.

A matrix X is called a least-squares g-inverse of A if it satisfies both 1) and 3) in the Penrose equations, and denoted by $A^{(1,3)}$; a matrix X is called a minimum norm g-inverse of A if it satisfies both 1) and 4) in the Penrose equations, and denoted by $A^{(1,4)}$. The general expression of $A^{(1,3)}$ and $A^{(1,4)}$ can be written as

$$A^{(1,3)} = A^{\dagger} + L_A V, \tag{1}$$

$$A^{(1,4)} = A^{\dagger} + W R_A \tag{2}$$

where $L_A = I - A^{\dagger}A$, $R_A = I - AA^{\dagger}$, the two matrices V and W is a arbitrary; see [[2], pp. 44-46].

For convenience of representation, we suppose

$$A = A_0 + A_1 i + A_2 j + A_3 k \tag{3}$$

and

$$A^{(1,3)} = B_0 + B_1 i + B_2 j + B_3 k,$$
$$A^{(1,4)} = C_0 + C_1 i + C_2 j + C_3 k \tag{4}$$

where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m \times n}$, $B_0, B_1, B_2, B_3 \in \mathbb{R}^{m \times n}$, $C_0, C_1, C_2, C_3 \in \mathbb{R}^{n \times m}$.

For an arbitrary quaternion matrix

$$M = M_1 + M_2 i + M_3 j + M_4 k,$$

we define a map $\phi(\cdot)$ from $\mathbb{H}^{m \times n}$ to $\mathbb{R}^{4m \times 4n}$ by

$$\phi(M) = \begin{bmatrix} M_4 & M_3 & M_2 & M_1 \\ -M_3 & M_4 & M_1 & -M_2 \\ M_2 & M_1 & -M_4 & -M_3 \\ M_1 & -M_2 & M_3 & -M_4 \end{bmatrix}. \tag{5}$$

By (5), it is easy to verify that $\phi(\cdot)$ satisfies the following properties:

a) $M = N \Leftrightarrow \phi(M) = \phi(N)$.

b) $\phi(M + N) = \phi(M) + \phi(N)$, $\phi(MN) = \phi(M)\phi(N)$,
$$\phi(kM) = k\phi(M), k \in \mathbb{R}.$$

c) $\phi(M) = -T_m^{-1} \phi(M) T_n = -R_m^{-1} \phi(M) R_n$
$$= S_m^{-1} \phi(M) S_n,$$

where

$$T_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix}, R_t = \begin{bmatrix} 0 & 0 & -I_t & 0 \\ 0 & 0 & 0 & -I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{bmatrix},$$

$$S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}, t = m, n.$$

d) $r\left[\phi(M)\right] = 4r(M)$.

The least-squares g-inverse and minimum norm g-inverse have a very wide range of applications in numerical analysis and mathematical statistics and have been examined by many authors(see, e.g., [3-6]). Haruo [3] developed some equivalent conditions on least-squares general inverse in 1990. Tian [4] presented the maximal and minimal ranks of the Schur complement to least-squares g-inverse and minimum norm g-inverse in 2004. Tian [5] establish necessary and sufficient conditions for a matrix to be the least-squares g-inverse and minimum norm g-inverse from rank formulas in 2005. Guo, Wei and Wang [6] derived structures of least squares g-inverses and minimum norm g-inverse of a bordered matrix in 2006.

Quaternion matrices play an important role in mechanics, computer science, quantum physics, signal and color image processing and so on. More and more interests of quaternion matrices have been witnessed recently (e.g. [7-14]).

Noticing that the properties of the real matrices in least-squares general inverse $A^{(1,3)}$ and minimum norm g-inverse $A^{(1,4)}$ (4) have not been considered so far in the literature. We in this paper use the real representations of quaternion matrices and matrix rank method to investigate (4) over $\mathbb{H}$. In Section 2, we first give the expression of the real matrices B_i and C_i $(i = 0,1,2,3)$ in (4), then determine the maximal and minimal ranks of the real matrices B_i and C_i $(i = 0,1,2,3)$ in (4). As applications, we establish necessary and sufficient conditions for a quaternion matrix has a pure real or pure imaginary $A^{(1,3)}$ and $A^{(1,4)}$. The necessary and sufficient conditions for all $A^{(1,3)}$ and $A^{(1,4)}$ are pure real or pure imaginary of a quaternion matrix are also presented.

2. Main Results

We begin with the following lemmas which proof just like those over the complex field.

