

An Existence Theorem of Solutions for the System of Generalized Vector Quasi-Variational-Like Inequalities

Shamshad Husain¹, Sanjeev Gupta^{1*}, Vishnu Narayan Mishra²

¹Department of Applied Mathematics, Faculty of Engineering & Technology, Aligarh Muslim University, Aligarh, India

²Department of Applied Mathematics & Humanities, Sardar Vallabhbhai National Institute of Technology, Surat, India

Email: s_husain68@yahoo.com, *guptasanmp@gmail.com, vishnunarayanmishra@gmail.com

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ABSTRACT

In this paper, we introduce and study the system of generalized vector quasi-variational-like inequalities in Hausdorff topological vector spaces, which include the system of vector quasi-variational-like inequalities, the system of vector variational-like inequalities, the system of vector quasi-variational inequalities, and several other systems as special cases. Moreover, a number of C-diagonal quasiconvexity properties are proposed for set-valued maps, which are natural generalizations of the g-diagonal quasiconvexity for real functions. Together with an application of continuous selection and fixed-point theorems, these conditions enable us to prove unified existence results of solutions for the system of generalized vector quasi-variational-like inequalities. The results of this paper can be seen as extensions and generalizations of several known results in the literature.

Keywords: The System of Generalized Vector Quasi-Variational-Like Inequalities; Fixed Point Theorem; Open Lower Section; Upper Semicontinuous; C-Diagonal Quasiconvexity

1. Introduction and Formulation

In recent years, the system of generalized vector quasi-variational-like inequality, which is a unified model for the system of vector quasi-variational-like inequalities, the system of vector variational-like inequalities, the system of vector equilibrium problems and the system of variational inequalities etc., has been studied (see [1-18] and references therein).

In this paper, we consider the systems of four kinds of generalized vector quasi-variational-like inequalities with set-valued mappings and discuss the existence of its solutions in locally convex topological vector space (l.c.s. in short), motivated and inspired by the recent works of Peng [1] and Ansari *et al.* [2].

Throughout this paper, unless otherwise specified, assume that I be an index set. For each $i \in I$, let Z_i be a locally convex topological vector space (l.c.s., in short) and K_i be a nonempty convex subset of Hausdorff topological vector space (t.v.s., in short) E_i . Let Y_i be a subset of continuous function space $L(E_i, Z_i)$ from E_i into Z_i , where $L(E_i, Z_i)$ is equipped with a σ -

topology. Let $\text{int}A$ and $\text{co}A$ denote the interior and convex hull of a set A respectively. Let $C_i : K \rightarrow 2^{Z_i}$ be a set-valued mapping such that $\text{int}C_i(x) \neq \emptyset$ for each $x \in K$. Denote that $K = \prod_{i \in I} K_i$ and $E = \prod_{i \in I} E_i$.

For each $i \in I$, let $\eta_i : K_i \times K_i \rightarrow E_i$ be a vector-valued mapping, $G_i : L(E, Z) \rightarrow 2^{L(E_i, Z_i)}$, $S_i : K \times K \rightarrow 2^{Z_i}$, $T_i : K \rightarrow 2^{Y_i}$ and $D_i : K \rightarrow 2^{K_i}$ be four set-valued mappings. Then,

1) Strong type I system of generalized vector quasi-variational-like inequalities which is to find $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in D_i(\bar{x})$, $\bar{t}_i \in T_i(\bar{x})$ and $\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \subseteq C_i(\bar{x})$, $\forall y_i \in D_i(\bar{x})$, (1.1)

2) Strong type II system of generalized vector quasi-variational-like inequalities which is to find $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in D_i(\bar{x})$, $\bar{t}_i \in T_i(\bar{x})$ and $\{ \langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \} \cap C_i(\bar{x}) \neq \emptyset$, $\forall y_i \in D_i(\bar{x})$, (1.2)

3) Weak type I system of generalized vector quasi-variational-like inequalities which is to find $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in D_i(\bar{x})$, $\bar{t}_i \in T_i(\bar{x})$ and

*Corresponding author.

$$\left\{ \langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \right\} \cap -\text{int } C_i(\bar{x}) \neq \emptyset, \quad (1.3)$$

$$\forall y_i \in D_i(\bar{x}),$$

4) Weak type II system of generalized vector quasi-variational-like inequalities which is to find $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in D_i(\bar{x}), \bar{t}_i \in T_i(\bar{x})$ and $\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \not\subseteq -\text{int } C_i(\bar{x}), \forall y_i \in D_i(\bar{x}),$ (1.4)

where $\langle l, x \rangle$ denotes the evaluation of $l \in L(E, Z)$ at $x \in E$. By the corollary of the Schaefer [3], $L(E, Z)$ becomes a l.c.s.. By Ding and Tarafdar [4], the bilinear map $\langle \cdot, \cdot \rangle : L(K, Z) \times K \rightarrow Z$ is continuous.

The following problems are the special cases of above four kinds of systems of generalized vector quasi-variational-like inequalities.

The above system of generalized vector quasi-variational-like inequalities encompass many models of system of variational inequalities. The following problems are the special cases of problem (1.4).

1) If for each $i \in I$, let G_i be an identity mapping, $S_i \equiv 0$, problem (1.4) reduces to the system of generalized quasi-variational-like inequalities of finding $\bar{x} \in K$ such that for each $i \in I, \bar{x}_i \in D_i(\bar{x})$ and

$$\forall y_i \in D_i(\bar{x}), \exists \bar{t}_i \in T_i(\bar{x}) : \langle \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle \not\subseteq -\text{int } C_i(\bar{x}), \quad (1.5)$$

which was introduced and studied by Peng [1].

2) If for each $i \in I$, let G_i be an identity mapping, $S_i \equiv 0$ and $D_i(x) = K_i$, problem (1.5) reduces to the system of generalized variational-like inequalities of finding $\bar{x} \in K$ such that for each $i \in I, \bar{x}_i \in K_i$ and

$$\forall y_i \in K_i, \exists \bar{t}_i \in T_i(\bar{x}) : \langle \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle \not\subseteq -\text{int } C_i(\bar{x}). \quad (1.6)$$

In addition, let $Z_i = \mathbb{R}$ and let $C_i(x) = \mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$ for all $x \in K$, then problem (1.5) reduces to the system of generalized vector quasi-variational inequalities studied by Ansari and Yao [5].

3) If for each $i \in I, G_i$ be an identity mapping, $S_i \equiv 0, \eta_i(y_i, \bar{x}_i) = y_i - \bar{x}_i$ and $D_i(x) = K_i$, then problem (1.5) reduces to the system of generalized vector variational inequalities of finding $\bar{x} \in K$ such that for each $i \in I, \bar{x}_i \in K_i$ and

$$\forall y_i \in K_i, \exists \bar{t}_i \in T_i(\bar{x}) : \langle \bar{t}_i, y_i - \bar{x}_i \rangle \not\subseteq -\text{int } C_i(\bar{x}). \quad (1.7)$$

4) If $I = \{1\}$, problem (1.4) reduces to generalized vector quasi-variational-like inequalities of finding $\bar{x} \in K$ such that $\bar{x} \in D(\bar{x})$ and

$$\langle G\bar{t}, \eta(y, \bar{x}) \rangle + S(\bar{x}, y) \not\subseteq -\text{int } C(\bar{x}), \forall y_i \in K, \quad (1.8)$$

such type of problem studied in [6-10].

5) If $I = \{1\}$ and $\eta(y, \bar{x}) = y - \bar{x}$, T is single valued mapping, G be an identity mapping, $S \equiv 0$, and $C(x) = \mathbb{R}^+$ for all $x \in K$, then problem (1.4) reduces to classical variational inequality problem of finding $\bar{x} \in K$ such that $\bar{x} \in D(\bar{x})$ and

$$\forall y \in D(\bar{x}), \exists \bar{t} \in T(\bar{x}) : \langle T(\bar{x}), (y - \bar{x}) \rangle \not\subseteq -\text{int } C(\bar{x}), \quad (1.9)$$

which was introduced and studied by Hartman and Stampacchia [11].

2. Preliminaries

Definition 2.1. [12] Let E and Z be two t.v.s. and K be a convex subset of t.v.s. E . Let $C : K \rightarrow 2^Z$ and $\theta : K \times K \rightarrow 2^Z$ be two set-valued mappings. Assume given any finite subset $\Lambda = \{x_1, x_2, \dots, x_n\}$ in K , any

$$x = \sum_{i=1}^n \alpha_i x_i, \text{ with } \alpha_i \geq 0 \text{ for } i=1, \dots, n, \text{ and } \sum_{i=1}^n \alpha_i = 1.$$

Then, 1) θ is said to be strong Type I C-diagonally quasiconvex (SIC-DQC, in short) in the second argument if for some $x_i \in \Lambda$,

$$\theta(x, x_i) \subseteq C(x);$$

2) θ is said to be strong Type II C-diagonally quasiconvex (SIIC-DQC, in short) in the second argument if for some $x_i \in \Lambda$,

$$\theta(x, x_i) \cap C(x) \neq \emptyset;$$

3) θ is said to be weak Type I C-diagonally quasiconvex (WIC-DQC, in short) in the second argument if for some $x_i \in \Lambda$,

$$\theta(x, x_i) \cap -\text{int } C(x) \neq C(x);$$

4) θ is said to be weak Type II C-diagonally quasiconvex (WIIC-DQC, in short) in the second argument if for some $x_i \in \Lambda$,

$$\theta(x, x_i) \not\subseteq -\text{int } C(x).$$

It is easy to verify that the following proposition, 1) SIC-DQC implies SIIC-DQC; 2) SIIC-DQC implies WIC-DQC; 3) WIC-DQC implies WIIC-DQC. The converse is not true. Following example shows that the converse is not true.

Example 2.1. Let $E = Z = \mathbb{R}$ and $\varphi(x_1, x_2) = \text{co}\{x_1, x_2\}$.

1) If $C(x) = [x + \epsilon, +\infty)$. Then φ is SIIC-DQC, but it is not SIC-DQC.

2) If $-\text{int } C(x) = (-\infty, x + \epsilon)$. Then φ is WIIC-DQC, but it is not WIC-DQC.

Definition 2.2. [13] Let E and Z be two t.v.s. and K be a convex subset of t.v.s. E . A mapping $\theta : K \times K \rightarrow (2^Z)Z$ is called (generalized) vector 0-

diagonally convex if for any finite subset

$\Lambda = \{x_1, x_2, \dots, x_n\}$ of K and any $x = \sum_{i=1}^n \alpha_i x_i$ with

$\alpha_i \geq 0$ for $i=1, \dots, n$, and $\sum_{i=1}^n \alpha_i = 1$,

$$\sum_{i=1}^n \alpha_i \theta(x, x_i) (\not\subseteq) \notin -\text{int } C(x).$$

Definition 2.3. [14] Let X and Y be two topological spaces and $T : X \rightarrow 2^Y$ be a set-valued mapping. Then,

1) T is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for every $y \in Y$;

2) T is said to be upper semicontinuous (u.s.c., in short) if for each $x_0 \in X$ and each open set U in Y with $T(x_0) \subset U$, there exists an open neighborhood V of x_0 in X such that $T(x) \subset U$ for each $x \in V$;

3) T is said to be lower semicontinuous (l.s.c., in short) if for each $x_0 \in X$ and each open set U in Y with $T(x_0) \cap U \neq \emptyset$, there exists an open neighborhood V of x_0 in X such that $T(x) \cap U \neq \emptyset$ for each $x \in V$;

4) T is said to be continuous if it is both upper and lower semicontinuous;

5) T is said to be closed if for any net $\{x^o\}$ in X such that $x^o \rightarrow x^*$ and any net $\{y^o\}$ in B such that $y^o \rightarrow y^*$ and $y^o \in T(x^o)$ for any o , we have $y^* \in T(x^*)$.

Lemma 2.1. [15] Let X and Y be two topological spaces. If $T : X \rightarrow 2^Y$ is u.s.c. set-valued mapping with closed values, then T is closed.

Lemma 2.2. [16] Let X and Y be two topological spaces and $T : X \rightarrow 2^Y$ is u.s.c. mapping with compact values. Suppose $\{x^o\}$ is a net in X such that $x^o \rightarrow x^*$. If $y^o \in T(x^o)$ for each o , then there are a $y^* \in T(x^*)$ and a subnet $\{y^n\}$ of $\{y^o\}$ such that $y^n \rightarrow y^*$.

Lemma 2.3. [17] Let X and Y be two topological spaces. Suppose that $T : X \rightarrow 2^Y$ and $K : X \rightarrow 2^Y$ are set-valued mappings having open lower sections, then

1) A set-valued mapping $F : X \rightarrow 2^Y$ defined by, for each $x \in X$, $F(x) = \text{co}T(x)$ has open lower sections;

2) A set-valued mapping $J : X \rightarrow 2^Y$ defined by, for each $x \in X$, $J(x) = T(x) \cap K(x)$ has open lower sections.

For each $i \in I$, E_i a Hausdorff t.v.s. Let $\{K_i\}$ be a family of nonempty compact convex subsets with each K_i in E_i . Let $K = \prod_{i \in I} K_i$ and $E = \prod_{i \in I} E_i$. The following system of fixed-point theorem is needed in this paper.

Lemma 2.4. [18] For each $i \in I$, let $T_i : K \rightarrow 2^{K_i}$ be

a set-valued mapping. Assume that the following conditions hold.

1) For each $i \in I$, T_i is convex set-valued mapping;

2) $K = \bigcup \{\text{int } T_i^{-1}(x_i) : x_i \in K_i\}$.

Then there exist $\bar{x} \in K$ such that

$\bar{x} \in T(\bar{x}) = \prod_{i \in I} T_i(\bar{x})$, that is, $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$, where \bar{x}_i is the projection of \bar{x} onto K_i .

3. Main Results

Theorem 3.1. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;

2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping $\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is WIIC-DQC;

3) For each $y_i \in K_i$, the set $\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x)\}$ is open.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \not\subseteq -\text{int } C_i(\bar{x}),$$

$$\forall y_i \in D_i(\bar{x}).$$

Proof. Define a set-valued mapping $P_i : K \times Y \rightarrow 2^{K_i}$ by

$$P_i(x, t) = \{y_i \in K_i : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x)\},$$

$$\forall (x, t) \in K \times Y.$$

We first prove that $x_i \notin \text{co}(P_i(x, t))$ for all $(x, t) \in K \times Y$. To see this, suppose, by way of contradiction, that there exist some $i \in I$ and some point $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in \text{co}(P_i(\bar{x}, \bar{t}))$. Then, there exist finite points $y_{i_1}, y_{i_2}, \dots, y_{i_n}$ in K_i and $\alpha_j \geq 0$

with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x}_i = \sum_{j=1}^n \alpha_j y_{i_j}$ and

$y_{i_j} \in P_i(\bar{x}, \bar{t})$ for all $j = 1, \dots, n$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_{i_j}, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_{i_j}) \subseteq -\text{int } C_i(\bar{x}), \quad j = 1, \dots, n,$$

which contradicts the hypothesis 2). Hence,

$x_i \notin \text{co}(P_i(x, t))$.

By hypothesis 3), for each $i \in I$ and each $y_i \in K_i$, we know that

$$Q_i^{-1}(y_i) = \{(x, t) \in K \times Y : \langle G_{t_i}, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x)\}$$

is open and so P_i has open lower sections.

For each $i \in I$, consider a set-valued mapping $Q_i : K \times Y \rightarrow 2^{K_i}$ defined by

$$Q_i(x, t) = \text{co}(P_i(x, t)) \cap D_i(x), \quad \forall (x, t) \in K \times Y.$$

Since D_i has open lower sections by hypothesis 1), we may apply Lemma 2.3 to assert that the set-valued mapping Q_i has also open lower sections. Let

$$W_i = \{(x, t) \in K \times Y : Q_i(x, t) \neq \emptyset\} \subset K \times Y.$$

There are two cases to consider. In the case $W_i = \emptyset$, we have

$$\text{co}(P_i(x, t)) \cap D_i(x) = \emptyset, \quad \forall (x, t) \in K \times Y.$$

This implies that, $\forall (x, t) \in K \times Y$,

$$P_i(x, t) \cap D_i(x) = \emptyset.$$

On the other hand, by condition 1), and the fact K_i is a compact convex subset of E_i , we can apply Lemma 2.4 to assert the existence of a fixed point $x_i^* \in D_i(x^*)$. Since $T_i(x^*) \neq \emptyset$, picking $t_i^* \in T_i(x^*)$, we have

$$P_i(x^*, t_i^*) \cap D_i(x^*) = \emptyset.$$

This implies $\forall y_i \in D_i(x^*), y_i \notin P_i(x^*, t_i^*)$. Hence, in this particular case, the assertion of the theorem holds.

We now consider the case $W_i \neq \emptyset$. Define a set-valued mapping $S_i : K \times Y \rightarrow 2^{K_i}$ by

$$S_i(x, t) = \begin{cases} Q_i(x, t), & (x, t) \in W_i \\ D_i(x), & (x, t) \in K_i \times Y_i \setminus W_i. \end{cases}$$

Then, $S_i(x, t)$ is a convex set-valued mapping and for each $u \in K$, $S_i^{-1}(u) = Q_i^{-1}(u) \cup (D_i^{-1}(u) \times Y_i)$ is open. For each $i \in I$, consider the set-valued mapping $H : K \times Y \rightarrow 2^{K \times Y}$ where $H = \Pi_{i \in I} H_i$ defined by

$$H_i(x, t) = (S_i(x, t), T_i(x)).$$

By condition 1) and the properties of $S_i(x, t)$, H_i satisfies all the conditions of Lemma 2.4. Therefore, there exists $(x^*, t^*) \in K \times Y$ such that

$(x_i^*, t_i^*) \in H_i(x^*, t^*)$. Suppose that $(x^*, t^*) \in W_i$, then

$$x_i^* \in \text{co}(P_i(x^*, t^*)) \cap D_i(x^*),$$

so that $x_i^* \in \text{co}(P_i(x^*, t^*))$. This is a contradiction.

Hence, $(x^*, t^*) \notin W_i$. Therefore,

$$(x_i^*, t_i^*) \in (D_i(x^*), T_i(x^*)), \text{ and } Q_i(x^*, t^*) = \emptyset.$$

Thus

$$x_i^* \in D_i(x^*), t_i^* \in T_i(x^*), \text{ co}(P_i(x^*, t^*)) \cap D_i(x^*) = \emptyset.$$

This implies

$$P_i(x^*, t^*) \cap D_i(x^*) = \emptyset.$$

Consequently, the assertion of the theorem holds in this case.

Corollary 3.2. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;

2) For all $y_i \in K_i$, the mapping

$\langle G_{t_i}, \eta_i(y_i, *) \rangle + S_i(*, y_i) : K \times Y \rightarrow 2^{Z_i}$ is an u.s.c. set-valued mapping;

3) $C_i : K \rightarrow 2^{Z_i}$ is a convex set-valued mapping with $\text{int } C_i(x) \neq \emptyset$ for all $x \in K$;

4) $\eta_i : K_i \times K_i \rightarrow E_i$ is affine in the first argument and for all $x_i \in K_i$, $\eta_i(x_i, x_i) = 0$;

5) $S_i : K \times K \rightarrow 2^{Z_i}$ is a generalized vector 0-diagonally convex set-valued mapping;

6) For a given $x_i \in K_i$, and a neighborhood U_i of x_i , for all $u \in U_i$, $\text{int } C_i(x) = \text{int } C_i(u)$.

Then there exists $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_{\bar{t}_i}, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \not\subseteq -\text{int } C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}).$$

Proof. Define a set-valued mapping $P_i : K \times Y \rightarrow 2^{K_i}$ by

$$P_i(x, t) = \{y_i \in K_i : \langle G_{t_i}, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x)\},$$

$$\forall (x, t) \in K \times Y.$$

We first prove that $x_i \notin \text{co}(P_i(x, t))$ for all $(x, t) \in K \times Y$. By contradiction, for each $i \in I$, suppose there exists some point $(\bar{x}, \bar{t}) \in K \times Y$ such that $\bar{x}_i \in \text{co}(P_i(\bar{x}, \bar{t}))$. Then, there exist finite points $y_{i_1}, y_{i_2}, \dots, y_{i_n}$ in K_i , such that

$$\langle G_{\bar{t}_i}, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \subseteq -\text{int } C_i(\bar{x}), \quad i = 1, 2, \dots, n.$$

Since $\eta_i(\cdot, x_i)$ is affine and $\text{int } C_i(\bar{x})$ is convex, for $\alpha_j \geq 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x}_i = \sum_{j=1}^n \alpha_j y_{i_j}$ and $y_{i_j} \in P_i(\bar{x}, \bar{t})$ for all $j = 1, \dots, n$ such that

$$\left\langle G_{\bar{t}_i}, \eta_i \left(\sum_{j=1}^n \alpha_j y_{i_j}, \bar{x}_i \right) \right\rangle + \sum_{j=1}^n \alpha_j S_i(\bar{x}_i, y_{i_j}) \subseteq -\text{int } C_i(\bar{x}), \quad j = 1, \dots, n.$$

Since $\eta_i(x_i, x_i) = 0$ for all $x_i \in K_i$

$$\sum_{j=1}^n \alpha_j S_j(\bar{x}_i, y_i) \subseteq -\text{int } C_i(\bar{x})$$

which contradicts the hypothesis 5). Therefore $x_i \notin \text{co}(P_i(x, t))$.

We now prove that for each

$$\begin{aligned} & y_i \in K_i, P_i^{-1}(y_i) \\ &= \{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \\ &\subseteq -\text{int } C_i(x)\} \end{aligned}$$

is open. Indeed, let $(\bar{x}, \bar{t}) \in P_i^{-1}(y_i)$, that is $\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \subseteq -\text{int } C_i(\bar{x})$. Since $\langle G_i t_i, \eta_i(y_i, *) \rangle + S_i(*, y_i) : K \times Y \rightarrow 2^{Z_i}$ is an u.s.c. set-valued mapping, there exists a neighborhood U_i of (\bar{x}, \bar{t}) such that

$$\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(\bar{x}), \forall (x, t) \in U_i.$$

By 6),

$$\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \forall (x, t) \in U_i.$$

Hence, $U_i \subset P_i^{-1}(y_i)$. This implies, $P_i^{-1}(y_i)$ is open for each $y_i \in K_i$, and so P_i have open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.1. This completes the proof.

Corollary 3.3. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For all $y_i \in K_i$, the mapping $\langle G_i, \eta_i(y_i, *) \rangle + S_i(*, y_i) : K \times Y \rightarrow 2^{Z_i}$ is an u.s.c. set-valued mapping;
- 3) $C_i : K \rightarrow 2^{Z_i}$ is a convex set-valued mapping such that for each $x \in K$, $C_i(x) = C_i$ is a convex cone with $\text{int } C_i(x) \neq \emptyset$;
- 4) $\eta_i : K_i \times K_i \rightarrow E_i$ is affine in the first argument and for all $x_i \in K_i$, $\eta_i(x_i, x_i) = 0$;
- 5) $S_i : K \times K \rightarrow 2^{Z_i}$ is a generalized vector 0-diagonally convex set-valued mapping;
- 6) For a given $x_i \in K_i$, and a neighborhood U_i of x , for all $u \in U_i$, $\text{int } C_i(x) = \text{int } C_i(u)$.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \not\subseteq -\text{int } C_i, \forall y_i \in D_i(\bar{x}).$$

Proof. By hypothesis 3), the condition 4) in Corollary 3.2 is satisfied. Hence, all the conditions are satisfied as

in Corollary 3.2.

Corollary 3.4. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that S_i and G_i are single valued mappings and the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For all $y_i \in K_i$, the mapping $\langle G_i, \eta_i(y_i, *) \rangle + S_i(*, y_i) : K_i \times Y_i \rightarrow Z_i$ is continuous;
- 3) $C_i : K \rightarrow 2^{Z_i}$ is a convex set-valued mapping with $\text{int } C_i(x) \neq \emptyset$ for all $x \in K$;
- 4) $\eta_i : K_i \times K_i \rightarrow E_i$ is affine in the first argument and for all $x_i \in K_i$, $\eta_i(x_i, x_i) = 0$;
- 5) $S_i : K \times K \rightarrow Z_i$ is a vector 0-diagonally convex mapping;
- 6) $Z_i \setminus \{-\text{int } C_i(x)\}$ is an u.s.c. set-valued mapping.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \not\subseteq -\text{int } C_i(\bar{x}), \forall y_i \in D_i(\bar{x}).$$

Proof. Define a set-valued mapping $P_i : K \times Y \rightarrow 2^{K_i}$ by

$$\begin{aligned} P_i(x, t) &= \{y_i \in K_i : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \\ &\subseteq -\text{int } C_i(x)\}, \\ &\forall (x, t) \in K \times Y. \end{aligned}$$

We now prove that for each

$$\begin{aligned} & y_i \in K_i, P_i^{-1}(y_i) \\ &= \{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \\ &\subseteq -\text{int } C_i(x)\} \end{aligned}$$

is open, that is, the set

$$\begin{aligned} & \{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \\ &\subseteq Z_i \setminus \{-\text{int } C_i(x)\}\} \end{aligned}$$

is closed. Indeed, let $\{(x^\circ, t^\circ)\}$ be a net in $K \times Y$ such that $(x^\circ, t^\circ) \rightarrow (x^*, t^*)$ and

$$\langle G_i t_i^\circ, \eta_i(y_i, x_i^\circ) \rangle \cap S_i(x_i^\circ, y_i) \in Z_i \setminus \{-\text{int } C_i(x^\circ)\}.$$

Since $\langle G_i, \eta_i(y_i, *) \rangle + S_i(*, y_i) : K \times Y \rightarrow 2^{Z_i}$ is continuous, hence

$$\begin{aligned} & \langle G_i t_i^\circ, \eta_i(y_i, x_i^\circ) \rangle \cap S_i(x_i^\circ, y_i) \\ &\rightarrow \langle G_i t_i^*, \eta_i(y_i, x_i^*) \rangle + S_i(x_i^*, y_i). \end{aligned}$$

Since $Z_i \setminus \{-\text{int } C_i(x)\}$ is an u.s.c. set-valued mapping with closed values, by Lemma 2.1, we have

$$\langle G_i t_i^*, \eta_i(y_i, x_i^*) \rangle + S_i(x_i^*, y_i) \in Z_i \setminus \{-\text{int } C_i(x^*)\},$$

and hence (x^*, t^*) in the set

$$\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \in -\text{int } C_i(x)\}.$$

This implies $P_i^{-1}(y_i)$ is open for each $y_i \in K_i$ and so P_i has open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.1 and Corollary 3.2. This completes the proof.

Theorem 3.5. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping $\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is WIC-DQC;
- 3) for each $y_i \in K_i$, the set

$$\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \in -\text{int } C_i(x) \neq \emptyset\}$$

is open.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \in -\text{int } C_i(\bar{x}) = \emptyset, \forall y_i \in D_i(\bar{x}).$$

Proof. Define a set-valued mapping $P_i : K \times Y \rightarrow 2^{K_i}$ by

$$P_i(x, t) = \{y_i \in K_i : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \in -\text{int } C_i(x) \neq \emptyset\},$$

$$\forall (x, t) \in K \times Y.$$

For the remainder proof, we just follow that of Theorem 3.1.

Corollary 3.6. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping $\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is WIC-DQC;

3) $Z_i \setminus \{-\text{int } C_i(x)\}$ is an u.s.c. set-valued mapping.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \in -\text{int } C_i(\bar{x}) = \emptyset, \forall y_i \in D_i(\bar{x}).$$

Proof. Let $P_i : K \times Y \rightarrow 2^{K_i}$ be a set-valued mapping define in Theorem 3.5. We just prove that for each

$$y_i \in K_i, P_i^{-1}(y_i) = \{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \in -\text{int } C_i(x) \neq \emptyset\}$$

is open, that is, the set

$$\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \in -\text{int } C_i(x) \neq \emptyset\}$$

is closed. Indeed, let $\{(x^\alpha, t^\alpha)\}$ be a net in $K \times Y$ such that $(x^\alpha, t^\alpha) \rightarrow (x^*, t^*)$ and

$$\langle G_i t_i^\alpha, \eta_i(y_i, x_i^\alpha) \rangle + S_i(x_i^\alpha, y_i) \in -\text{int } C_i(x^\alpha) = \emptyset.$$

This implies

$$\langle G_i t_i^\alpha, \eta_i(y_i, x_i^\alpha) \rangle + S_i(x_i^\alpha, y_i) \in Z_i \setminus \{-\text{int } C_i(x^\alpha)\}.$$

We now prove that

$$\langle G_i t_i^*, \eta_i(y_i, x_i^*) \rangle + S_i(x_i^*, y_i) \in Z_i \setminus \{-\text{int } C_i(x^*)\}.$$

If it is not true, then there exists a

$w^* \in \langle G_i t_i^*, \eta_i(y_i, x_i^*) \rangle + S_i(x_i^*, y_i)$ such that $w^* \notin Z_i \setminus \{-\text{int } C_i(x^*)\}$. Since Z_i is Hausdorff t.v.s. (l.c.s. is Hausdorff space) and $Z_i \setminus \{-\text{int } C_i(x^*)\}$ is closed, there exists two open sets $U_i, V_i \subset Z_i$ such that $w^* \in U_i, Z_i \setminus \{-\text{int } C_i(x^*)\} \subset V_i$ and $U_i \cap V_i = \emptyset$.

Since $\langle G_i \cdot, \eta_i(y_i, *) \rangle \cap S_i(*, y_i) : P \times Y \rightarrow 2^{Z_i}$ is an l.s.c. set-valued mapping and $Z_i \setminus \{-\text{int } C_i(x)\}$ is an u.s.c. set-valued mapping, there exists a neighborhood $U_i(x^*, y^*)$ of (x^*, y^*) such that

$$\langle G_i \cdot, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \in U_i \neq \emptyset, \forall (x, t) \in U_i(x^*, y^*)$$

and a neighborhood $U_i(x^*)$ of x^* such that $Z_i \setminus \{-\text{int } C_i(x)\} \subset V_i, \forall x \in U_i(x^*)$.

Hence, for all $(x^o, t^o) \in U_i(x^*, y^*) \cap \{U_i(x^*) \times Y_i\}$, there exists $w^o \in \{\langle G_i t_i^o, \eta_i(y_i, x_i^o) \rangle + S_i(x_i^o, y_i)\}$ such that $w^o \notin Z_i \setminus \{-\text{int } C_i(x^o)\}$, which is contradiction. Therefore, the set

$$\{(x, t) \in K \times Y : \{\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i)\} \cap -\text{int } C_i(x) = \emptyset\}$$

is closed. Hence, all the conditions of Theorem 3.5 satisfied. Consequently, the assertion of the theorem holds.

Theorem 3.7. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping $\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is SIIC-DQC;
- 3) for each $y_i \in K_i$, the set

$$\{(x, t) \in K \times Y : \{\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i)\} \cap C_i(x) = \emptyset\}$$

is open.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that $\{\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i)\} \cap C_i(\bar{x}) \neq \emptyset, \forall y_i \in D_i(\bar{x})$.

Proof. Define a set-valued mapping $P_i : K \times Y \rightarrow 2^{K_i}$ by

$$P_i(x, t) = \{y_i \in K_i : \{\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i)\} \cap C_i(x) = \emptyset\},$$

$$\forall (x, t) \in K \times Y.$$

For the remainder proof, we just follow that of Theorem 3.1.

Corollary 3.8. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping $\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is SIIC-DQC;
- 3) For all $x \in K$, $C_i(x)$ is closed convex cone C_i .

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\{\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i)\} \cap C_i = \emptyset, \forall y_i \in D_i(\bar{x}).$$

Proof. Let $P_i : K \times Y \rightarrow 2^{K_i}$ be a set-valued mapping defined in Theorem 3.7. We prove that for each

$$\begin{aligned} & y_i \in K_i, P_i^{-1}(y_i) \\ &= \{(x, t) \in K \times Y : \{\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i)\} \\ & \quad \cap C_i = \emptyset\} \end{aligned}$$

is open, that is, the set

$$\{(x, t) \in K \times Y : \{\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i)\} \subset Z_i \setminus C_i\}$$

is open. If $(\bar{x}, \bar{t}) \in P_i^{-1}(y_i)$, since $Z_i \setminus C_i$ is open set and for all

$y_i \in K_i, \langle G_i \cdot, \eta_i(y_i, *) \rangle + S_i(*, y_i) : K \times Y \rightarrow 2^{Z_i}$, an u.s.c. set-valued mapping, there exists a neighborhood U_i of (\bar{x}, \bar{t}) , for all $(x, t) \in U_i$,

$$\{\langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i)\} \subset Z_i \setminus C_i.$$

This implies $P_i^{-1}(y_i)$ is open for each $y_i \in K_i$. Therefore, all the conditions of Theorem 3.7 are satisfied. Consequently the assertion of the theorem holds.

Theorem 3.9. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping $\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is SIC-DQC;
- 3) for each $y_i \in K_i$, the set $\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle \cap S_i(x_i, y_i) \not\subseteq C_i(x)\}$ is open.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \not\subseteq C_i(\bar{x}), \forall y_i \in D_i(\bar{x}).$$

Proof. Define a set-valued mapping $P_i : K \times Y \rightarrow 2^{K_i}$ by

$$P_i(x, t) = \{y_i \in K_i : \langle G_i t_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \not\subseteq C_i(x)\},$$

$$\forall (x, t) \in K \times Y.$$

The rest of the proof is similar to that of Theorem 3.1.

Corollary 3.10. For each $i \in I$, let Z_i be a l.c.s., K_i a nonempty compact convex subset of Hausdorff t.v.s. E_i , Y_i a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a σ -topology. For each $i \in I$, assume that the following conditions are satisfied.

- 1) $D_i : K \rightarrow 2^{K_i}$ and $T_i : K \rightarrow 2^{Y_i}$ are two nonempty convex set-valued mappings and have open lower sections;
- 2) For each $t_i \in Y_i$ and $x_i \in \text{co}\Lambda_i$, the mapping

$\langle G_i t_i, \eta_i(\cdot, x_i) \rangle + S_i(x_i, \cdot) : K \rightarrow 2^{Z_i}$ is SIC-DQC;

3) $C_i(x)$ is an u.s.c. mapping with closed values.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$\langle G_i \bar{t}_i, \eta_i(y_i, \bar{x}_i) \rangle + S_i(\bar{x}_i, y_i) \subseteq C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}).$$

Proof. Let $P_i : K \times Y \rightarrow 2^{K_i}$ a set-valued mapping defined in Theorem 3.9. We prove that for each $y_i \in K_i$, the set

$\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle \cap S_i(x_i, y_i) \not\subseteq C_i(x)\}$ is open, that is, the set

$\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle \cap S_i(x_i, y_i) \subseteq C_i(x)\}$ is

closed. Indeed, let $\{(x^o, t^o)\}$ be a net in $K \times Y$ such that $(x^o, t^o) \rightarrow (x^*, t^*)$ and

$$\langle G_i t_i^o, \eta_i(y_i, x_i^o) \rangle \cap S_i(x_i^o, y_i) \subseteq C_i(x^o).$$

We claim that

$$\langle G_i t_i^*, \eta_i(y_i, x_i^*) \rangle \cap S_i(x_i^*, y_i) \subseteq C_i(x^*).$$

To prove this assertion, we can just follow that of Corollary 3.6. Hence, the set

$\{(x, t) \in K \times Y : \langle G_i t_i, \eta_i(y_i, x_i) \rangle \cap S_i(x_i, y_i) \not\subseteq C_i(x)\}$ is open. Therefore, all the conditions of Theorem 3.9 are satisfied. Consequently, the assertion of the corollary hold.

REFERENCES

- [1] J.-W. Peng, "System of Generalized Set-Valued Quasi-Variational-Like Inequalities," *Bulletin of the Australian Mathematical Society*, Vol. 68, No. 3, 2003, pp. 501-515. [doi:10.1017/S0004972700037904](https://doi.org/10.1017/S0004972700037904)
- [2] Q. H. Ansari, S. Schaible and J. C. Yao, "The System of Generalized Vector Equilibrium Problems with Applications," *Journal of Global Optimization*, Vol. 22, No. 1, 2002, pp. 3-16. [doi:10.1023/A:1013857924393](https://doi.org/10.1023/A:1013857924393)
- [3] H. H. Schaefer, "Topological Vector Space," 2nd Edition, *Graduate Texts in Mathematics*, Springer, Berlin, 1999.
- [4] X. P. Ding and E. Tarafdar, "Generalized Variational-Like Inequalities with Pseudomonotone Set-Valued Mappings," *Archive der Mathematik*, Vol. 74, No. 4, 2000, pp. 302-313. [doi:10.1007/s000130050447](https://doi.org/10.1007/s000130050447)
- [5] Q. H. Ansari and J. C. Yao, "System of Generalized Variational Inequalities and Their Applications," *Applicable Analysis*, Vol. 76, No. 3-4, 2000, pp. 203-217. [doi:10.1080/00036810008840877](https://doi.org/10.1080/00036810008840877)
- [6] G. Xiao and S. Liu, "Existence of Solutions for Generalized Vector Quasi-Variational-Like Inequalities without Monotonicity," *Computers and Mathematics with Applications*, Vol. 58, No. 8, 2009, pp. 1550-1557. [doi:10.1016/j.camwa.2009.05.021](https://doi.org/10.1016/j.camwa.2009.05.021)
- [7] G. Xiao, Z. