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A Different Approach to Cone-Convex Optimization

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ABSTRACT

In classical convex optimization theory, the Karush-Kuhn-Tucker (KKT) optimality conditions are necessary and sufficient for optimality if the objective as well as the constraint functions involved is convex. Recently, Lassere [1] considered a scalar programming problem and showed that if the convexity of the constraint functions is replaced by the convexity of the feasible set, this crucial feature of convex programming can still be preserved. In this paper, we generalize his results by making them applicable to vector optimization problems (VOP) over cones. We consider the minimization of a cone-convex function over a convex feasible set described by cone constraints that are not necessarily cone-convex. We show that if a Slater-type cone constraint qualification holds, then every weak minimizer of (VOP) is a KKT point and conversely every KKT point is a weak minimizer. Further a Mond-Weir type dual is formulated in the modified situation and various duality results are established.

Keywords: Convex Optimization; Cone-Convex Functions; KKT Conditions; Duality

1. Introduction

Convex programming deals with the minimization of a convex objective function over a convex set usually described by convex constraint functions. In the past various attempts have been made to weaken the convexity hypothesis [2-4] by replacing convex objective as well as constraint functions with more general ones and thus exploring the extent of optimality conditions applicability.

As a breakthrough to this, Lassere [1] showed that as far as KKT optimality conditions are concerned, the convexity (or any of its generalization) of the constraint functions can be replaced by the convexity of the feasible set described by the constraints. More precisely, Lassere considered the following convex optimization problem (CP):

$$\text{(CP) minimize} \quad f(x)$$

subject to

$$g_j(x) \le 0, \ \ j = 1, \cdots, m$$

where $f : R^n \to R$ is a differentiable convex function and the feasible set

$$F_0 = \left\{ x \in R^n : g_j(x) \le 0, \ j = 1, \cdots, m \right\}$$

is a convex set while the $g_j's: \ R^n \to R$ are differentiable but not necessarily convex functions. To prove the necessity and sufficiency of KKT conditions in this framework Lassere considered the following non-degeneracy condition (ND_1): For all $j = 1, \cdots, m$,

$$\nabla g_j(x) \ne 0 \text{, whenever } x \in F_0 \text{ and } g_j(x) = 0 \quad (\text{ND}_1)$$

He showed that if the Slater constraint qualification[1] and the above non-degeneracy condition (ND_1) hold, then a feasible point x^* of (CP) is a global minimizer if and only if it is a KKT point, that is,

$$\nabla f(x^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) = 0,$$

and

$$\lambda_j g_j(x^*) = 0, \ j = 1, \cdots, m \quad (\text{KKT}_1)$$

for some non-negative vector $\lambda = (\lambda_1, \cdots, \lambda_m) \in R^m$.

This work of Lassere [1] has been carried forward to the non-smooth case by Dutta and Lalitha [5]. They considered the same problem (CP) with the only difference being that the function *f* is a non-differentiable convex function and the convex set F_0 is described by local

[1]The Slater constraint qualification is said to hold for the problem (CP) if there exists $\hat{x} \in R^n$ such that $g_j(\hat{x}) < 0$ for all $j = 1, \cdots, m$.

Lipschitz constraint functions g_j which are not necessarily differentiable or convex. In terms of Dutta and Laltha [5] a point $x^* \in F_0$ is said to be a KKT point for the problem (CP) if there exist scalars $\lambda_j \geq 0, j = 1, \cdots, m$, such that

$$0 \in \partial f\left(x^*\right) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j\left(x^*\right),$$

and

$$\lambda_j g_j\left(x^*\right) = 0, \quad j = 1, \cdots, m \quad (\mathrm{KKT}_2)$$

where

$$\partial f\left(x^*\right) = \left\{\xi \in R^n : f\left(y\right) - f\left(x^*\right) \geq \left\langle \xi, y - x^* \right\rangle, \forall y \in R^n\right\}$$

denotes the sub-differential of f at x^* and

$$\partial^0 g_j\left(x^*\right) = \left\{\xi \in R^n : g_j^0\left(x, d\right) \geq \left\langle \xi, d \right\rangle, \forall d \in R^n\right\}$$

denotes the Clarke sub-differential of the function g_j at x^*.

Further, Dutta and Lalitha [5] introduced the following non-smooth version (ND_2) of Lassere's non-degeneracy condition:

For all $j = 1, \cdots, m$

$$0 \notin \partial^0 g_j\left(x\right), \text{ whenever } x \in F_0 \text{ and } g_j\left(x\right) = 0 \quad (\mathrm{ND}_2)$$

In this modified setting Dutta and Lalitha [5] concluded that if each g_j is assumed to be regular in the sense of Clarke [6] and if the Slater constraint qualification and the non-degeneracy condition (ND_2) hold, then a feasible point x^* is a global minimizer of *f* over F_0 if and only if it is a KKT point.

The overall aim of this paper is to extend Lassere's [1] results to a vector optimization problem over cones.

2. Preliminaries and Problem Formulation

We consider the following vector optimization problem (VOP) over cones:

$$(\mathrm{VOP})\ K - \text{minimize} \quad f\left(x\right)$$

subject to $g\left(x\right) \in Q$

where $f : R^n \to R^p$ and $g : R^n \to R^m$ are differentiable functions, K and Q are closed convex cones with non-empty interiors in R^p and R^m respectively.

Let $F = \left\{x \in R^n : -g\left(x\right) \in Q\right\}$ be the set of feasible solutions of (VOP).

The positive dual cone K^* and the strict positive dual cone K^{s*} of K are respectively defined as

$$K^* = \left\{z \in R^p : x^t z \geq 0 \text{ for all } x \in K\right\}$$

and

$$K^{s*} = \left\{z \in R^p : x^t z > 0 \text{ for all } x \in K \setminus \{0\}\right\}.$$

We begin by defining the notion of a KKT point in terms of (VOP).

Definition 2.1: A point $x^* \in F$ is said to be a KKT-point if there exist $\lambda \in K^* \setminus \{0\}$ and $\mu \in Q^*$ such that

$$\lambda^t \nabla f\left(x^*\right) + \mu^t \nabla g\left(x^*\right) = 0 \text{ and } \mu^t g\left(x^*\right) = 0.$$

For the problem (VOP), the solutions are defined in the following sense:

Definition 2.2 [7]: A point $x^* \in F$ is called

1) a weak minimum of (VOP) if for all $x \in F$

$$f\left(x^*\right) - f\left(x\right) \notin \operatorname{int} K;$$

2) a Pareto-minimum of (VOP) if for all $x \in F$

$$f\left(x^*\right) - f\left(x\right) \notin K \setminus \{0\};$$

3) a Strong minimum of (VOP) if for all $x \in F$

$$f\left(x\right) - f\left(x^*\right) \in K.$$

Let F^w denote the set of weak minimum solutions of (VOP).

The forthcoming optimality and duality results are based on suitable generalized convexity assumptions over cones, thus we recall some known definitions in the literature.

Definition 2.3 [8,9]: A function $f : R^n \to R^p$ is said to be

1) K-convex at a point $x^* \in R^n$ if for every $x \in R^n$

$$f\left(x\right) - f\left(x^*\right) - \nabla f\left(x^*\right)\left(x - x^*\right) \in K.$$

2) K-pseudoconvex at $x^* \in R^n$ if for every $x \in R^n$

$$-\nabla f\left(x^*\right)\left(x - x^*\right) \notin \operatorname{int} K \Rightarrow f\left(x^*\right) - f\left(x\right) \notin \operatorname{int} K.$$

3) strongly *K*-pseudoconvex at $x^* \in R^n$ if for every $x \in R^n$

$$-\nabla f\left(x^*\right)\left(x - x^*\right) \notin \operatorname{int} K \Rightarrow f\left(x\right) - f\left(x^*\right) \in K.$$

4) strictly K-pseudoconvex at $x^* \in R^n$ if for every $x \in R^n$

$$-\nabla f\left(x^*\right)\left(x - x^*\right) \notin \operatorname{int} K \Rightarrow f\left(x^*\right) - f\left(x\right) \notin K.$$

If *f* is *K*-convex (*K*-pseudoconvex, strongly *K*-pseudoconvex, strictly *K*-pseudoconvex) at every $x^* \in R^n$ then *f* is said to be *K*-convex (*K*-pseudoconvex, strongly *K*-pseudoconvex, strictly *K*-pseudoconvex) on R^n.

On the lines of Jahn [10] we define Slater-type cone constraint qualification as follows:

Definition 2.4: The problem (VOP) is said to satisfy Slater-type cone constraint qualification at $x^* \in F$ if there exists $\hat{x} \in R^n$ such that

$$g\left(x^*\right) + \nabla g\left(x^*\right)\left(\hat{x} - x^*\right) \in -\operatorname{int} Q.$$

Note that if g is Q-convex at x^* and the problem (VOP) satisfies Slater constraint qualification, that is, there exists $\hat{x} \in R^n$ such that $-g(\hat{x}) \in \operatorname{int} Q$, then (VOP) satisfies Slater-type cone constraint qualification at x^*.

Also, it is worth noticing that following the steps of Lassere [1] and Dutta and Lalitha [5] we can define the analogous non-degeneracy condition (ND_3) for (VOP) as follows:

For all $\mu \in Q^* \setminus \{0\}$, $\mu^t \nabla g(x) \neq 0$, whenever $x \in F$ and $\mu^t g(x) = 0$.

But if we assume that Slater-type cone constraint qualification holds at a point $x^* \in F$, then there exists $\hat{x} \in R^n$ such that

$$g(x^*) + \nabla g(x^*)(\hat{x} - x^*) \in -\operatorname{int} Q$$

which means that for all $\mu \in Q^* \setminus \{0\}$ for which $\mu^t g(x^*) = 0$, we have $\mu^t \nabla g(x^*)(\hat{x} - x^*) < 0$ which itself implies that $\mu^t \nabla g(x^*) \neq 0$ and hence the non-degeneracy condition holds.

Thus in the paper, we shall extend Lassere's [1] results to the vector optimization problem (VOP) over cones but, unlike Lassere, to prove our results we need to assume only Slater-type cone constraint qualification at a point.

3. Optimality Conditions

In this section we prove several classical optimality results by taking generalized convexity assumptions over cones on the objective function and assuming the feasible set to be convex and with no convexity type restriction on the constraint function. It is clear that if the constraint function g in (VOP) is *Q*-convex then the feasible set F is convex, so we begin by exemplifying the fact that F can be convex without g being *Q*-convex.

Example 3.1: Consider $g: R \to R^2$ defined as

$$g(x) = (x,\ 6 - x^2)$$

and

$$Q = \{(x, y) : y \geq x, x \leq 0\}.$$

Here g is not *Q*-convex, because if we take $x = 5/2$ and $x^* = 3$ then

$$g(x) - g(x^*) - \nabla g(x^*)(x - x^*) = \left(0, -\frac{1}{4}\right) \notin Q.$$

But the feasible set $F = \{x \in R : x \geq 2\}$ is convex. We have the following lemma.

Lemma 3.1: If the feasible set *F* of (VOP) is convex then

$$\left(\mu^t \nabla g(x^*)\right)(x - x^*) \leq 0, \text{for all } x \in F \tag{1}$$

where $x^* \in F, \mu \in Q^* \setminus \{0\}$ satisfy $\mu^t g(x^*) = 0$.

Proof: Let *F* be convex and suppose

$$x^* \in F, \mu \in Q^* \setminus \{0\},$$

satisfy $\mu^t g(x^*) = 0$.

Assume that

$$\left(\mu^t \nabla g(x^*)\right)(x - x^*) > 0 \text{ for some } x \in F. \tag{2}$$

Now, for $0 < \alpha < 1$, we have

$$\begin{aligned} &g\left(x^* + \alpha(x - x^*)\right) \\ &= g(x^*) + \nabla g(x^*)\alpha(x - x^*) + o(\alpha) \end{aligned}$$

where

$$\lim_{\alpha \to 0^+} o(\alpha) = 0.$$

This implies that.

$$\begin{aligned} &\mu^t g\left(x^* + \alpha(x - x^*)\right) \\ &= \mu^t g(x^*) + \alpha\left(\mu^t \nabla g(x^*)\right)(x - x^*) + \mu^t o(\alpha) \end{aligned}$$

Using (2) together with $\mu^t g(x^*) = 0$ for α sufficiently small, $\alpha > 0$, we get

$$\mu^t g\left(x^* + \alpha(x - x^*)\right) > 0. \tag{3}$$

Since *F* is convex, therefore $x^* + \alpha(x - x^*) \in F$, that is,

$$-g\left(x^* + \alpha(x - x^*)\right) \in Q,$$

so that

$$\mu^t g\left(x^* + \alpha(x - x^*)\right) \leq 0.$$

This contradicts (3). Hence the result.

