

An Inventory Model for Deteriorating Items with Generalised Exponential Decreasing Demand, Constant Holding Cost and Time-Varying Deterioration Rate

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Abstract

In this present paper, an inventory model with a generalised exponential decreasing demand is considered. A numerical example is used to illustrate the application of the model. Sensitivity analysis of the optimal solution with respect to various parameters is carried out to see the effect of parameter changes on the solution.

Keywords

Linear Deterioration, Generalize Exponential Decreasing Demand, Inventory, Constant Holding Cost

1. Introduction

In real life, the effect of deterioration is very important in many inventory systems. In general, deterioration is defined as decay, damage, spoilage, evaporation, obsolescence, pilferage, loss of utility, or loss of marginal value of a commodity that results in decreasing usefulness, Wee [1]. Most of physical goods undergo decay or deterioration over time. The proposed model in this paper is for the deteriorating item which has a time-dependent generalised exponential decreasing demand rate and time dependent, linear deterioration rate and a constant holding cost. The items that exhibit the above phenomenon are food items, photographic films, drugs, chemicals, pharmaceuticals, electronic components, blood kept in blood banks and so on. Therefore, the effect of deterioration on these items cannot be disregarded in their inventory systems.

Consequently, the production and inventory problem of deteriorating items has been extensively studied by researchers. Some of the researchers include Ghare and Schrader [2] who are the first researchers to derive an economic order quantity model by assuming exponential decay for the item. Later, Covert and Philip [3] extended Ghare and Schrader's [2] model by considering the deterioration rate to be a two-parameter weibull distribution. Later, Shah and Jaiswal [4] presented an order-level inventory model for deteriorating items with a constant rate of deterioration. Aggarwal [5] corrected the analysis in Shah and Jaiswal's model [4]. Dave and Patel [6] considered an inventory model for deteriorating items with time-proportional demand when shortages were not allowed. Authors such as Hollier and Mark [7], Hariga and Benkherouf [8], Wee [9] [10] all developed their models by considering demand to be an exponential demand.

Some recent works on deteriorating items include the work of Goyal and Giri [11], in which they presented a very good survey on the recent trends in modeling of deteriorating inventory. Also Ouyang *et al.* [12] developed an Economic Order Quantity (EOQ) inventory model for deteriorating items in which demand function is exponentially declining and with partial backlogging. Also Shah and Pandey [13] in their study developed an optimal ordering policy for time dependent deterioration with associated salvage value where delay in payments is permissible. Another recent work is He and He [14], who made an extension to consider the fact that some products may deteriorate during storage. They developed a production inventory model for deteriorating items with production disruptions. The inventory plans and optimal production were provided, in such a way that the manufacturer can minimize the loss caused by disruptions. Kumar *et al.* [15] in their research, developed a deterministic inventory model for deteriorating items, where they considered their demand as a quadratic function of time, no shortages are allowed and the effect of inflation rate in the model was assumed to be over a finite planning horizon taking a variable holding cost. Singh and Pattnayk [16] also in their work presented an Economic Order Quantity (EOQ) model for deteriorating items with time-dependent quadratic demand and variable deterioration, under permissible delay in payment. Dash *et al* [17] also developed an inventory model for deteriorating items having a time-dependent exponential declining demand rate and time-varying holding cost as a linear function of time. Shortages were not allowed. Aliyu and Sani [18] developed an inventory model for deteriorating items with generalised exponential decreasing demand and linear time-varying holding cost. The rate of deterioration was considered to be a constant. Shortages were not allowed.

In this paper, we consider the same generalised exponential decreasing demand as in Aliyu and Sani [18] but the holding cost is assumed to be a constant while the deterioration rate is assumed to be a linear function of time.

2. Assumptions and Notation

In formulating the mathematical model, the following notation and assumptions

are employed.

2.1. Assumptions

The inventory system considers a single item only.

The demand rate is deterministic and is a generalised exponential decreasing function of time.

The deterioration rate is considered to be a linear function of time.

Lead time is zero.

There are no shortages.

The inventory system is considered over an infinite time horizon.

The holding cost is assumed to be a constant.

2.2. Notation

N_0 : The fixed ordering cost per order

$I(t)$: The inventory at any time t , $0 \leq t \leq T$

$D(t)$: The exponential demand rate, where $D(t) = Ke^{h-\beta t}$, $K > 0, \beta > 0, h > 0$, are all constants.

The deterioration rate is a linear function of time given as $a + bt$.

The holding cost which is a constant is given as iC where C is the unit cost of an item and i is the inventory carrying charge.

C : The cost of each deteriorated unit.

T : The length of the ordering cycle.

I_0 : Initial stock.

TC : The total cost per unit time.

T^* : The optimal length of the cycle.

I_o^* : The economic order quantity

TC^* : The minimum total cost per unit time.

The difference between this work and that of Aliyu and Sani [18] is the fact that in Aliyu and Sani [18] a linear holding cost *i.e.* $h_0(t) = h_1 + h_2t$ was considered while the deterioration rate was a constant but in this paper, we consider the deterioration rate to be a linear function of time given as $a + bt$, while the holding cost is a constant.

3. Mathematical Model and Analysis

Applying the above assumptions, we obtained a typical cycle for the variation of inventory level with time as shown in **Figure 1**.

As we can observe from **Figure 2**, the inventory level gradually decreases from initial stage due to the effect of both demand and deterioration. The differential equation which describes the state of inventory level $I(t)$ in the interval $[0, T]$, is given by;

$$\frac{dI(t)}{dt} + (a + bt)I(t) = -D(t), \quad 0 \leq t \leq T \quad (1)$$

where $D(t) = Ke^{h-\beta t}$

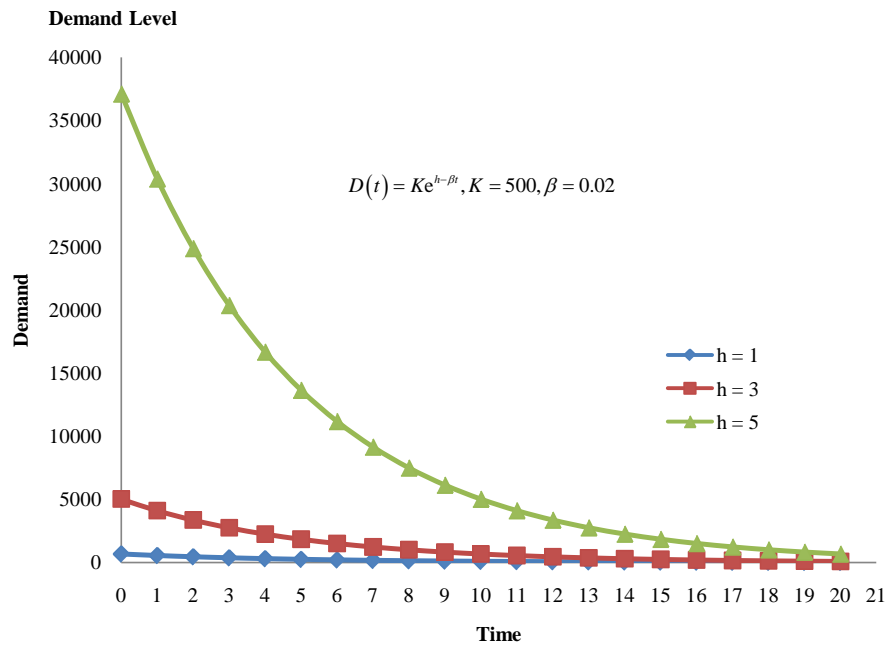


Figure 1. Graphical representation of various demand levels having different h values.

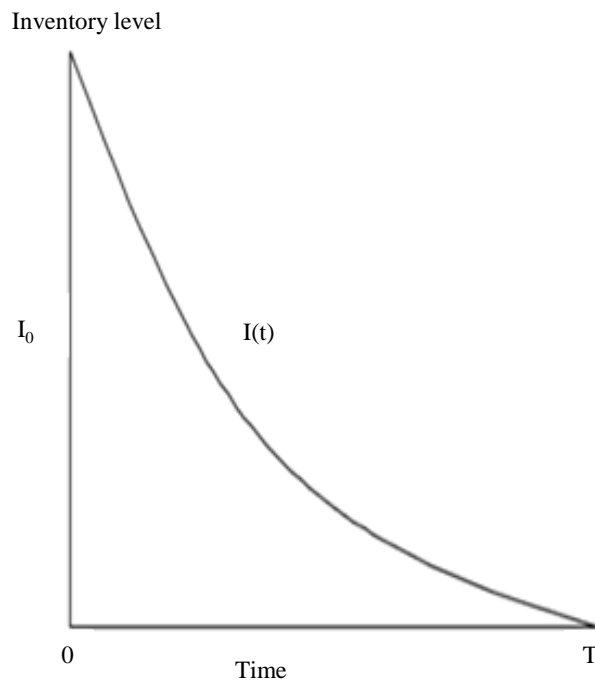


Figure 2. The graphical representation for the inventory system.

The solution of Equation (1) is

$$I(t) = \frac{-K}{-\beta + a + bt} e^{h-\beta t} + Ce^{-at - \frac{1}{2}bt^2} \tag{2}$$

(see **Appendix**).

Applying the boundary condition $I(t) = 0$ when $t = T$ in Equation (2), where T is the length of ordering cycle and t is the current time we are con-

cerned with. Therefore,

$$I(0) = 0 = \frac{-K}{-\beta + a + bT} e^{h-\beta T} + C e^{-aT - \frac{1}{2}bT^2}$$

$$\Rightarrow \frac{K}{-\beta + a + bT} e^{h-\beta T} = C e^{-aT - \frac{1}{2}bT^2} \text{ or } C = \frac{K}{-\beta + a + bT} e^{h-\beta T} \cdot e^{aT + \frac{1}{2}bT^2}$$