Lemma 2.1 (see [15]) *Let A, B and C be $m \times n, m \times k, l \times n$ matrices over $\mathbb{H}$. Then*

a) $r[A,B] = r(A) + r(R_A B) = r(B) + r(R_B A)$,

b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CL_A) = r(C) + r(AL_C)$,

c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(R_B A L_C)$.

Lemma 2.2 (see [16]) *Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times p}$, $C \in \mathbb{H}^{q \times n}$ be given. Then*

a) *The maximal rank of $A - BXC$ with respect to X is*

$$\max_X r(A - BXC) = \min \left\{ r[A \quad B], r\begin{bmatrix} A \\ C \end{bmatrix} \right\}; \quad (6)$$

b) *The minimal rank of $A - BXC$ with respect to X is*

$$\min_X r(A - BXC) = r[A \quad B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (7)$$

Theorem 2.3 *Suppose $\left[X_{ij} \right]_{4 \times 4} (i,j = 1,2,3,4)$ is a least-squares g-inverse of $\phi(A)$, where $X_{ij} \in \mathbb{R}^{n \times m}$, A and $A^{(1,3)}$ are defined as (3) and (4). Then B_i $(i = 0,1,2,3)$ in (4) can be written as*

$$B_0 = \frac{1}{4}\left(X_{11} + X_{22} - X_{33} - X_{44}\right),$$

$$B_1 = \frac{1}{4}\left(X_{12} - X_{21} + X_{43} - X_{34}\right),$$

$$B_2 = \frac{1}{4}\left(X_{13} + X_{31} - X_{24} - X_{42}\right), \quad (8)$$

$$B_3 = \frac{1}{4}\left(X_{41} + X_{14} + X_{32} + X_{23}\right).$$

Written in an explicit form, B_i $(i = 0,1,2,3)$ in (4) are

$$B_0 = \frac{1}{4}P_1\phi\left(A^\dagger\right)Q_1 + \frac{1}{4}P_2\phi\left(A^\dagger\right)Q_2$$

$$\quad - \frac{1}{4}P_3\phi\left(A^\dagger\right)Q_3 - \frac{1}{4}P_4\phi\left(A^\dagger\right)Q_4 \quad (9)$$

$$\quad + \left[P_1 L_{\phi(A)}, P_2 L_{\phi(A)}, P_3 L_{\phi(A)}, P_4 L_{\phi(A)}\right]\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix},$$

$$B_1 = \frac{1}{4}P_1\phi\left(A^\dagger\right)Q_2 - \frac{1}{4}P_2\phi\left(A^\dagger\right)Q_1$$

$$\quad + \frac{1}{4}P_4\phi\left(A^\dagger\right)Q_3 - \frac{1}{4}P_3\phi\left(A^\dagger\right)Q_4 \quad (10)$$

$$\quad + \left[P_1 L_{\phi(A)}, P_2 L_{\phi(A)}, P_4 L_{\phi(A)}, P_3 L_{\phi(A)}\right]\begin{bmatrix} V_2 \\ -V_1 \\ V_3 \\ -V_4 \end{bmatrix},$$

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$$B_2 = \frac{1}{4} P_1 \phi\left(A^\dagger\right) Q_3 - \frac{1}{4} P_2 \phi\left(A^\dagger\right) Q_4$$
$$+ \frac{1}{4} P_3 \phi\left(A^\dagger\right) Q_1 - \frac{1}{4} P_4 \phi\left(A^\dagger\right) Q_2 \qquad (11)$$
$$+ \left[P_1 L_{\phi(A)}, P_3 L_{\phi(A)}, -P_2 L_{\phi(A)}, -P_4 L_{\phi(A)} \right] \begin{bmatrix} V_3 \\ V_1 \\ -V_4 \\ -V_2 \end{bmatrix},$$

$$B_3 = \frac{1}{4} P_1 \phi\left(A^\dagger\right) Q_4 + \frac{1}{4} P_2 \phi\left(A^\dagger\right) Q_3$$
$$+ \frac{1}{4} P_3 \phi\left(A^\dagger\right) Q_2 + \frac{1}{4} P_4 \phi\left(A^\dagger\right) Q_1 \qquad (12)$$
$$+ \frac{1}{4} \left[P_1 L_{\phi(A)}, P_4 L_{\phi(A)}, P_3 L_{\phi(A)}, P_2 L_{\phi(A)} \right] \begin{bmatrix} V_4 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

where

$$P_1 = [I_n, 0, 0, 0], P_2 = [0, I_n, 0, 0],$$
$$P_3 = [0, 0, I_n, 0], P_4 = [0, 0, 0, I_n],$$
$$Q_1 = [I_m, 0, 0, 0]^T, Q_2 = [0, I_m, 0, 0]^T,$$
$$Q_3 = [0, 0, I_m, 0]^T, Q_4 = [0, 0, 0, I_m]^T,$$

and V_1, V_2, V_3 and V_4 are arbitrary real matrices with compatible sizes.