The above lemma plays a pivotal role throughout the rest of the paper, thus we illustrate it by means of an example.

Example 3.2: Consider $g: R \to R^2$ and *Q* as defined in Example 3.1. Then we have already seen that g is not *Q*-convex whereas the feasible set *F* is convex.

Now, if we take $x^* = 2 \in F$, then $\mu^t g(x^*) = 0$ if and only if $\mu = (-\alpha, \alpha), \alpha \geq 0$, and for this choice of μ,

$$\left(\mu^t \nabla g(x^*)\right)(x - x^*) = -5\alpha(x - 2) \leq 0 \text{ for all } x \in F$$

Also, for any other $x^* > 2$, there does not exist any $\mu \in Q^* \setminus \{0\}$ for which $\mu^t g(x^*) = 0$.

Hence the lemma holds.

The following theorem serves the main purpose of the paper.

Theorem 3.1: Consider a feasible solution x^* of the vector optimization problem (VOP) and assume that Slater-type cone constraint qualification holds at x^*. If *f* is *K*-convex at x^* and the feasible set *F* is convex then x^* is a weak minimum of (VOP) if and only if it is a KKT-point.

Proof: Let $x^* \in F$ be a weak minimum of (VOP). By Lemma 1 [11], there exist $\lambda \in K^*$ and $\mu \in Q^*$ not both zero such that

$$\left(\lambda^t \nabla f\left(x^*\right)+\mu^t \nabla g\left(x^*\right)\right)\left(x-x^*\right) \geq 0, \forall x \in R^n \tag{4}$$

and

$$\mu^t g\left(x^*\right)=0 . \tag{5}$$

If possible, let $\lambda = 0$, then $\mu \neq 0$ so that from (4), we get

$$\left(\mu^t \nabla g\left(x^*\right)\right)\left(x-x^*\right) \geq 0, \forall x \in R^n . \tag{6}$$

Since Slater-type cone constraint qualification holds at x^*, there exists $\hat{x} \in R^n$ such that

$$g\left(x^*\right)+\nabla g\left(x^*\right)\left(\hat{x}-x^*\right) \in -\operatorname{int} Q ,$$

which gives that

$$\mu^t\left\{g\left(x^*\right)+\nabla g\left(x^*\right)\left(\hat{x}-x^*\right)\right\}<0 .$$

This together with (5) implies

$$\left(\mu^t \nabla g\left(x^*\right)\right)\left(\hat{x}-x^*\right)<0 ,$$

which contradicts (6). Therefore $\lambda \neq 0$.

Since the inequality (4) holds for every $x \in R^n$, we conclude that

$$\lambda^t \nabla f\left(x^*\right)+\mu^t \nabla g\left(x^*\right)=0 \tag{7}$$

and

$$\mu^t g\left(x^*\right)=0 . \tag{8}$$

Hence x^* is a KKT-point.

Conversely, let $x^* \in F$ be a KKT-point, that is, there exist $\lambda \in K^* \setminus \{0\}$ and $\mu \in Q^*$ such that (7) and (8) hold.

Suppose x^* is not a weak minimum of (VOP), so there exists $\hat{x} \in F$ such that

$$f\left(x^*\right)-f(\hat{x}) \in \operatorname{int} K . \tag{9}$$

Since *f* is *K*-convex at x^*,

$$f(\hat{x})-f\left(x^*\right)-\nabla f\left(x^*\right)\left(\hat{x}-x^*\right) \in K . \tag{10}$$

By (9) and (10),

$$-\nabla f\left(x^*\right)\left(\hat{x}-x^*\right) \in \operatorname{int} K ,$$

which implies

$$\left(\lambda^t \nabla f\left(x^*\right)\right)\left(\hat{x}-x^*\right)<0 .$$

This, by (7), gives

$$\left(\mu^t \nabla g\left(x^*\right)\right)\left(\hat{x}-x^*\right)>0 .$$

But this contradicts Lemma 3.1 as $\mu^t g\left(x^*\right)=0$.