We substitute C in (2) to obtain

$$I(t) = \frac{-K}{-\beta + a + bt} e^{h-\beta t} + \frac{K}{-\beta + a + bT} e^{h-\beta T} \cdot e^{aT + \frac{1}{2}bT^2} \cdot e^{-at - \frac{1}{2}bt^2}$$

$$= Ke^h \left[\frac{e^{-\beta t}}{-\beta + a + bt} + \frac{e^{-\beta T + aT + \frac{1}{2}bT^2 - at - \frac{1}{2}bt^2}}{-\beta + a + bT} \right] \tag{3}$$

$$= \frac{Ke^h}{(-\beta + a + bt)(-\beta + a + bT)} \left[-e^{-\beta t} (-\beta + a + bT) + (-\beta + a + bt) e^{-\beta T + aT + \frac{1}{2}bT^2 - at - \frac{1}{2}bt^2} \right],$$

$0 \leq t \leq T$

The initial order quantity can be obtained by putting the boundary condition $I(0) = I_0$ into Equation (3) as follows:

$$I(0) = I_0 = \frac{Ke^h}{(-\beta + a + b(0))(-\beta + a + bT)}$$

$$\times \left[-e^{-\beta(0)} (-\beta + a + bT) + (-\beta + a + b(0)) e^{-\beta T + aT + \frac{1}{2}bT^2 - a(0) - \frac{1}{2}b(0)^2} \right] \tag{4}$$

$$= \frac{Ke^h}{(-\beta + a)(-\beta + a + bT)} \left[-(-\beta + a + bT) + (-\beta + a) e^{-\beta T + aT + \frac{1}{2}bT^2} \right]$$

The total demand during the cycle period $[0, T]$ is given as follows:

$$\int_0^T D(t) dt = \int_0^T Ke^{h-\beta t} dt = \frac{K}{-\beta} \left[e^{h-\beta t} \right]_0^T$$

$$= \frac{K}{-\beta} \left[e^{h-\beta T} - e^h \right] = \frac{Ke^h}{-\beta} \left[e^{-\beta T} - 1 \right] \tag{5}$$

The number of deteriorated units is given as initial order quantity minus the total demand in the cycle period $[0, T]$. Thus the number of deteriorated units is

$$I_0 - \int_0^T D(t) dt$$

$$= \frac{Ke^h}{(-\beta + a)(-\beta + a + bT)} \left[-(-\beta + a + bT) + (-\beta + a) e^{-\beta T + aT + \frac{1}{2}bT^2} \right] - \frac{Ke^h}{-\beta} \left[e^{-\beta T} - 1 \right]$$

$$= Ke^h \left[\left\{ \frac{-(-\beta + a + bT)}{(-\beta + a)(-\beta + a + bT)} + \frac{(-\beta + a)}{(-\beta + a)(-\beta + a + bT)} e^{-\beta T + aT + \frac{1}{2}bT^2} \right\} - \frac{1}{-\beta} \{ e^{-\beta T} - 1 \} \right]$$

$$= \frac{Ke^h}{-\beta(-\beta + a)(-\beta + a + bT)}$$

$$\times \left[\beta(-\beta + a + bT) - \beta(-\beta + a) e^{-\beta T + aT + \frac{1}{2}bT^2} - (-\beta + a)(-\beta + a + bT) \{ e^{-\beta T} - 1 \} \right]$$

$$\begin{aligned}
 &= \frac{Ke^h}{-\beta(-\beta+a)(-\beta+a+bT)} \\
 &\left[\beta^2 e^{-\beta T+aT+\frac{1}{2}bT^2} - a\beta e^{-\beta T+aT+\frac{1}{2}bT^2} - \beta^2 e^{-\beta T} + 2a\beta e^{-\beta T} \right. \\
 &\left. + b\beta T e^{-\beta T} - a^2 e^{-\beta T} - abT e^{-\beta T} - a\beta + a^2 + abT \right] \tag{6}
 \end{aligned}$$

Deterioration cost (DC) for the cycle $[O, T] = Cx$ (the number of deteriorated units)

$$\begin{aligned}
 &= \frac{CKe^h}{-\beta(-\beta+a)(-\beta+a+bT)} \\
 &\left[\beta^2 e^{-\beta T+aT+\frac{1}{2}bT^2} - a\beta e^{-\beta T+aT+\frac{1}{2}bT^2} - \beta^2 e^{-\beta T} + 2a\beta e^{-\beta T} \right. \\
 &\left. + b\beta T e^{-\beta T} - a^2 e^{-\beta T} - abT e^{-\beta T} - a\beta + a^2 + abT \right] \tag{7}
 \end{aligned}$$

The total inventory holding cost (IHC) for the cycle $[O, T]$ is as follows

$$\begin{aligned}
 &= \int_0^T iCI(t) dt \\
 &= \frac{iCKe^h}{(-\beta+a+bt)(-\beta+a+bT)} \\
 &\times \left[\int_0^T \left\{ (-\beta+a+bt) e^{-\beta T+aT+\frac{1}{2}bT^2-at-\frac{1}{2}bt^2} - e^{-\beta t} (-\beta+a+bT) \right\} dt \right] \tag{8} \\
 &= \frac{iCKe^h}{(-\beta+a+bT)} \\
 &\times \left[\int_0^T \left\{ e^{-\beta T+aT+\frac{1}{2}bT^2-at-\frac{1}{2}bt^2} - e^{-\beta t} (-\beta+a+bT)(-\beta+a+bt)^{-1} \right\} dt \right]
 \end{aligned}$$

To solve Equation (8), we can divide the square bracket into two parts, the first part we denote it by D and the second part by E, *i.e.*

$$\int_0^T e^{-\beta T+aT+\frac{1}{2}bT^2-at-\frac{1}{2}bt^2} dt \quad (\text{D}) \quad \text{and} \quad \int_0^T e^{-\beta t} (-\beta+a+bT)(-\beta+a+bt)^{-1} dt \quad (\text{E})$$

The solution of Equation (8) will then be

$$\int_0^T iCI(t) dt = \frac{iCKe^h}{(-\beta+a+bT)} \left[\text{Solution of D} - \text{Solution of E} \right]$$

Solution of D

$$D = \int_0^T e^{-\beta T+aT+\frac{1}{2}bT^2-at-\frac{1}{2}bt^2} dt$$

Using integration by part,

$$\begin{aligned}
 \int_0^T e^{-\beta T+aT+\frac{1}{2}bT^2-at-\frac{1}{2}bt^2} dt &= \frac{1}{-a-bt} e^{-\beta T+aT+\frac{1}{2}bT^2-at-\frac{1}{2}bt^2} \Bigg|_0^T \\
 &= \frac{1}{-a-bT} e^{-\beta T} + \frac{1}{a} e^{-\beta T+aT+\frac{1}{2}bT^2}
 \end{aligned}$$

Solution of E

$$\begin{aligned}
 E &= \int_0^T e^{-\beta t} (-\beta + a + bT)(-\beta + a + bt)^{-1} dt \\
 &= (-\beta + a + bT) \int_0^T e^{-\beta t} (-\beta + a + bt)^{-1} dt \\
 &= (-\beta + a + bT) \left[\frac{1}{b} \ln(-\beta + a + bT) e^{-\beta t} - \frac{1}{b} \ln(-\beta + a + bT) e^{-\beta t} + \frac{E}{(-\beta + a + bT)} \right]_0^T \\
 &= (-\beta + a + bT) \left[\frac{K}{(-\beta + a + bT)} \right]_0^T = 0 \\
 \therefore \int_0^T iCI(t) dt &= \frac{iCKe^h}{(-\beta + a + bT)} [\text{Solution of D-Solution of E}] \\
 &= \frac{iCKe^h}{(-\beta + a + bT)} \left[-\frac{1}{a + bT} e^{-\beta T} + \frac{1}{a} e^{-\beta T + aT + \frac{1}{2} bT^2} - 0 \right] \tag{9} \\
 &= \frac{iCKe^h}{(-\beta + a + bT)} \left[-\frac{1}{a + bT} e^{-\beta T} + \frac{1}{a} e^{-\beta T + aT + \frac{1}{2} bT^2} \right]
 \end{aligned}$$

Total variable cost = Ordering cost (OC) + Deterioration cost + Inventory Holding cost (IHC).

The total variable Cost per unit time TC(T) is

$$\begin{aligned}
 TC(T) &= \frac{N_0}{T} + \frac{CKe^h}{-\beta(-\beta + a)(-\beta + a + bT)T} \\
 &\quad \left[\beta^2 e^{-\beta T + aT + \frac{1}{2} bT^2} - a\beta e^{-\beta T + aT + \frac{1}{2} bT^2} - \beta^2 e^{-\beta T} + 2a\beta e^{-\beta T} \right. \\
 &\quad \left. + b\beta T e^{-\beta T} - a^2 e^{-\beta T} - abT e^{-\beta T} - a\beta + a^2 + abT \right] \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{iCKe^h}{(-\beta + a + bT)T} \left[-\frac{1}{a + bT} e^{-\beta T} + \frac{1}{a} e^{-\beta T + aT + \frac{1}{2} bT^2} \right] \\
 &= \frac{N_0}{T} + \frac{CKe^h}{-\beta(-\beta + a)} \\
 &\quad \times \left[\frac{\beta^2 e^{-\beta T + aT + \frac{1}{2} bT^2}}{(-\beta T + aT + bT^2)} - \frac{a\beta e^{-\beta T + aT + \frac{1}{2} bT^2}}{(-\beta T + aT + bT^2)} - \frac{\beta^2 e^{-\beta T}}{(-\beta T + aT + bT^2)} \right. \\
 &\quad \left. + \frac{2a\beta e^{-\beta T}}{(-\beta T + aT + bT^2)} + \frac{b\beta e^{-\beta T}}{(-\beta T + aT + bT^2)} - \frac{a^2 e^{-\beta T}}{(-\beta T + aT + bT^2)} \right. \\
 &\quad \left. - \frac{abe^{-\beta T}}{(-\beta + a + bT)} - \frac{a\beta}{(-\beta T + aT + bT^2)} + \frac{a^2}{(-\beta T + aT + bT^2)} + \frac{ab}{(-\beta + a + bT)} \right] \\
 &+ iCKe^h \left[-\frac{e^{-\beta T}}{(a + bT)(-\beta T + aT + bT^2)} + \frac{e^{-\beta T + aT + \frac{1}{2} bT^2}}{(-\beta T + aT + bT^2)} \right] \tag{11}
 \end{aligned}$$

The main objective is to find the minimum variable cost per unit time. The necessary and sufficient conditions to minimize TC(T) are respectively,

$$\frac{dTC(T)}{dT} = 0 \text{ and } \frac{d^2TC(T)}{dT^2} > 0$$

Therefore to satisfy the necessary condition we have to differentiate equation (11) with respect to T, as follows

$$\begin{aligned} \frac{dTC(T)}{dT} = & -\frac{N_0}{T^2} + \frac{CKe^h}{-\beta(-\beta+a)} \\ & \left[\frac{\beta^2 e^{-\beta T+aT+\frac{1}{2}bT^2}}{T} - \frac{\beta^2(-\beta+a+2bT)e^{-\beta T+aT+\frac{1}{2}bT^2}}{(-\beta T+aT+bT^2)^2} - \frac{a\beta e^{-\beta T+aT+\frac{1}{2}bT^2}}{T} \right. \\ & + \frac{a\beta(-\beta+a+2bT)e^{-\beta T+aT+\frac{1}{2}bT^2}}{(-\beta T+aT+bT^2)^2} + \frac{\beta^3 e^{-\beta T}}{(-\beta T+aT+bT^2)} + \frac{\beta^2(-\beta+a+2bT)e^{-\beta T}}{(-\beta T+aT+bT^2)^2} \\ & - \frac{2a\beta^2 e^{-\beta T}}{(-\beta T+aT+bT^2)} - \frac{2a\beta(-\beta+a+2bT)e^{-\beta T}}{(-\beta T+aT+bT^2)^2} - \frac{b\beta^2 e^{-\beta T}}{(-\beta+a+bT)} \\ & - \frac{b^2\beta e^{-\beta T}}{(-\beta+a+bT)^2} + \frac{a^2\beta e^{-\beta T}}{(-\beta T+aT+bT^2)} + \frac{a^2(-\beta+a+2bT)e^{-\beta T}}{(-\beta T+aT+bT^2)^2} \\ & + \frac{ab\beta e^{-\beta T}}{(-\beta+a+bT)} + \frac{ab^2 e^{-\beta T}}{(-\beta+a+bT)^2} + \frac{a\beta(-\beta+a+2bT)}{(-\beta T+aT+bT^2)^2} \\ & \left. - \frac{a^2(-\beta+a+2bT)}{(-\beta T+aT+bT^2)^2} - \frac{ab^2}{(-\beta+a+bT)^2} \right] \\ & + icke^h \left[+ \frac{\beta e^{-\beta T}}{(a+bT)(-\beta T+aT+bT^2)} + \frac{be^{-\beta T}}{(a+bT)^2(-\beta T+aT+bT^2)} \right. \\ & + \frac{(-\beta+a+2bT)e^{-\beta T}}{(a+bT)(-\beta T+aT+bT^2)^2} + \frac{(-\beta+a+bT)e^{-\beta T+aT+\frac{1}{2}bT^2}}{a(-\beta T+aT+bT^2)} \\ & \left. - \frac{(-\beta+a+2bT)e^{-\beta T+aT+\frac{1}{2}bT^2}}{a(-\beta T+aT+bT^2)^2} \right]. \end{aligned} \tag{12}$$

We now equate Equation (12) to zero and simplify by multiplying with $-T^2 a\beta(-\beta+a)(-\beta T+aT+bT^2)^2(-\beta+a+bT)^2(a+bT)^2$ on both sides in order to determine the T which minimizes the variable cost per unit time as follows:

We equate Equation (12) to zero simply because we want to determine the minimum cost. This is the necessary condition for getting the roots of an equation which optimise the equation. Thus this is the necessary condition for getting the turning points of the equation.

$$\begin{aligned}
& N_0 a \beta (-\beta + a) (-\beta T + aT + bT^2)^2 (-\beta + a + bT)^2 (a + bT)^2 \\
& + CK e^h a T \beta^2 (-\beta T + aT + bT^2)^2 (-\beta + a + bT)^2 (a + bT)^2 e^{-\beta T + aT + \frac{1}{2} bT^2} \\
& - CK e^h a T^2 \beta^2 (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T + aT + \frac{1}{2} bT^2} \\
& - CK e^h a^2 \beta T (-\beta T + aT + bT^2)^2 (-\beta + a + bT)^2 (a + bT)^2 e^{-\beta T + aT + \frac{1}{2} bT^2} \\
& + e^h a^2 \beta T^2 (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T + aT + \frac{1}{2} bT^2} \\
& + CK e^h a T^2 \beta^3 (-\beta T + aT + bT^2) (-\beta + a + bT)^2 (a + bT)^2 e^{-\beta T} \\
& + CK e^h a T^2 \beta^2 (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T} \\
& - 2CK e^h a^2 T^2 \beta^2 (-\beta T + aT + bT^2) (-\beta + a + bT)^2 (a + bT)^2 e^{-\beta T} \\
& - 2CK e^h a^2 T^2 \beta (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T} \\
& - CK e^h a b T^2 \beta^2 (-\beta T + aT + bT^2)^2 (a + bT)^2 (-\beta + a + bT) e^{-\beta T} \\
& - CK e^h a b^2 T^2 \beta (-\beta T + aT + bT^2)^2 (a + bT)^2 e^{-\beta T} \\
& + CK e^h a^3 T^2 \beta (-\beta T + aT + bT^2) (-\beta + a + bT)^2 (a + bT)^2 e^{-\beta T} \\
& + CK e^h a^3 T^2 (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T} \\
& + CK e^h a^2 b T^2 \beta (-\beta T + aT + bT^2)^2 (-\beta + a + bT) (a + bT)^2 e^{-\beta T} \\
& + CK e^h a^2 b^2 T^2 (-\beta T + aT + bT^2)^2 e^h a^2 b^2 T^2 (-\beta T + aT + bT^2)^2 \\
& + CK e^h a^2 \beta T^2 (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) \\
& - CK e^h a^3 T^2 (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T} \\
& - CK e^h a^2 b^2 T^2 (-\beta T + aT + bT^2)^2 (a + bT)^2 \\
& - iCK e^h a T^2 \beta^2 (-\beta + a) (-\beta T + aT + bT^2) (-\beta + a + bT)^2 (a + bT) e^{-\beta T} \\
& - iCK e^h T^2 a b \beta (-\beta + a) (-\beta T + aT + bT^2) (-\beta + a + bT)^2 e^{-\beta T} \\
& - iCK e^h T^2 a \beta (-\beta + a) (-\beta + a + bT)^2 (a + bT) (-\beta + a + 2bT) e^{-\beta T} \\
& - iCK e^h T^2 \beta (-\beta + a) (-\beta T + aT + bT^2) (-\beta + a + bT)^3 (a + bT)^2 e^{-\beta T + aT + \frac{1}{2} bT^2} \quad (13) \\
& + iCK e^h T^2 \beta (-\beta + a) (-\beta + a + bT)^2 (a + bT)^2 (-\beta + a + 2bT) e^{-\beta T + aT + \frac{1}{2} bT^2} = 0.
\end{aligned}$$

The value of T which we obtain, gives the minimum cost once it satisfies the following condition

$$\frac{d^2 TC(T)}{dT^2} > 0. \quad (14)$$

Equation (13) is highly nonlinear and therefore difficult to solve by any analytic method. Likewise the same problem will exist in trying to check the inequality in (14) above. However, in all our examples below, we use *direct search method* to obtain the root of the equation and also confirm that the sufficient condi-

tion (14) is satisfied.

4. Numerical Example

Example 1

To illustrate the model developed an example is considered based on the following values of parameters: $N_0 = \text{₦}5000$ per order, $K = 500$, $C = \text{₦}200$ per unit, $\beta = 0.02$, $a = 0.2$, $b = 0.01$, $i = 0.1$ per Naira per unit time, and $h = 2$. Substituting and simplify the above parameters into Equation (13), gives $T^* = 0.254794521$ (93 days). On substitution of this optimal value T^* in equations (11) and (4), we obtain the minimum total cost per unit time $TC^* = \text{₦}323947.1376$ and economic order quantity $I_0^* = 670.2162846$ units. Note that the T^* value satisfies $\frac{d^2TC(T)}{dT^2} > 0$.

5. Sensitivity Analysis

We now study the effect of changes in the values of the system parameters N_0 , K , β , a , b , C , i , and h on the optimal length of the cycle (T^*), the economic order quantity (I_0^*) and the minimum total cost per unit time (TC). The sensitivity analysis is performed by changing each of the parameters by 50%, 25%, -25%, -50%, and keeping the remaining parameters at their original values. Furthermore, to see the changes around the original values, we also carry the same sensitivity analysis by changing the parameters by 5%, 2%, -2%, -5%. This shows the condition at which T^* , TC^* and I_0^* change from their original values. The corresponding changes in the cycle time, total cost per unit and the economic order quantity are shown in **Table 2**.

Example 2

Applying the same values as in example 1, with h changed to 3, the solutions are $T^* = 0.156164384$ (57 days), $TC^* = \text{₦}838647.5434$ and $I_0^* = 1103.942979$ units

Example 3

Also using the same values as in example 1, with h changed to 4, the solutions, are $T^* = 0.095890411$ (35 days), $TC^* = \text{₦}2211152.042$ and $I_0^* = 1829.849334$ units.

Table 1 shows a summary of the results for the three examples above.

From the table, we see that as we increase the value of h , then TC^* and I_0^* increase while T^* decreases as it is expected. This is because as the demand increases the economic order quantity also increases, hence the total variable cost,

Table 1. Summary of the results of Examples 1, 2 & 3.

| h | T^* | TC^* | I_0^* |
|-----|-----------------------|-------------|-------------|
| 2 | 0.254794521 (93 days) | 323947.1376 | 670.2162846 |
| 3 | 0.156164384 (57 days) | 838647.5434 | 1103.942979 |
| 4 | 0.095890411 (35 days) | 2211152.042 | 1829.849334 |

TC also increases. On the other hand however the cycle period decreases as a result of higher demand.

6. Discussion of Results

Observing **Table 2** carefully, we can make the following deductions.

Table 2. Sensitivity Analysis on example 1 to see changes in the values of T^* , TC^* and I_0^* with changes in other parameters.

| Parameter | % change in parameter | T^* | TC^* | I_0^* |
|-----------|-----------------------|-----------------------------|-------------------|-------------------|
| <i>a</i> | 50 | 0.19452055 (71 days) | 250028.049 | 645.081859 |
| | 25 | 0.216438356 (79 days) | 278918.783 | 666.3989667 |
| | 5 | 0.243835616 (89 days) | 312959.059 | 669.963952 |
| | 2 | 0.24931507 (91 days) | 319397.937 | 667.974543 |
| | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.25753425 (94 days) | 328720.274 | 663.769862 |
| | -5 | 0.26575343 (97 days) | 336342.561 | 662.292235 |
| | -25 | 0.3890411 (142 days) | 408122.89 | 627.651103 |
| | -50 | 2.78082192 (1015 days) | 537835.34 | -1686.2455 |
| <i>b</i> | 50 | 0.27945206 (102 days) | 257382.088 | 580.006427 |
| | 25 | 0.26575343 (97 days) | 290665.441 | 624.875763 |
| | 5 | 0.25479452 (93 days) | 317291.386 | 655.778458 |
| | 2 | 0.25479452 (93 days) | 321284.462 | 664.438736 |
| | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.25479452 (93 days) | 326823.506 | 672.713076 |
| | -5 | 0.2520548 (92 days) | 330602.505 | 677.102485 |
| | -25 | 0.24383562 (89 days) | 357222.223 | 709.95673 |
| | -50 | 0.23287671 (85 days) | 390487.278 | 743.972984 |
| <i>c</i> | 50 | 0.20821918 (76 days) | 475097.937 | 544.758003 |
| | 25 | 0.22739726 (83 days) | 399739.474 | 596.254924 |
| | 5 | 0.24657534 (90 days) | 339147.519 | 647.979022 |
| | 2 | 0.2520548 (92 days) | 330029.853 | 662.799202 |
| | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.25753425 (94 days) | 317860.573 | 677.638033 |
| | -5 | 0.26027397 (95 days) | 308722.515 | 685.064449 |
| | -25 | 0.29041096 (106 days) | 247543.088 | 767.063993 |
| | -50 | 0.35616438 (130 days) | 170181.966 | 947.953168 |

Continued

| | | | | |
|---------|----------|-----------------------------|-------------------|-------------------|
| | 50 | 0.30958904 (113 days) | 332831.938 | 819.541618 |
| | 25 | 0.28219178 (103 days) | 328606.542 | 744.6442 |
| | 5 | 0.26027397 (95 days) | 324920.514 | 685.064449 |
| | 2 | 0.25753425 (94 days) | 324339.599 | 677.638033 |
| N_0 | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.2520548 (92 days) | 323550.9 | 662.799202 |
| | -5 | 0.24657534 (90 days) | 322949.356 | 647.979022 |
| | -25 | 0.21917808 (80 days) | 318676.391 | 574.157056 |
| | -50 | 0.18082192 (66 days) | 312448.222 | 471.583081 |
| | 50 | 0.20821918 (76 days) | 475097.937 | 817.137005 |
| | 25 | 0.22739726 (83 days) | 399739.474 | 745.318655 |
| | 5 | 0.24657534 (90 days) | 339147.519 | 680.377973 |
| | 2 | 0.2520548 (92 days) | 330029.853 | 676.055186 |
| K | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.25753425 (94 days) | 317860.573 | 664.085272 |
| | -5 | 0.26027397 (95 days) | 308722.515 | 650.811227 |
| | -25 | 0.29041096 (106 days) | 247543.088 | 575.297994 |
| | -50 | 0.35616438 (130 days) | 170181.966 | 473.976584 |
| | 50 | 0.2630137 (96 days) | 325042.242 | 655.254553 |
| | 25 | 0.25753425 (94 days) | 324679.539 | 660.160486 |
| | 5 | 0.25479452 (93 days) | 324119.819 | 666.869160 |
| | 2 | 0.25479452 (93 days) | 324017.657 | 668.883823 |
| β | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.2520548 (92 days) | 323874.621 | 0.2520548 |
| | -5 | 0.2520548 (92 days) | 323761.6 | 666.057181 |
| | -25 | 0.24931507 (91 days) | 322905.133 | 671.000678 |
| | -50 | 0.24657534 (90 days) | 321605.821 | 677.707789 |
| | 50 | 0.24657534 (90 days) | 581557.808 | 647.979022 |
| | 25 | 0.24931507 (91 days) | 452756.617 | 655.386782 |
| | 5 | 0.2520548 (92 days) | 349709.65 | 662.799202 |
| | 2 | 0.2520548 (92 days) | 334252.444 | 662.799202 |
| i | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.25479452 (93 days) | 313641.821 | 670.216285 |
| | -5 | 0.25479452 (93 days) | 298183.845 | 670.216285 |
| | -25 | 0.25753425 (94 days) | 195128.418 | 677.638033 |
| | -50 | 0.26027397 (95 days) | 66301.3478 | 685.064449 |
| | 50 | 0.15616438 (57 days) | 838647.543 | 1103.94298 |
| | 25 | 0.2 (73 days) | 519767.475 | 861.880901 |
| | 5 | 0.24109589 (88 days) | 355893.61 | 699.769324 |
| | 2 | 0.24931507 (91 days) | 336355.27 | 682.133624 |
| h | 0 | 0.25479452 (93 days) | 323947.138 | 670.216285 |
| | -2 | 0.25753425 (94 days) | 312010.258 | 651.067465 |
| | -5 | 0.26575343 (97 days) | 294949.053 | 633.32403 |
| | -25 | 0.32328767 (118 days) | 203299.537 | 519.89826 |
| | -50 | 0.4109589 (150 days) | 128661.259 | 404.9594 |

1) With increase in the value of the parameter a , the values of T^* , TC^* and I_0^* decreases. This is probably because when a increases, deterioration increases and so the model forces a reduction in T^* to reduce deterioration. This makes both TC^* and I_0^* to also reduce. The decreases in the values are low hence the decision variables are not very sensitive to changes in a .

2) With increase in the value of parameter b , the values of I_0^* and TC^* decrease while T^* increases. This is also probably because when b increases, deterioration increases depending upon the value of t and so a suitable T^* will be selected by the model which makes both TC^* and I_0^* reduce. The decreases/increase in the values are moderate hence the decision variables are moderately sensitive to changes in b .

3) With increase in the value of parameter C , the value of TC^* increases while T^* and I_0^* decrease. This is expected since when C increases the stockholding cost and deterioration cost increase. Thus TC^* increases as a result of which the model selects smaller values of T^* and I_0^* . The increase/decreases in the values are moderate hence the variables are moderately sensitive to changes in C .

4) With increase in the value of the parameter N_0 , the values of T^* , TC^* and I_0^* increase. This is also expected since when ordering cost increases then the model will avoid more orders and so both T^* and I_0^* increase. TC^* will however increase due to increase in stockholding cost. The increases in the values are moderate hence the decision variables T^* , TC^* and I_0^* are moderately sensitive to changes in N_0 .

5) With increase in the value of parameter K , the values of TC^* and I_0^* increase while T^* decreases. This is also expected because when K increases, the demand in that case will also increase which results in increase in the optimal total cost and economic order quantity and hence the cycle period will decrease due to higher demand. The increases/decrease in the values are high hence the decision variables are highly sensitive to changes in K .

6) With increase in the value of the parameter β , the values of T^* and TC^* increase while I_0^* decreases. This is probably because when β increases, the demand decreases making T^* to be longer. As a result of this, deterioration increases and this increases TC^* . The model then decreases I_0^* to avoid much deterioration. The increases/decrease in the values are low hence the decision variables are not very sensitive to changes in β .

7) With increase in the value of parameter i , the value of TC^* increases while T^* and I_0^* decrease. This is expected because when the inventory carrying charge, i is increased there will be more stockholding cost so the model will avoid that by increasing more orders, *i.e.* by reducing T^* which will eventually reduce I_0^* . TC^* will increase due to more ordering cost. The increase/decreases in the values are high hence the decision variables are highly sensitive to changes in i .

8) With increase in the value of parameter h , the values of TC^* and I_0^* increase while T^* decreases. This is because as the demand increases the economic

order quantity also increases, hence the total variable cost, TC^* also increases. On the other hand however the cycle period decreases as a result of higher demand. The increase/decrease in the values is high hence the decision variables are highly sensitive to changes in h .

7. Conclusion

In this paper, an inventory model is developed which determines the optimal order quantity of an on-hand inventory due to a generalised exponential decreasing demand rate. The deterioration rate is time varying linear function of time and the stockholding cost is a constant. The model has been solved analytically by minimizing the total inventory cost. A numerical example has been given to show the application of the model. Later, a sensitivity analysis is carried out to see the effect of changes in the parameter values. The analysis shows that T^* , TC^* and I_0^* are sensitive to changes in the parameters, N_0 , K , b , C , i and h . However T^* , TC^* and I_0^* are not very sensitive to changes in the parameters a and β . Moreover, it has been shown that the values T^* , TC^* and I_0^* all increase with increase in the parameter N_0 , but all of them decrease with increase in the parameter a .

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Appendix

$$\frac{dI(t)}{dt} + (a+bt)I(t) = -D(t), 0 \leq t \leq T \quad (1)$$

where $D(t) = Ke^{h-\beta t}$

Solving Equation (1) with boundary conditions $I(0) = I_0$ and $I(T) = 0$ we obtain the solution as follows:

$$\frac{dI(t)}{dt} + I(t)(a+bt) = -Ke^{h-\beta t}$$

The integrating factor, $IF = e^{\int(a+bt)dt} = e^{at+\frac{1}{2}bt^2}$ and so we get

$$\begin{aligned} I(t)e^{at+\frac{1}{2}bt^2} &= -K \int e^{h-\beta t} \cdot e^{at+\frac{1}{2}bt^2} dt \\ &= \frac{-K}{-\beta+a+bt} e^{h-\beta t+at+\frac{1}{2}bt^2} + C \\ \therefore I(t) &= \frac{-K}{-\beta+a+bt} e^{h-\beta t} + Ce^{-at-\frac{1}{2}bt^2} \end{aligned} \quad (2)$$