Proof. Suppose $\left[X_{ij} \right]_{4\times4} (i, j = 1, 2, 3, 4)$ is a least-squares g-inverse of $\phi(A)$, where $X_{ij} \in \mathbb{R}^{n\times m}$, *i.e.*

$$\phi(A) \left[X_{ij} \right]_{4\times4} \phi(A) = \phi(A),$$
$$\left(\phi(A) \left[X_{ij} \right]_{4\times4} \right)^* = \phi(A) \left[X_{ij} \right]_{4\times4}.$$

Then applying property (c) of $\phi(\cdot)$ above to them yields

$$T_m^{-1} \phi(A) T_n \left[X_{ij} \right]_{4\times4} T_m^{-1} \phi(A) T_n = -T_m^{-1} \phi(A) T_n,$$
$$\left(T_m^{-1} \phi(A) T_n \left[X_{ij} \right]_{4\times4} \right)^* = T_m^{-1} \phi(A) T_n \left[X_{ij} \right]_{4\times4};$$
$$R_m^{-1} \phi(A) R_n \left[X_{ij} \right]_{4\times4} R_m^{-1} \phi(A) R_n = -R_m^{-1} \phi(A) R_n,$$
$$\left(R_m^{-1} \phi(A) R_n \left[X_{ij} \right]_{4\times4} \right)^* = R_m^{-1} \phi(A) R_n \left[X_{ij} \right]_{4\times4};$$
$$S_m^{-1} \phi(A) S_n \left[X_{ij} \right]_{4\times4} S_m^{-1} \phi(A) S_n = S_m^{-1} \phi(A) S_n,$$
$$\left(S_m^{-1} \phi(A) S_n \left[X_{ij} \right]_{4\times4} \right)^* = S_m^{-1} \phi(A) S_n \left[X_{ij} \right]_{4\times4}.$$

Hence,

$$\phi(A) T_n \left[X_{ij} \right]_{4\times4} T_m^{-1} \phi(A) = \phi(A),$$
$$\left(\phi(A) T_n \left[X_{ij} \right]_{4\times4} T_m^{-1} \right)^* = \phi(A) T_n \left[X_{ij} \right]_{4\times4} T_m^{-1};$$

$$\phi(A) R_n \left[X_{ij} \right]_{4\times4} R_m^{-1} \phi(A) = \phi(A),$$
$$\left(\phi(A) R_n \left[X_{ij} \right]_{4\times4} R_m^{-1} \right)^* = \phi(A) R_n \left[X_{ij} \right]_{4\times4} R_m^{-1};$$
$$\phi(A) S_n \left[X_{ij} \right]_{4\times4} S_m^{-1} \phi(A) = \phi(A),$$
$$\left(\phi(A) S_n \left[X_{ij} \right]_{4\times4} S_m^{-1} \right)^* = \phi(A) S_n \left[X_{ij} \right]_{4\times4} S_m^{-1},$$

which implies that $T_n \left[X_{ij} \right]_{4\times4} T_m^{-1}, R_n \left[X_{ij} \right]_{4\times4} R_m^{-1}$ and $S_n \left[X_{ij} \right]_{4\times4} S_m^{-1}$ are also least-squares g-inverses of $\phi(A)$. Thus,

$$\frac{1}{4} \left(\left[X_{ij} \right]_{4\times4} - T_n \left[X_{ij} \right]_{4\times4} T_m^{-1} - R_n \left[X_{ij} \right]_{4\times4} R_m^{-1} \right.$$
$$\left. + S_n \left[X_{ij} \right]_{4\times4} S_m^{-1} \right)$$

is also a least-squares g-inverse of $\phi(A)$, where

$$\left[X_{ij} \right]_{4\times4} - T_n \left[X_{ij} \right]_{4\times4} T_m^{-1} - R_n \left[X_{ij} \right]_{4\times4} R_m^{-1}$$
$$+ S_n \left[X_{ij} \right]_{4\times4} S_m^{-1} = \left[\widehat{X}_{ij} \right]_{4\times4}$$