Hence x^* is a weak minimum for (VOP).

Theorem 3.2: Let f be *K*-pseudoconvex at $x^* \in F$ and suppose that F is convex. Further assume that Slater-type cone constraint qualification holds at x^*. Then x^* is a weak minimum of (VOP) if and only if it is a KKT-point.

Proof: Proof follows on similar lines as Theorem 3.1.

Now we obtain sufficient optimality conditions for (VOP).

Theorem 3.3: Let *f* be *K*-convex at $x^* \in F$ and the feasible set F be convex and suppose that there exist $\lambda \in K^{s*}$ and $\mu \in Q^*$ such that (7) and (8) hold. Then x^* is a Pareto minimum of (VOP).

Proof: Let if possible, x^* be not a Pareto minimum of (VOP). Then there exists $\hat{x} \in F$ such that

$$f\left(x^*\right)-f(\hat{x}) \in K \setminus \{0\} . \tag{11}$$

Since f is *K*-convex at x^*, we have

$$f(\hat{x})-f\left(x^*\right)-\nabla f\left(x^*\right)\left(\hat{x}-x^*\right) \in .$$

Using (11), we get

$$-\nabla f\left(x^*\right)\left(\hat{x}-x^*\right) \in K \setminus \{0\} K .$$

Since $\lambda \in K^{s*}$, we have

$$\left(\lambda^t \nabla f\left(x^*\right)\right)\left(\hat{x}-x^*\right)<0 .$$

Now proceeding as in the converse part of Theorem 3.1, we get a contradiction to Lemma 3.1. Hence x^* is a Pareto minimum of (VOP).

We now give an example to illustrate Theorem 3.3.

Example 3.3: Consider the problem

$$\text{(VOP) K-Minimize} \quad f(x)$$

Subject to $-g(x) \in Q$

where $g: R \rightarrow R^2$ and *Q* are as defined in Example 3.1 and $f: R \rightarrow R^2$ and K are given by

$$f(x)=\left(x^2+2x-3,-x-3\right), K=\{(x, y):-x \leq y \leq x\} .$$

Then, as shown in Example 3.1, *g* is not *Q*-convex. while the feasible set $F=\{x \in R: x \geq 2\}$ of (VOP) is convex. Also *f* is *K*-convex at $x^*=2$.

It can be seen that for

$$\lambda=\left(1, \frac{3}{4}\right) \in K^{s*} \text{ and } \mu=\left(\frac{-21}{20}, \frac{21}{20}\right) \in Q^* ,$$

$$\lambda^t \nabla f\left(x^*\right)+\mu^t \nabla g\left(x^*\right)=0, \text{ and } \mu^t g\left(x^*\right)=0 .$$

Thus by Theorem 3.3, $x^*=2$ is a Pareto minimum of (VOP).

Remark 3.1: Example 3.3 describes a vector optimization problem in which a Pareto minimum is obtained by applying Theorem 3.3 whereas it is impossible to do so using Lassere's [1] results.

Theorem 3.4: Let *f* be strictly *K*-pseudoconvex at $x^* \in F$ and the feasible set *F* be convex and suppose that there exist $\lambda \in K^* \setminus \{0\}$ and $\mu \in Q^*$ such that (7) and (8) hold. Then x^* is a Pareto minimum of (VOP).

Proof: Let if possible, x^* be not a Pareto minimum of (VOP).

Then there exists $\hat{x} \in F$ such that

$$f(x^*) - f(\hat{x}) \in K \setminus \{0\}.$$

Since f is strictly *K*-pseudoconvex at x^*, we get

$$-\nabla f(x^*)(\hat{x} - x^*) \in \operatorname{int} K.$$

As $\lambda \in K^* \setminus \{0\}$, we have

$$(\lambda^t \nabla f(x^*))(\hat{x} - x^*) < 0.$$

Now proceeding as in the converse part of Theorem 3.1, we get a contradiction to Lemma 3.1. Hence x^* is a Pareto minimum of (VOP).

