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A Parametric Approach to Non-Convex Optimal Control Problem

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Abstract

In this paper we have considered a non convex optimal control problem and presented the weak, strong and converse duality theorems. The optimality conditions and duality theorems for fractional generalized minimax programming problem are established. With a parametric approach, the functions are assumed to be pseudo-invex and v-invex.

Keywords

Non Convex Programming; Pseudo-Invex Functions; V-Invex Functions; Fractional Minimax Programming

1. Introduction

Parametric nonlinear programming problems are important in optimal control and design optimization problems. The objective functions are usually multi objective. The constraints are convex, concave or non convex in nature. In [1]-[3], the authors have established both theoretical and applied results involving such functions. Here we have considered a generalized non-convex programming problem where the objective and/or constraints are non-convex in nature. Under non-convexity assumption [4] on the functions involved, the weak, strong and converse duality theorems are proved. Mond and Hanson [5] [6] extended the Wolfe-duality results of mathematical programming to a class of functions subsequently called invex functions. Many results in mathematical programming previously established for convex functions also hold for invex functions. Jeyakumar and Mond [7] introduced v-invex functions and established the sufficient optimality criteria and duality results in multi objective problem [8] in the static case. In [9] under v-invexity assumptions and continuity, the sufficient optimality and duality results for a class of multi objective variational problems are established. Here we extend some of these results to generalized minimax fractional programming problems. The parametric approach is also used in

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[10] by Baotic *et al*.

2. Preliminaries

Consider the real scalar function $f(t,x,u)$, where $t \in (t_0, t_f)$, $x \in R^n$ and $u \in R^n$. Here t is the independent variable, $u(t)$ is the control variable and $x(t)$ is the state variable. u is related to x by the state equations $G(t,x,u) = \dot{x}$, Where $\cdot$ denotes the derivative with respect to t.

If $x = (x^1, x^2, \cdots, x^n)^T$, the gradient vector f with respect to x is denoted by

$f_x = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \cdots, \frac{\partial f}{\partial x^n}\right)^T$ where T denotes the transpose of a matrix.

For a r-dimensional vector function、the gradient with respect to x is

$$R_x = \begin{pmatrix} \frac{\partial R^1}{\partial x^1} & \cdots & \frac{\partial R^r}{\partial x^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial R^1}{\partial x^n} & \cdots & \frac{\partial R^r}{\partial x^n} \end{pmatrix}.$$

Gradient with respect to u is defined similarly. It is assumed that f, G and R have continuous second derivatives with the arguments. The control problem is to transfer the state variable from an initial state x_0 at t_0 to a final state x_f at t_f so as to optimize (maximize or minimize) a given functional subject to constraints on the control and state variables.

Definition 1. A vector function $F = (F_1, F_2, \cdots, F_n)$ is said to be v-invex [8] if there exist differentiable vector functions $\eta(t,x,\bar{x}): I \times X_0 \times X_0 \to R^n$ with $\eta(t,x,\bar{x}) = 0$ such that for each $x, \bar{x} \in X_0$ and to $i = 1, 2, \cdots, p$,

$$F_i(x) - F_i(\bar{x}) \geq \int_{t_0}^{t_f} \left[f_{ix}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \eta\left(t, x(t), \bar{x}(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t} \eta_i\left(t, x(t), \bar{x}(t)\right) f_{i\bar{x}}\left(t, x(t), \bar{x}(t)\right) \right] \mathrm{d}t$$

Definition 2. We define the vector function $F = (F_1, F_2, \cdots, F_n)$ to be v-pseudo invex if there exist functions $\eta: I \times X_0 \times X_0 \to R^p$ with $\eta = 0$ for each $x, \bar{x} \in X_0$ [4] [9] [11] [12].

Definition 3. Let S be a non-empty subset of a normed linear space X. The positive dual or positive conjugate core of S (denoted S^+) is defined by $S^+ = \{x^+ \in X^+ : x^+(x) \geq 0, \forall x \in X\}$ (where X^+ denotes the space of all continuous linear functionals on X, and $x^+(x) = (x^+, x)$) is the value of the functional x^+ at x.

3. The Optimal Control Problem

Problem P (Primal):

Minimize $F_i(x) = \int_{t_0}^{t_f} f_i\left(t, x(t), u(t)\right) \mathrm{d}t$

subject to

$$x(t_0) = x_0,\ x(t_f) = x_f \quad (1)$$

$$G(t,x,u) = \dot{x} \quad (2)$$

$$R(t,x,u) \geq 0 \quad (3)$$

The corresponding dual problem is given by:

Problem D (Dual):

Maximize $F_i(x)-\int_{t_0}^{t_f}\left[\lambda(t)^{\mathrm{T}}[G-\dot{x}]-\mu(t)^{\mathrm{T}}R\right]\mathrm{d}t$

subject to

$$x(t_0)=x_0,\ x(t_f)=x_f,\ \ f_{ix}-G_x\lambda(t)-R_x\mu(t)=\dot{\lambda}(t),\ \ f_{iu}-G_u\lambda(t)-R_u\mu(t)=0,\ \mu(t)\geq 0$$

where $\lambda:[t_0,t_f]\to R^n$ and e $\mu:[t_0,t_f]\to R^r$

$x(t)$ and $u(t)$ are required to be piecewise smooth functions on $[t_0,t_f]$, their derivatives are continuous except perhaps at points of discontinuity of $u(t)$, which has piecewise continuous first and second derivatives. [13] [14].

4. Previous Results

Theorem 1: (Weak Duality)

If $\int_{t_0}^{t_f}\left[f_i-\lambda^{\mathrm{T}}(G-\dot{x})-\mu^{\mathrm{T}}R\right]\mathrm{d}t$, for any $\lambda\in R^n$ and $\mu\in R^r$ with $\mu(t)\geq 0$, is pseudo invex with respect to η then $\inf(\mathrm{P})\geq\mathrm{Sup}(\mathrm{D})$ [3] [6] [9] [11].

Theorem 2: (Strong Duality)

Under the pseudo invexity condition of theorem 1, if (x^*,u^*) is an optimal solution of (P) then there exist $\lambda(t)$ and $\mu(t)$ such that (x^*,u^*,λ,μ) is optimal for (D) and corresponding objective values are equal. [1] [2] [5] [6].

Theorem 3: (Converse duality)

If $(x^*,u^*,\lambda^*,\mu^*)$ is optimal for (D), and if $\begin{pmatrix} f_{ixx}-(G_x\lambda)_x-(R_x\mu)_x & f_{iux}-(G_x\lambda)_u-(R_x\mu)_u \\ f_{ixu}-(G_u\lambda)_x-(R_u\mu)_x & f_{iuu}-(G_u\lambda)_u-(R_u\mu)_u \end{pmatrix}$ is non-singular for all $t\in[t_0,t_f]$ then (x^*,u^*) is optimal for (P), and the corresponding objective values are equal [1] [2] [5] [6].

Sufficiency:

It can be shown that, pseudo-convex functions together with positive dual conditions are sufficient for optimality [11] [12].

5. Main Result

Optimality conditions and duality for generalized fractional minimax programming problem:

We consider the following generalized fractional minimax programming problem:

$(GP)\,\lambda^*(t)=\min_{x\in X}\max_{1<i\leq s}\int_{t_0}^{t_f}\frac{f_i(t,x,u)}{h_i(t,x,u)}\mathrm{d}t$, $h_i=G_i-\dot{x}$, where

1) $X=\{x\in R^n, u\in R^n, R_j(t,x,u)\leq 0, j=1,2,\cdots,m\}$ is non empty and complete set in R^n.

2) $f_i, h_i, i=1,2,\cdots,s,\ R_j,\ j=1,2,\cdots,m$ be differentiable functions.

3) $h_i(t,x,u)>0, i=1,2,\cdots,s$.

4) If h_i is not affine then $f_i\geq 0$ for all $i=1,2,\cdots,s$ and $x\in X$.

Consider the following minimax nonlinear parametric programming problem.

$$(P_\lambda)\,\phi(\lambda)=\min_{x\in X}\max_{1<i\leq s}\int_{t_0}^{t_f}\left[f_i(t,x,u)-\lambda^*(t)h_i(t,x,u)\right]\mathrm{d}t.$$

Lemma 1: If (GP) has an optimal solution (x^*,u^*) with an optimal value of (GP)-objective function as λ^*, then $\phi(\lambda^*)=0$. Conversely, if $\phi(\lambda^*)=0$ at t_0 and t_f, then (GP) and (P_λ^*) have some optimal solution.

Lemma 2: In relation to P_λ we have an equivalent programming problem for given $\lambda(t)$

EP_λ Minimize $\int_{t_0}^{t_f} f(t,x,u)\mathrm{d}t$

subject to $\left[f_i(t,x,u)-\lambda^*(t)h_i(t,x,u)\right]\le\dot{\lambda}(t)$, $R_j(t,x,u)\le 0$.

Lemma 3: If (t,x,u,λ) is (EP_λ)-feasible, then (t,x,u) is (*GP*)-feasible. If (t,x,u) is (*GP*)-feasible then there exist $\lambda(t)$ and $\dot{\lambda}(t)$ such that (t,x,u) is (EP_λ)-feasible.

Lemma 4: (t,x^*,u^*) is (GP)-optimal with corresponding optimal value of the (GP)-objective equal to λ^* if and only if $(t,x,x^*,\lambda^*,\dot{\lambda}^*)$ is (EP_λ)-optimal with corresponding optimal value of the (EP_λ)-objective equal to zero *i.e.* $\dot{\lambda}^*(t)=0$.

Theorem 4: (Necessary conditions)

Let (t,x^*,u^*) be an optimal solution of (GP) with an optimal value of (GP)-objective equal to λ^*. Let the conditions of lemma 1 be satisfied *i.e.* (x^*,u^*) be a feasible solution for P and $B(x^*,u^*)$ be the set of binding constraints. *i.e.* $j\in R(x^*,u^*)$ if and only if $R_j(t,x^*,u^*)=0$

Then $R_{jx}<0$ for

$$j\in B(x^*,u^*) \tag{4}$$

and $R_{ju}<\infty$ for

$$j\in B(x^*,u^*) \tag{5}$$

Hence from (4) and (5) $x\in F$

Then there exist $\lambda^*\in R^n$, $\mu^*\in R^n$, $\xi^*\in R^s$, $t\in[t_0,t_f]$ such that $(t,x^*,u^*,\lambda^*,\xi^*\mu^*)$ satisfy

$$\int_{t_0}^{t_f}\left[\xi^*\left[f_i(t,x^*,u^*)-\lambda^* h_i(t,x^*,u^*)\right]-(\mu^*)^{\mathrm{T}} R_j(t,x^*,u^*)\right]\mathrm{d}t\ge 0$$

$$\left.\begin{array}{l}\xi^* f_i(t,x^*,u^*)-\lambda^* h_i(t,x^*,u^*)=0\\ G(t,x^*,u^*)=0\\ R(t,x^*,u^*)\ge 0\\ \mu^*(t)\ge 0\\ \xi^*\ge 0,\sum_{j=1}^{m}\xi_j^*=1,\lambda^*(t)=0\end{array}\right\} \tag{6}$$

Theorem 5: (Sufficient conditions)

For some $\xi\in R^s$, $\mu\in R^m$, $\lambda^*\in R^r$, let $\xi^{\mathrm{T}}\left[f(\bullet)-\lambda h(\bullet)\right]+\mu^{\mathrm{T}}R(\bullet)$ be proper v-pseudo invex. At $x^*\in R^n$ and $u^*\in R^m$ let $\xi^{\mathrm{T}}\left[f(t,x^*,u^*)-\lambda h(t,x^*,u^*)\right]+\mu^{\mathrm{T}}(t)R(t,x^*,u^*)$ be finite and conditions (6) be satisfied. Then (x^*,u^*) is an optimal solution for (GP) with corresponding value of the objective function λ^*.

Two duals (GP) are introduced Wolfe-type dual.

$$(\mathrm{D}_1)\quad \text{Max}\quad \int_{t_0}^{t_f}\left[\xi^{\mathrm{T}}\left[f(t,x,u)-\lambda h(t,x,u)\right]+\mu^{\mathrm{T}}(t)R(t,x,u)\right]\mathrm{d}t$$

subject to

$$\xi^T\left[f_x(t,x,u)-\lambda h_x(t,x,u)\right]-R_x(t,x,u)\mu^{\mathrm{T}}(t)\geq 0,\ \xi^{\mathrm{T}}\left[f_u(t,x,u)-\lambda h_u(t,x,u)\right]-R_u(t,x,u)\mu^{\mathrm{T}}(t)=0$$

$$\xi\in R^s,\ \xi(t)\geq 0,\ \sum_{i=1}^{s}\xi_i=1,\ \mu(t)\in R^m,\ \mu(t)\geq 0,\ (t,x,u)\in R^n,\ \lambda\in R$$

Weir and Mond type dual.

$$(\mathrm{D}_2)\ \ \mathrm{Max}\ \int_{t_0}^{t_f}\left[\xi^{\mathrm{T}}f(t,x,u)-\lambda(t)h(t,x,u)\right]\mathrm{d}t$$

subject to

$$\xi^{\mathrm{T}}\left[f_x(w)-\lambda h_x(w)\right]-\mu^{\mathrm{T}}R_x(w)\geq 0,\ \mu^{\mathrm{T}}R_x\geq 0;\ \xi^{\mathrm{T}}\left[f_u(w)-\lambda h_u(w)\right]-\mu^{\mathrm{T}}R_u(w)\geq 0,\ \mu^{\mathrm{T}}R_u\geq 0$$

$$\xi\in R^s,\ \xi(t)\geq 0,\ \sum_{i=1}^{s}\xi_i=1,\ (t,x,u)\in w\in R^n,\ \mu^{\mathrm{T}}\in R^m,\ \mu(t)\geq 0,\ \lambda\in R$$

Proof of the corresponding duality results for the above two duals follow the same lines as the proofs of the theorems 2, 3, 4.

7. Conclusion

Here in this presentation we have considered a non convex optimal control problem in parametric form and established the weak duality theorem, the strong duality theorem and the converse duality theorem. The results which are available in literature for v-invex functions are hereby extended to v-pseudo invex functions in a minimax fractional non convex optimal control problem.

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