and

$$\widehat{X}_{11} = X_{11} + X_{22} - X_{33} - X_{44},$$
$$\widehat{X}_{12} = X_{12} - X_{21} + X_{43} - X_{34},$$
$$\widehat{X}_{13} = X_{13} + X_{31} - X_{24} - X_{42},$$
$$\widehat{X}_{14} = X_{41} + X_{14} + X_{32} + X_{23},$$
$$\widehat{X}_{21} = X_{21} - X_{12} + X_{34} - X_{43},$$
$$\widehat{X}_{22} = X_{11} + X_{22} - X_{33} - X_{44},$$
$$\widehat{X}_{23} = X_{41} + X_{14} + X_{32} + X_{23},$$
$$\widehat{X}_{24} = X_{24} + X_{42} - X_{13} - X_{31},$$
$$\widehat{X}_{31} = X_{13} + X_{31} - X_{24} - X_{42},$$
$$\widehat{X}_{32} = X_{41} + X_{14} + X_{32} + X_{23},$$
$$\widehat{X}_{33} = X_{33} + X_{44} - X_{11} - X_{22},$$
$$\widehat{X}_{34} = X_{21} - X_{12} + X_{34} - X_{43},$$
$$\widehat{X}_{41} = X_{41} + X_{14} + X_{32} + X_{23},$$
$$\widehat{X}_{42} = X_{24} + X_{42} - X_{13} - X_{31},$$
$$\widehat{X}_{43} = X_{12} - X_{21} + X_{43} - X_{34},$$
$$\widehat{X}_{44} = X_{33} + X_{44} - X_{11} - X_{22}.$$

Let

$$\widetilde{X} = \frac{1}{4}\left(X_{11} + X_{22} - X_{33} - X_{44}\right)$$

$$+ \frac{1}{4}\left(X_{12} - X_{21} + X_{43} - X_{34}\right)i$$

$$+ \frac{1}{4}\left(X_{13} + X_{31} - X_{24} - X_{42}\right)j$$

$$+ \frac{1}{4}\left(X_{41} + X_{14} + X_{32} + X_{23}\right)k,$$

Then, by (5),

$$\phi\left(\widetilde{X}\right) = \frac{1}{4}\left(\left[X_{ij}\right]_{4\times4} - T_n\left[X_{ij}\right]_{4\times4}T_m^{-1}\right.$$

$$\left. - R_n\left[X_{ij}\right]_{4\times4}R_m^{-1} + S_n\left[X_{ij}\right]_{4\times4}S_m^{-1}\right),$$

is a least-squares g-inverse of $\phi(A)$. Hence, by the property (a) of $\phi(\cdot)$ we know that $\hat{X}$ a least-squares g-inverse of A. The above discussion shows that the least-squares g-inverse of $\phi(A)$ and the least-squares g-inverse of A are equivalent. Observe that X_{ij}, $i, j = 1, 2, 3, 4$ in (8) can be written as

$$X_{ij} = P_i \hat{X} Q_j.$$

From (1), the least-squares g-inverse of $\phi(A)$ can be written as

$$\widetilde{X} = \phi\left(A^{\dagger}\right) + L_{\phi(A)}V$$

where $V = \left(V_1, V_2, V_3, V_4\right)$ and $V_1, V_2, V_3, V_4 \in \mathbb{R}^{4p\times q}$ are arbitrary.

Substituting them into (8) yields the four real matrices B_0, B_1, B_2 and B_3 in (9)-(12).

According to Lemma 2 and Theorem 2, we can get the following extreme ranks formulas for the real matrices in the least-squares g-inverses.

Theorem 2.4 *Suppose that A and $A^{(1,3)}$ are defined as* (3) *and* (4). *Then*

a) $\max r\left(B_0\right) = \min\left\{r\left[\overline{A}_0 \quad \widetilde{A}\right] - 4r(A) + n, m\right\};$

$\min r\left(B_0\right) = r\left[\overline{A}_0 \quad \widetilde{A}\right] - r\left(\widetilde{A}\right).$

b) $\max r\left(B_1\right) = \min\left\{r\left[\overline{A}_1 \quad \widetilde{A}\right] - 4r(A) + n, m\right\};$

$\min r\left(B_1\right) = r\left[\overline{A}_1 \quad \widetilde{A}\right] - r\left(\widetilde{A}\right).$

c) $\max r\left(B_2\right) = \min\left\{r\left[\overline{A}_2 \quad \widetilde{A}\right] - 4r(A) + n, m\right\};$