Theorem 3.5: Let f be strongly K-pseudoconvex at $x^* \in F$ and the feasible set F be convex and suppose that there exist $\lambda \in K^* \setminus \{0\}$ and $\mu \in Q^*$ such that (7) and (8) hold. Then x^* is a strong minimum of (VOP).

Proof: Let if possible, x^* be not a strong minimum of (VOP).

Then there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(x^*) \notin K.$$

Since f is strongly *K*-pseudoconvex at x^*, we get

$$-\nabla f(x^*)(\hat{x} - x^*) \in \operatorname{int} K.$$

As $\lambda \in K^* \setminus \{0\}$, we have

$$(\lambda^t \nabla f(x^*))(\hat{x} - x^*) < 0.$$

Again proceeding as in the converse part of Theorem 3.1, we get a contradiction. Hence x^* is a strong minimum of (VOP).

4. Duality

With the primal problem (VOP), we associate the following Mond-Weir type dual program (MDP):

$$\text{(MDP) K-maximize} \quad f(y)$$

subject to

$$(\lambda^t \nabla f(y) + \mu^t \nabla g(y))(x - y) \geq 0, \forall x \in F \tag{12}$$

$$\mu^t g(y) = 0, \tag{13}$$

$$y \in F, \lambda \in K^* \setminus \{0\}, \mu \in Q^*.$$

Let F^D denote the set of feasible solutions of (MDP).

Definition 4.1: A point $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in F^D$ is said to be a weak maximum of (MDP) if

$$f(y) - f(\bar{y}) \notin \operatorname{int} K, \text{for all } (y, \lambda, \mu) \in F^D.$$

Let F_W^D denote the set of weak maximum solutions of (MDP).

Theorem 4.1: (Weak Duality) Let $x \in F$ and $(y, \lambda, \mu) \in F^D$. Assume that f is K-pseudoconvex at y and the feasible set F is convex, then

$$f(y) - f(x) \notin \operatorname{int} K.$$

Proof: Let $x \in F$ and $(y, \lambda, \mu) \in F^D$. Suppose to the contrary that

$$f(y) - f(x) \in \operatorname{int} K. \tag{14}$$

Since f is *K*-pseudoconvex at y, (14) implies

$$-\nabla f(y)(x - y) \in \operatorname{int} K.$$

As $\lambda \in K^* \setminus \{0\}$, we get

$$(\lambda^t \nabla f(y))(x - y) < 0. \tag{15}$$

Since $\mu^t g(y) = 0$, therefore by Lemma 3.1,

$$(\mu^t \nabla g(y))(x - y) \leq 0. \tag{16}$$

Adding (15) and (16), we have

$$(\lambda^t \nabla f(y) + \mu^t \nabla g(y))(x - y) < 0,$$

which contradicts (12). Hence, $f(y) - f(x) \notin \operatorname{int} K$.

Theorem 4.2: (Strong Duality) Let $x^* \in F^W$. Assume that Slater-type cone constraint qualification holds at x^*. If f is *K*-pseudoconvex at x^* and the feasible set F is convex, then there exist $\lambda^* \in K^* \setminus \{0\}$ and $\mu^* \in Q^*$ such that $(x^*, \lambda^*, \mu^*) \in F^D$. Further, if the conditions of Weak Duality Theorem 4.1 hold for all $x \in F$ and $(y, \lambda, \mu) \in F^D$, then $(x^*, \lambda^*, \mu^*) \in F_W^D$.

Proof: Since all the conditions of Theorem 3.2 hold, therefore there exist $\lambda^* \in K^* \setminus \{0\}$ and $\mu^* \in Q^*$ such that

$$\lambda^{*t} \nabla f(x^*) + \mu^{*t} \nabla g(x^*) = 0$$

and

$$\mu^{*t} g(x^*) = 0.$$

Thus $(x^*, \lambda^*, \mu^*) \in F^D$. Further if $(x^*, \lambda^*, \mu^*) \notin F_W^D$, then there exists $(y, \lambda, \mu) \in F^D$ such that

$$f(y) - f(x^*) \in \operatorname{int} K,$$

which contradicts Theorem 4.1.

Hence, $(x^*, \lambda^*, \mu^*) \in F_W^D$.

Theorem 4.3: (Converse Duality) Let

$$(y^*, \lambda^*, \mu^*) \in F_W^D.$$

Assume that f is K-pseudoconvex at y^* and the feasible set F is convex. Then $y^* \in F^W$.

Proof: Suppose $y^* \notin F^W$. Then there exists $\hat{x} \in F$ such that

$$f(y^*) - f(\hat{x}) \in \operatorname{int} K.$$

Since f is K-pseudoconvex at y^*, we get

$$-\nabla f\left(y^*\right)\left(\hat{x}-y^*\right)\in \operatorname{int} K,$$

so that,

$$\left(\lambda^{*t}\nabla f\left(y^*\right)\right)\left(\hat{x}-y^*\right)<0. \tag{17}$$

Also, $\mu^{*t}g\left(y^*\right)=0$, so that by Lemma 3.1,

$$\left(\mu^{*t}\nabla g\left(y^*\right)\right)\left(\hat{x}-y^*\right)\le 0. \tag{18}$$

Adding (17) and (18), we have

$$\left(\lambda^{*t}\nabla f\left(y^*\right)+\mu^{*t}\nabla g\left(y^*\right)\right)\left(\hat{x}-y^*\right)<0,$$

which contradicts (12). Hence, $y^*\in F^W$.

5. Conclusion

This paper gives a new direction to the search for solution of a vector optimization problem over cones. We have shown that, with Slater-type cone constraint qualification, convexity of the feasible set can replace the cone-convexity (or any of its generalization) of the constraint functions, and then we just need to assume the cone-convexity (or a suitable generalization) of the objective function to prove the necessity and sufficiency of the KKT optimality conditions. Moreover, a Mond-Weir type dual has been formulated in the modified situation and various duality results have been established.

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REFERENCES

[1] J. B. Lasserre, "On Representations of the Feasible Set in Convex Optimization," *Optimization Letters*, Vol. 4, No. 1, 2010, pp. 1-5. http://dx.doi.org/10.1007/s11590-009-0153-6

[2] C. R. Bector, S. Chandra and M. K. Bector, "Sufficient Optimality Conditions and Duality for a Quasiconvex Programming Problem," *Journal of Optimization Theory and Applications*, Vol. 59, No. 2, 1988, pp. 209-221.

[3] O. L. Mangasian, "Non-Linear Programming," McGraw Hill, New York, 1969.

[4] S. Nobakhtian, "Sufficiency in Nonsmooth Multiobjective Programming Involving Generalized (F, ρ)-Convexity," *Journal of Optimization Theory and Applications*, Vol. 130, No. 2, 2006, pp. 359-365. http://dx.doi.org/10.1007/s10957-006-9105-9

[5] J. Dutta and C. S. Lalitha, "Optimality Conditions in Convex Optimization Revisited," *Optimization Letters*, Vol. 7, No. 2, 2013, pp. 221-229. http://dx.doi.org/10.1007/s11590-011-0410-3

[6] F. H. Clarke, "Optimization and Non-smooth Analysis," Wiley, New York, 1983.

[7] L. Coladas, Z. Li and S. Wang, "Optimality Conditions for Multiobjective and Nonsmooth Minimization in Abstract Spaces," *Bulletin of Australian Mathematical Society*, Vol. 50, No. 2, 1994, pp. 205-218. http://dx.doi.org/10.1017/S0004972700013678

[8] S. Aggarwal, "Optimality and Duality in Mathematical Programming Involving Generalized Convex Functions," Ph.D. Thesis, University of Delhi, Delhi, 1998.

[9] T. Weir, B. Mond and B. D. Craven, "Weak Minimization and Duality," *Numerical Functional Analysis and Optimization*, Vol. 9, No. 1-2 ,1987, pp. 181-192.

[10] J. Jahn, "Vector Optimization," Springer Verlag, New York, 2003.

[11] S. K. Suneja, S. Aggarwal and S. Davar, "Multiobjective Symmetric Duality Involving Cones," *European Journal of Operational Research*, Vol. 141, No. 3, 2002, pp. 471-479. http://dx.doi.org/10.1016/S0377-2217(01)00258-2