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Minimizing the Loss Probability in M/M/2/1 Queueing System with Ordered Entry

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Abstract

This study analyzed the M/M/2/1 queueing model with queue of length one (waiting room of capacity just one), heterogeneous servers and ordered entry using the method of semi-Markov process. The customers who arrive in the system enter the free server; if the two servers are free, the customers enter the first server. If the two servers are busy, just one customer can wait at the waiting room. If the two servers are busy and the waiting room has a customer, the following customers will leave the system without receiving any service. Such a customer is called LOST COSTOMER. The probability of lost customers in the queueing system under examination was computed. Furthermore, by using inequality $f(s) \ge e^{-as}$ obtained from Jensen's inequality, it was shown that the loss probability was minimum when inter-arrival times fit deterministic distribution [1] [2].

Keywords

Loss Probability, Heterogeneous Servers, Semi-Markov Process, Laplace-Stieltjes Transform

1. Introduction

The fundamental M/M/2/1 queueing models with waiting room of capacity one and with identical two servers have been examined. The most effective measurement of the system is the loss probability, and the loss probability occurs when the two servers are busy and the waiting room has a customer. The steady state probabilities of this system are obtained by the formula: [3] [4]

$$P_k = \frac{\rho^k / k!}{\sum_{k=0}^{n} \rho^k / k!}, \quad 0 \le k \le n, \rho = \frac{\lambda}{\mu} \tag{1}$$

where k is number of the occupied servers in the system; n is the number of servers in the system, in this model $n = 2$; λ is the inter-arrival rate μ is the service rate; λ^{-1}, μ^{-1} are the mean of the inter-arrival time and the mean of the service time respectively. The probability of lost customers in the system can be computed by the following formula: [5]

$$\frac{1}{P_3} = \sum_{k=0}^{3} C_k^3 h_k \tag{2}$$

$$h_k = \prod_{k=1}^{2} \frac{1-\left[f(k\mu)\right]}{f(k\mu)} \tag{3}$$

$$h_0 = 1;\ h_1 = \frac{1-\left[f(\mu)\right]}{f(\mu)} \cdot \frac{1-\left[f(2\mu)\right]}{f(2\mu)};$$

$$h_2 = \frac{1-\left[f(2\mu)\right]}{f(2\mu)};\ h_3 = \frac{1-\left[f(2\mu)+f(\lambda)\right]}{f(2\mu)+f(\lambda)}$$

Methods: In this paper, we get the steady state probabilities of this system by formula (1), get the total probability by formula (7), get Laplace-Stieltjes transform by formula (8), get steady state probabilities π_j by formula $j = 0, 1, 2, 3$ by formulas at (IV).

2. M/M/2/1 Queueing Model with Ordered Entry [6]

M/M/2/1 Queueing system with finite capacity and heterogeneous servers were analyzed in this study. In this model, inter-arrival times with a finite expected value

$$a = \int_0^{\infty} \left[1 - F(t)\right] \mathrm{d}t \tag{4}$$

inter-arrival times are independent of each other and have distribution function $F(t)$.

There are two servers in the system. Their mean service times are assumed to be different from each other. The service time of each customer in server k is a random variable b_k and has an exponential distribution with parameter μ_k; $k = 1, 2$

$$P(b_k \le t) = 1 - \mathrm{e}^{-\mu_k t}, t > 0 \tag{5}$$

The service discipline takes place with the principle of "ordered entry". That is, the customer, who arrives in the system, enters the server with the lowest index among the free server. If the two servers are busy, he wait in the waiting room, if the two servers are busy and the waiting room not empty the customer leaves the system without receiving any service. Such customers are termed "lost customers". Thus, the main problem herein is to compute the probability of lost customers in the system and minimize this probability.

3. Analyzing the Model with Semi-Markov Process [7] [8]

Let $t_0, t_1, t_2, \cdots, t_n$ be the arrival time of the customers in the system and $0 = t_0 < t_1 < t_2 < \cdots$.