$\min r\left(B_2\right) = r\left[\overline{A}_2 \quad \widetilde{A}\right] - r\left(\widetilde{A}\right).$

d) $\max r\left(B_3\right) = \min\left\{r\left[\overline{A}_3 \quad \widetilde{A}\right] - 4r(A) + n, m\right\};$

$$\min r\left(B_3\right) = r\left[\overline{A}_3 \quad \widetilde{A}\right] - r\left(\widetilde{A}\right),$$

where

$$\overline{A}_0 = \begin{bmatrix} -\hat{A}_0 \\ -\hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} -\hat{A}_1 \\ \hat{A}_0 \\ -\hat{A}_2 \\ \hat{A}_3 \end{bmatrix}, \overline{A}_2 = \begin{bmatrix} -\hat{A}_2 \\ -\hat{A}_0 \\ \hat{A}_3 \\ \hat{A}_1 \end{bmatrix}, \overline{A}_3 = \begin{bmatrix} \hat{A}_3 \\ \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \end{bmatrix},$$

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_0 & 0 & 0 & 0 \\ 0 & \widetilde{A}_1 & 0 & 0 \\ 0 & 0 & \widetilde{A}_2 & 0 \\ 0 & 0 & 0 & \widetilde{A}_3 \end{bmatrix}, \widetilde{A}_0 = \begin{bmatrix} -A_1 & -A_2 & -A_3 \\ A_0 & A_3 & -A_2 \\ -A_3 & A_0 & A_1 \\ A_2 & -A_1 & A_0 \end{bmatrix},$$

and

$$\hat{A}_0 = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} -A_1 \\ A_0 \\ -A_3 \\ A_2 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} -A_2 \\ A_3 \\ A_0 \\ -A_1 \end{bmatrix}, \hat{A}_3 = \begin{bmatrix} -A_3 \\ -A_2 \\ A_1 \\ A_0 \end{bmatrix},$$

$$\widetilde{A}_1 = \begin{bmatrix} A_0 & -A_2 & -A_3 \\ A_1 & A_3 & -A_2 \\ A_2 & A_0 & A_1 \\ A_3 & -A_1 & A_0 \end{bmatrix}, \widetilde{A}_2 = \begin{bmatrix} A_0 & -A_1 & -A_3 \\ A_1 & A_0 & -A_2 \\ A_2 & -A_3 & A_1 \\ A_3 & A_2 & A_0 \end{bmatrix},$$

$$\widetilde{A}_3 = \begin{bmatrix} A_0 & -A_1 & -A_2 \\ A_1 & A_0 & A_3 \\ A_2 & -A_3 & A_0 \\ A_3 & A_2 & -A_1 \end{bmatrix}.$$

Proof. Applying (6) and (7) to B_0 in (9), we get the following

$$\max r\left(B_0\right) = \min\left\{r\left[\hat{P} \quad \widetilde{P}\right], r\begin{bmatrix} \hat{P} \\ I_m \end{bmatrix}\right\};$$

$$\min r\left(B_0\right) = r\left[\hat{P} \quad \widetilde{P}\right] + r\begin{bmatrix} \hat{P} \\ I_m \end{bmatrix} - r\begin{bmatrix} \hat{P} & \widetilde{P} \\ I_m & 0 \end{bmatrix},$$

where

$$\hat{P} = \frac{1}{4}P_1\phi\left(A^{\dagger}\right)Q_1 + \frac{1}{4}P_2\phi\left(A^{\dagger}\right)Q_2$$

$$- \frac{1}{4}P_3\phi\left(A^{\dagger}\right)Q_3 - \frac{1}{4}P_4\phi\left(A^{\dagger}\right)Q_4,$$

$$\widetilde{P} = \left[P_1L_{\phi(A)}, P_2L_{\phi(A)}, P_3L_{\phi(A)}, P_4L_{\phi(A)}\right].$$

By Lemma 1, it is not difficult to find that

$$r\left[\hat{P} \quad \tilde{P}\right]$$

$$= r\begin{bmatrix} \hat{P} & P_1 & P_2 & P_3 & P_4 \\ 0 & \phi(A) & 0 & 0 & 0 \\ 0 & 0 & \phi(A) & 0 & 0 \\ 0 & 0 & 0 & \phi(A) & 0 \\ 0 & 0 & 0 & 0 & \phi(A) \end{bmatrix} - 4r\left[\phi(A)\right]$$

$$= r\begin{bmatrix} 0 & P_1 & P_2 & P_3 & P_4 \\ \hat{A}_0 & \phi(A) & 0 & 0 & 0 \\ \hat{A}_1 & 0 & \phi(A) & 0 & 0 \\ \hat{A}_2 & 0 & 0 & \phi(A) & 0 \\ \hat{A}_3 & 0 & 0 & 0 & \phi(A) \end{bmatrix} - 4r\left[\phi(A)\right]$$

$$= r\left[\overline{A}_0 \quad \tilde{A}\right] - 4r(A) + n,$$

where $\hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{A}_3$ and $\tilde{A}$ are defined as above. By the same manner, we can get extreme ranks of B_1, B_2 and B_3.

As one of important applications of the maximal and minimal ranks to real matrices, Theorem 2 can help to get the necessary and sufficient conditions for the existence of some special least-squares g-inverses. We show them in the following.

Corollary 2.5 *Suppose* $A = A_0 + A_1 i + A_2 j + A_3 k$. *Then*

a) *A quaternion matrix A has a real least-squares g-inverse if and only if*

$$r\left[\overline{A}_1 \quad \tilde{A}\right] = r\left[\overline{A}_2 \quad \tilde{A}\right] = r\left[\overline{A}_3 \quad \tilde{A}\right] = r(\tilde{A}).$$

b) *All least-squares g-inverses of quaternion matrix A are real matrices if and only if*

$$r\left[\overline{A}_1 \quad \tilde{A}\right] = r\left[\overline{A}_2 \quad \tilde{A}\right] = r\left[\overline{A}_3 \quad \tilde{A}\right] = 4r(A) - n,$$

where $\overline{A}_1, \overline{A}_2, \overline{A}_3$ *and* $\tilde{A}$ *are defined as Theorem* 2.

Corollary 2.6 *Suppose* $A = A_0 + A_1 i + A_2 j + A_3 k$. *Then*

a) *A quaternion matrix A has a pure imaginary least-squares g-inverse if and only if*

$$r\left[\overline{A}_0 \quad \tilde{A}\right] = r(\tilde{A}).$$

b) *All least-squares g-inverses of quaternion matrix A are pure imaginary matrices if and only if*

$$r\left[\overline{A}_0 \quad \tilde{A}\right] = 4r(A) - n,$$

where $\overline{A}_0$ *and* $\tilde{A}$ *are defined as Theorem* 2.

The following several theorems of minimum norm g-inverse can be shown by a similar approach, and their proofs are omitted here.

Theorem 2.7 *Suppose* $\left[Y_{ij}\right]_{4\times4} (i, j = 1, 2, 3, 4)$ *is a minimum norm g-inverse of* $\phi(A)$, *where* $Y_{ij} \in \mathbb{R}^{n\times m}$,

A *and* $A^{(1,4)}$ *are defined as* (3) *and* (4) *Then* $C_i (i = 0, 1, 2, 3)$ *in* (4) *can be written as*

$$C_0 = \frac{1}{4}\left(Y_{11} + Y_{22} - Y_{33} - Y_{44}\right),$$

$$C_1 = \frac{1}{4}\left(Y_{12} - Y_{21} + Y_{43} - Y_{34}\right),$$

$$C_2 = \frac{1}{4}\left(Y_{13} + Y_{31} - Y_{24} - Y_{42}\right),$$

$$C_3 = \frac{1}{4}\left(Y_{41} + Y_{14} + Y_{32} + Y_{23}\right).$$

Written in an explicit form, $C_i (i = 0, 1, 2, 3)$ *in* (4) *are*

$$C_0 = \frac{1}{4}P_1\phi\left(A^\dagger\right)Q_1 + \frac{1}{4}P_2\phi\left(A^\dagger\right)Q_2$$
$$- \frac{1}{4}P_3\phi\left(A^\dagger\right)Q_3 - \frac{1}{4}P_4\phi\left(A^\dagger\right)Q_4$$
$$+ \frac{1}{4}\left[W_1, W_2, -W_3, -W_4\right]\begin{bmatrix} R_{\phi(A)}Q_1 \\ R_{\phi(A)}Q_2 \\ R_{\phi(A)}Q_3 \\ R_{\phi(A)}Q_4 \end{bmatrix},$$

$$C_1 = \frac{1}{4}P_1\phi\left(A^\dagger\right)Q_2 - \frac{1}{4}P_2\phi\left(A^\dagger\right)Q_1$$
$$+ \frac{1}{4}P_4\phi\left(A^\dagger\right)Q_3 - \frac{1}{4}P_3\phi\left(A^\dagger\right)Q_4$$
$$+ \frac{1}{4}\left[W_1, -W_2, W_4, -W_3\right]\begin{bmatrix} R_{\phi(A)}Q_2 \\ R_{\phi(A)}Q_1 \\ R_{\phi(A)}Q_3 \\ R_{\phi(A)}Q_4 \end{bmatrix},$$

$$C_2 = \frac{1}{4}P_1\phi\left(A^\dagger\right)Q_3 - \frac{1}{4}P_2\phi\left(A^\dagger\right)Q_4$$
$$+ \frac{1}{4}P_3\phi\left(A^\dagger\right)Q_1 - \frac{1}{4}P_4\phi\left(A^\dagger\right)Q_2$$
$$+ \frac{1}{4}\left[W_1, W_3, -W_2, -W_4\right]\begin{bmatrix} R_{\phi(A)}Q_3 \\ R_{\phi(A)}Q_1 \\ R_{\phi(A)}Q_2 \\ R_{\phi(A)}Q_4 \end{bmatrix},$$

$$C_3 = \frac{1}{4}P_1\phi\left(A^\dagger\right)Q_4 + \frac{1}{4}P_2\phi\left(A^\dagger\right)Q_3$$
$$+ \frac{1}{4}P_3\phi\left(A^\dagger\right)Q_2 + \frac{1}{4}P_4\phi\left(A^\dagger\right)Q_1$$
$$+ \frac{1}{4}\left[W_1, W_4, W_3, W_2\right]\begin{bmatrix} R_{\phi(A)}Q_4 \\ R_{\phi(A)}Q_1 \\ R_{\phi(A)}Q_2 \\ R_{\phi(A)}Q_3 \end{bmatrix},$$

where

$$P_1 = [I_n, 0, 0, 0], P_2 = [0, I_n, 0, 0],$$

$$P_3 = [0, 0, I_n, 0], P_4 = [0, 0, 0, I_n],$$

$$Q_1 = [I_m, 0, 0, 0]^T, Q_2 = [0, I_m, 0, 0]^T,$$

$$Q_3 = [0, 0, I_m, 0]^T, Q_4 = [0, 0, 0, I_m]^T,$$

and W_1, W_2, W_3 and W_4 are arbitrary real matrices with compatible sizes.

According to Lemma 2 and Theorem 3, we can get the following extreme ranks formulas for the real matrices in minimum norm g-inverse.

Theorem 2.8 *Suppose that A and $A^{(1,4)}$ are defined as* (3) *and* (4)*. Then*

a) $\max r(C_0) = \min \left\{ r \begin{bmatrix} \overline{\overline{A}}_0 \\ \widetilde{} \\ \widetilde{A} \end{bmatrix} - 4r(A) + m, 4n \right\};$

$\min r(C_0) = r \begin{bmatrix} \overline{\overline{A}}_0 \\ \widetilde{\widetilde{A}} \end{bmatrix} - r(\widetilde{\widetilde{A}}).$

b) $\max r(C_1) = \min \left\{ r \begin{bmatrix} \overline{\overline{A}}_1 \\ \widetilde{\widetilde{A}} \end{bmatrix} - 4r(A) + m, 4n \right\};$

$\min r(C_1) = r \begin{bmatrix} \overline{\overline{A}}_1 \\ \widetilde{\widetilde{A}} \end{bmatrix} - r(\widetilde{\widetilde{A}}).$

c) $\max r(C_2) = \min \left\{ r \begin{bmatrix} \overline{\overline{A}}_2 \\ \widetilde{\widetilde{A}} \end{bmatrix} - 4r(A) + m, 4n \right\};$

$\min r(C_2) = r \begin{bmatrix} \overline{\overline{A}}_2 \\ \widetilde{\widetilde{A}} \end{bmatrix} - r(\widetilde{\widetilde{A}}).$

d) $\max r(C_3) = \min \left\{ r \begin{bmatrix} \overline{\overline{A}}_3 \\ \widetilde{\widetilde{A}} \end{bmatrix} - 4r(A) + m, 4n \right\};$

$\min r(C_3) = r \begin{bmatrix} \overline{\overline{A}}_3 \\ \widetilde{\widetilde{A}} \end{bmatrix} - r(\widetilde{\widetilde{A}}),$

where

$$\overline{\overline{A}}_0 = [A_0 \quad -A_1 \quad -A_2 \quad -A_3],$$

$$\overline{\overline{A}}_1 = [A_1 \quad A_3 \quad A_3 \quad -A_2],$$

$$\overline{\overline{A}}_2 = [A_2 \quad -A_3 \quad A_0 \quad A_1],$$

$$\overline{\overline{A}}_3 = [A_3 \quad A_2 \quad -A_1 \quad A_0],$$

$$\widetilde{\widetilde{A}} = \begin{bmatrix} \widetilde{\widetilde{A}}_0 & 0 & 0 & 0 \\ 0 & \widetilde{\widetilde{A}}_1 & 0 & 0 \\ 0 & 0 & \widetilde{\widetilde{A}}_2 & 0 \\ 0 & 0 & 0 & \widetilde{\widetilde{A}}_3 \end{bmatrix},$$

$$\widetilde{\widetilde{A}}_0 = \begin{bmatrix} -A_2 & A_3 & A_0 & -A_1 \\ A_1 & A_0 & -A_3 & -A_2 \\ A_0 & -A_1 & A_2 & -A_3 \end{bmatrix},$$

$$\widetilde{\widetilde{A}}_1 = \begin{bmatrix} A_3 & A_2 & A_1 & A_0 \\ A_1 & A_0 & -A_3 & -A_2 \\ A_0 & -A_1 & A_2 & -A_3 \end{bmatrix},$$

$$\widetilde{\widetilde{A}}_2 = \begin{bmatrix} A_3 & A_2 & A_1 & A_0 \\ -A_2 & A_3 & A_0 & -A_1 \\ A_0 & -A_1 & A_2 & -A_3 \end{bmatrix},$$

$$\widetilde{\widetilde{A}}_2 = \begin{bmatrix} A_3 & A_2 & A_1 & A_0 \\ -A_2 & A_3 & A_0 & -A_1 \\ A_1 & A_0 & -A_3 & -A_2 \end{bmatrix}.$$

As one of important applications of the maximal and minimal ranks to real matrices, Theorem 4 can help to get the necessary and sufficient conditions for the existence of some special minimum norm g-inverse. We show them in the following.

Corollary 2.9 *Suppose $A = A_0 + A_1 i + A_2 j + A_3 k$. Then*

a) *A quaternion matrix A has a real minimum norm g-inverse if and only if*

$$r \begin{bmatrix} \overline{\overline{A}}_1 \\ \widetilde{\widetilde{A}} \end{bmatrix} = r \begin{bmatrix} \overline{\overline{A}}_2 \\ \widetilde{\widetilde{A}} \end{bmatrix} = r \begin{bmatrix} \overline{\overline{A}}_3 \\ \widetilde{\widetilde{A}} \end{bmatrix} = r(\widetilde{\widetilde{A}}).$$

b) *All minimum norm g-inverse of quaternion matrix A are real matrices if and only if*

$$r \begin{bmatrix} \overline{\overline{A}}_1 \\ \widetilde{\widetilde{A}} \end{bmatrix} = r \begin{bmatrix} \overline{\overline{A}}_2 \\ \widetilde{\widetilde{A}} \end{bmatrix} = r \begin{bmatrix} \overline{\overline{A}}_3 \\ \widetilde{\widetilde{A}} \end{bmatrix} = 4r(A) - m,$$

where $\overline{\overline{A}}_1, \overline{\overline{A}}_2, \overline{\overline{A}}_3$ and $\widetilde{\widetilde{A}}$ are defined as Theorem 4.

Corollary 2.10 *Suppose $A = A_0 + A_1 i + A_2 j + A_3 k$. Then*

a) *A quaternion matrix A has a pure imaginary minimum norm g-inverse if and only if*

$$r \begin{bmatrix} \overline{\overline{A}}_0 \\ \widetilde{\widetilde{A}} \end{bmatrix} = r(\widetilde{\widetilde{A}}).$$

b) *All minimum norm g-inverse of quaternion matrix A are pure imaginary matrices if and only if*

$$r \begin{bmatrix} \overline{\overline{A}}_0 \\ \widetilde{\widetilde{A}} \end{bmatrix} = 4r(A) - m,$$

where $\overline{\overline{A}}_0$ and $\widetilde{\widetilde{A}}$ are defined as Theorem 4.

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