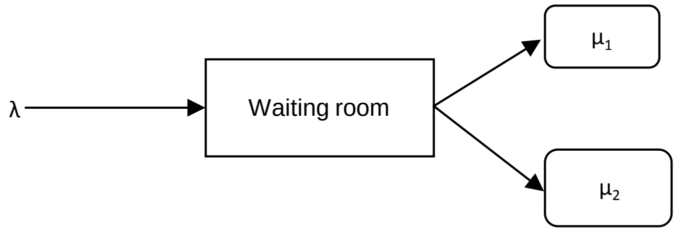
Let $X(t)$ be the number of customers in the system at time t and $X_n = X(t_n)$, $n \ge 1$; where the number of customers staying in the system immediately before the time of arrival of the n^{th} customer in the system is denoted by X_n. Let the semi-Markov process representing the system be defined as follows

$$\eta(t) = X_n, t_n \le t < t_{n+1}, n \ge 1 \tag{6}$$

The function defined as:

$$Q_{ij}(x) = P\{(X_{n+1} = j, t_{n+1} - t_n < x) | X_n = i\} \tag{7}$$

For $x > 0$ and $0 \le i, j \le 3$ is called the kernel of process. Based on assumption of Formula (6) and the total probability formula, functions of Formula (7) are computed as follows: [9]



$$Q_{00}(x) = F(x) - Q_{01}(x)$$

$$Q_{01}(x) = \int_0^x \left(e^{-\mu_1 t} + e^{-\mu_2 t} - e^{-(\mu_1+\mu_2)t}\right) dF(t),$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$Q_{02}(x) = Q_{03}(x) = 0$$

$$Q_{10}(x) = \int_0^x \left[1 - \left(e^{-\mu_1 t} + e^{-\mu_2 t} - e^{-(\mu_1+\mu_2)t}\right)\right] dF(t)$$

$$Q_{11}(x) = F(x) - Q_{10}(x) - Q_{12}(x)$$

$$Q_{12}(x) = \int_0^x e^{-(\mu_1+\mu_2)t} dF(t)$$

$$Q_{13}(x) = 0$$

$$Q_{20}(x) = \int_0^x \left[1 - e^{-(\mu_1+\mu_2)t}\right] dF(t)$$

$$Q_{21}(x) = \int_0^x \left(e^{-\mu_1 t} + e^{-\mu_2 t} - e^{-(\mu_1+\mu_2)t}\right) dF(t)$$

$$Q_{22}(x) = \int_0^x e^{-(\mu_1+\mu_2)t} dF(t)$$

$$Q_{23}(x) = \int_0^x e^{-(\mu_1+\mu_2+\lambda)t} dF(t)$$

$$Q_{30}(x) = 0$$

$$Q_{31}(x) = \int_0^x \left(1 - e^{-(\mu_1+\mu_2)t}\right) dF(t)$$

$$Q_{32}(x) = \int_0^x \left[1 - \left(e^{-\mu_1 t} + e^{-\mu_2 t} - e^{-(\mu_1+\mu_2)t}\right)\right] dF(t)$$

$$Q_{33}(x)=\int_0^x \mathrm{e}^{-(\mu_1+\mu_2+\lambda)t}\mathrm{d}F(t)$$

$q_{ij}(s)$: is the LS transforms of functions $Q_{ij}(x)$

$$q_{ij}(s)=\int_0^\infty \mathrm{e}^{-sx}\mathrm{d}Q_{ij}(x);\ 0\le i,j\le 3,\quad f(s)=\int_0^\infty \mathrm{e}^{-sx}\mathrm{d}F(x) \tag{8}$$

$$q_{00}(s)=f(s)-f(s+\mu_1)-f(s+\mu_2)+f(s+\mu_1+\mu_2) \tag{9}$$

$$q_{01}(s)=f(s+\mu_1)+f(s+\mu_2)-f(s+\mu_1+\mu_2) \tag{10}$$

$$q_{02}(s)=q_{03}(s)=0 \tag{11}$$

$$q_{10}(s)=f(s)-f(s+\mu_1)-f(s+\mu_2)+f(s+\mu_1+\mu_2) \tag{12}$$

$$q_{11}(s)=f(s+\mu_1)+f(s+\mu_2)-2f(s+\mu_1+\mu_2) \tag{13}$$

$$q_{12}(s)=f(s+\mu_1+\mu_2) \tag{14}$$

$$q_{13}(s)=0 \tag{15}$$

$$q_{20}(s)=f(s)-f(s+\mu_1+\mu_2) \tag{16}$$

$$q_{21}(s)=f(s+\mu_1)+f(s+\mu_2)-f(s+\mu_1+\mu_2) \tag{17}$$

$$q_{22}(s)=f(s+\mu_1+\mu_2) \tag{18}$$

$$q_{23}(s)=f(s+\mu_1+\mu_2+\lambda) \tag{19}$$

$$q_{30}(s)=0 \tag{20}$$

$$q_{31}(s)=f(s)-f(s+\mu_1+\mu_2) \tag{21}$$

$$q_{32}(s)=f(s)-f(s+\mu_1)-f(s+\mu_2)+f(s+\mu_1+\mu_2) \tag{22}$$

$$q_{33}(s)=f(s+\mu_1+\mu_2+\lambda) \tag{23}$$

$$p_{ij}=P(X_{n+1}=j|X_n=i);\ \ p_{ij}=q_{ij}(0);\ i,j=1,2,3 \tag{24}$$

The probabilities are obtained from (17) for the model M/M/2/1 with heterogeneous as the follows: [10] [11]

$$q_{00}(0)=p_{00}=f(0)-f(0+\mu_1)-f(0+\mu_2)+f(0+\mu_1+\mu_2)$$

$$p_{00}=1-f(\mu_1)-f(\mu_2)+f(\mu_1+\mu_2)=1-f_1-f_2+f_{12} \tag{25}$$

$$p_{01}=f_1+f_2-f_{12} \tag{26}$$

$$p_{02}=p_{03}=0 \tag{27}$$

$$p_{10}=1-f_1-f_2+f_{12} \tag{28}$$

$$p_{11}=f_1+f_2-2f_{12} \tag{29}$$

$$p_{12}=f_{12} \tag{30}$$

$$p_{13}=0 \tag{31}$$

$$p_{20}=1-f_{12} \tag{32}$$

$$p_{21}=f_1+f_2-f_{12} \tag{33}$$

$$p_{22}=f_{12} \tag{34}$$

$$p_{23} = f(\mu_1 + \mu_2 + \lambda) = f_{12\lambda} \quad (35)$$

$$p_{30} = 0 \quad (36)$$

$$p_{31} = 1 - f_{12} \quad (37)$$

$$p_{32} = f_1 + f_2 - f_{12} \quad (38)$$

$$p_{33} = f_{12\lambda} \quad (39)$$

4. Steady State Probabilities π_j Satisfy the Following Equations [12] [13] [14]

$$\pi_j = \sum_{i=0}^{3} \pi_i p_{ij}; \ \sum_j \pi_j = 1$$

$$\pi_0 = \sum_{i=0}^{3} \pi_i p_{i0} = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30} \quad (40)$$

$$(p_{00} - 1)\pi_0 + \pi_1 p_{10} + \pi_2 p_{20} = 0; \ p_{30} = 0$$

$$a_0 \pi_0 + \pi_1 a_1 + \pi_2 a_2 = 0 \quad (40.1)$$

$$a_0 = (p_{00} - 1) = 1 - f_1 - f_2 + f_{12} - 1 = -f_1 - f_2 + f_{12}$$

$$a_1 = p_{10} = 1 - f_1 - f_2 + f_{12}$$

$$a_2 = p_{20} = 1 - f_{12}$$

$$\pi_1 = \sum_{i=0}^{3} \pi_i p_{i1} = \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} \quad (41)$$

$$\pi_0 p_{01} + \pi_1 (p_{11} - 1) + \pi_2 p_{21} + \pi_3 p_{31} = 0$$

$$\pi_0 b_0 + \pi_1 b_1 + \pi_2 b_2 + \pi_3 b_3 = 0$$

$$b_0 = p_{01} = f_1 + f_2 - f_{12}$$

$$b_1 = (p_{11} - 1) = f_1 + f_2 - 2 f_{12} - 1$$

$$b_2 = p_{21} = f_1 + f_2 - f_{12}$$

$$b_3 = p_{31} = 1 - f_{12}$$

$$\pi_2 = \sum_{i=0}^{3} \pi_i p_{i2} = \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} \quad (42)$$

$$\pi_1 p_{12} + \pi_2 (p_{22} - 1) + \pi_3 p_{32} = 0; \ p_{02} = 0$$

$$\pi_1 c_1 + \pi_2 c_2 + \pi_3 c_3 = 0 \quad (42.1)$$

$$c_1 = p_{12} = f_{12}$$

$$c_2 = (p_{22} - 1) = f_{12} - 1$$

$$c_3 = p_{32} = 1 - f_1 - f_2 + f_{12}$$

$$\pi_3 = \sum_{i=0}^{3} \pi_i p_{i3} = \pi_0 p_{03} + \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} \quad (43)$$

$$\pi_2 p_{23} + \pi_3 (p_{33} - 1) = 0; \ p_{03} = p_{13} = 0$$

$$\pi_2 d_2 + \pi_3 d_3 = 0 \quad (43.1)$$

$$d_2 = p_{23} = f_{12\lambda}$$

$$d_3 = (p_{33} - 1) = f_{12\lambda} - 1$$

$$\sum_j \pi_j = 1$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \tag{44}$$

From Equation (43.1):

$$\pi_2 d_2 + \pi_3 d_3 = 0$$

$$\pi_2 = -\frac{f_{12\lambda} - 1}{f_{12\lambda}} \pi_3 \tag{45}$$

From Equation (42.1):

$$\pi_1 c_1 + \pi_2 c_2 + \pi_3 c_3 = 0$$

$$\pi_1 c_1 - \frac{d_3}{d_2} \pi_3 c_2 + \pi_3 c_3 = 0$$

$$\pi_1 = \frac{c_2 d_3 - c_3 d_2}{c_1 d_2} \pi_3 \tag{46}$$

From Equation (40.1):

$$a_0 \pi_0 + \pi_1 a_1 + \pi_2 a_2 = 0$$

$$a_0 \pi_0 + \left(\frac{c_2 d_3 - c_3 d_2}{c_1 d_2} \pi_3 \right) a_1 + \left(-\frac{d_3}{d_2} \pi_3 \right) a_2 = 0$$

$$a_0 \pi_0 = -\left(\frac{c_2 d_3 - c_3 d_2}{c_1 d_2} \pi_3 \right) a_1 + \left(\frac{d_3}{d_2} \pi_3 \right) a_2 = \frac{a_2 c_1 d_3 - a_1 c_2 d_3 + a_1 c_3 d_2}{c_1 d_2} \pi_3$$

$$\pi_0 = \frac{a_2 c_1 d_3 - a_1 c_2 d_3 + a_1 c_3 d_2}{a_0 c_1 d_2} \pi_3 \tag{47}$$

Use Equations (45:47) in Equation (44):

$$\pi_3 = \frac{-a_0 c_1 c_2}{a_1 c_1 d_3 - a_1 c_2 d_3 - a_1 c_1 c_2 + a_0 c_2 d_3 - a_0 c_1 d_3}$$

$$a_0 c_1 c_2 = \left(-f_1 - f_2 + f_{12} \right) f_{12} \left(f_{12} - 1 \right) = f_{12} \left(1 - f_{12} \right) \left(f_1 + f_2 - f_{12} \right)$$

$$a_1 c_1 c_2 = f_{12} \left(1 - f_{12} \right) \left(f_1 + f_2 - f_{12} \right)$$

$$d_3 \left(c_1 - c_2 \right) \left(a_1 - a_0 \right) = \left(f_{12\lambda} - 1 \right)$$

π_0: the probability that the system's probability of being free

π_1: the probability that only one server is busy in the system

π_2: the probability that the two servers are busy in the system

π_3: the probability that the two servers are busy in the system and the waiting room has a customer.

$$\pi_3 = \frac{f_{12} \left(f_{12} - 1 \right) \left(f_1 + f_2 - f_{12} \right)}{\left(f_{12\lambda} - 1 \right) - f_{12} \left(1 - f_{12} \right) \left(f_1 + f_2 - f_{12} \right)} \tag{48}$$

Loss probability and its minimization [15] [16] [17] [18]

As there is a waiting room with just one place is available in the system, the probability that the two servers are busy and the waiting room has a customer is equivalent to the probability of loss of customers in the system. That is, Formula (48) is equal to the loss probability. If we symbolize the probability of loss of customers in the system with P_{LOSS} and is written as

$$P_{\text{LOSS}} = \pi_3$$

Under $\mu_1 = \mu_2 = \mu \geq \lambda > 0$ the formula of loss probability presented by Formula (48) satisfies Palm's loss Formula (2) for n = 3. Optimization of the queueing system given above to create more efficient systems in real life appears an important problem. For the M/M/2/1 queueing model with a waiting line of just one place and with heterogeneous servers, and $\mu_1 + \mu_2 = \mu \geq 0$ under, Nath and Enns (1981) assigned the arriving customer to the server with the lowest mean service time among the free servers, thereby minimizing the probability of lost customers in the system.

Let A_m be a class of distribution functions F of the inter-arrival times, the mean of which is a constant m, $0 < m < t$. Let $P_{\text{LOSS}}(F)$ be the loss probability for the M/M/2/1 queueing system with heterogeneous servers and ordered entry, and $F \in A_m$. Assume that $D(t)$ is the deterministic distribution, in which $D(t) = 1$ for $t \leq m$ and $D(t) = 0$ for $t > m$.

The Laplace transform of Dirace Delta :

$$\mathcal{L}\left\{\delta(t-m)\right\} = \int_0^\infty e^{-st}\delta(t-m)\,dt = e^{-mt};\ m > 0 \tag{49}$$

Theorem:

When the distribution of inter-arrival times fits the deterministic distribution D among all distribution functions included in class A_m, loss probability $P_{\text{LOSS}}(F)$ becomes minimum, that is, $\min\limits_{F \in A_m} P_{\text{LOSS}}(F) = P_{\text{LOSS}}(D)$ [19]

Proof:

For minimizing the loss probability, let Formula (48) be :

$$P_{\text{LOSS}}(F) = \pi_3 = \frac{-f_{12}(1-f_{12})(f_1 + f_2 - f_{12})}{(f_{12\lambda} - 1) - f_{12}(1-f_{12})(f_1 + f_2 - f_{12})} \tag{50}$$

By using Jensen's inequality $f(s) \geq e^{-ms}$ which mean that the probability of lost customers in the system is minimum when under-arrival times fit the deterministic distribution.

The numerator:

$$f_{12}(1-f_{12})(f_1 + f_2 - f_{12}) \geq e^{-m(\mu_1+\mu_2)}\left(1 - e^{-m(\mu_1+\mu_2)}\right)\left(e^{-m\mu_1} + e^{-m\mu_2} - e^{-m(\mu_1+\mu_2)}\right)$$

$$f_{12}(f_{12} - 1)(f_1 + f_2 - f_{12}) \leq -e^{-m(\mu_1+\mu_2)}\left(1 - e^{-m(\mu_1+\mu_2)}\right)\left(e^{-m\mu_1} + e^{-m\mu_2} - e^{-m(\mu_1+\mu_2)}\right)$$

The denominator:

$$(f_{12\lambda} - 1) - f_{12}(1-f_{12})(f_1 + f_2 - f_{12}) \geq \left(e^{-m(\mu_1+\mu_2+\lambda)} - 1\right) - e^{-m(\mu_1+\mu_2)} * \left(1 - e^{-m(\mu_1+\mu_2)}\right) * \left(e^{-m\mu_1} + e^{-m\mu_2} - e^{-m(\mu_1+\mu_2)}\right)$$

From (48) and use (50):

$$P_{\text{LOSS}}(F) \geq \frac{f_{12}(f_{12}-1)(f_1+f_2-f_{12})}{(f_{12\lambda}-1)-f_{12}(1-f_{12})(f_1+f_2-f_{12})}$$
$$\geq \frac{e^{-m(\mu_1+\mu_2)}\left(e^{-m(\mu_1+\mu_2)}-1\right)\left[e^{-m\mu_1}+e^{-m\mu_2}-e^{-m(\mu_1+\mu_2)}\right]}{\left[e^{-m(\mu_1+\mu_2+\lambda)}-1\right]-e^{-m(\mu_1+\mu_2)}*\left[1-e^{-m(\mu_1+\mu_2)}\right]*\left[e^{-m\mu_1}+e^{-m\mu_2}-e^{-m(\mu_1+\mu_2)}\right]} \tag{51}$$

The right side of (51) has the value of $P_{\text{LOSS}}(D)$ thus $\min_{F\in A_m} P_{\text{LOSS}}(F) = P_{\text{LOSS}}(D)$ (Q.E.D)

Numerical example:

Use Equation (51)

Case 1: At $m=1; \mu_1=1; \mu_2=2; \lambda=1$

$$P_{\text{LOSS}}(D) = 0.0505 \tag{52}$$

Case 2: At $m=1; \mu_1=2; \mu_2=2; \lambda=2$

$$P_{\text{LOSS}}(D) = 0.0159 \tag{53}$$

Case 3: At $m=1; \mu_1=2; \mu_2=2; \lambda=1$

$$P_{\text{LOSS}}(D) = 0.0164 \tag{54}$$

It is cleared that case 2 is the smallest probability of losing customers.

5. Discussion

In this study, the M/M/2/1 model with recurrent entries, finite capacity and ordered entry was analyzed and the steady-state probabilities of the system and the probability of lost customers in the system were obtained. Optimization was performed according to the arrival processes and it was shown that the loss probability was minimum with probability 0.0159 in the M/M/2/1 queueing system with heterogeneous servers and ordered entry when the deterministic distribution was selected among the distributions of inter-arrival times with identical means.

6. Conclusion

The loss probability is minimum in M/M/2/1 model where $\mu_1 = \mu_2 = \lambda$.

7. Recommendations

It can be studied that the model M/M/n/r, $1 \leq r \leq n$ and the distribution of stream of overflows can be obtained by analyzing the stream of overflows. With the distribution to be obtained, the loss probability can be generalized for n services. Similar analyses can be made for "Random" service discipline or for different suggested disciplines instead of "Ordered Entry" discipline.

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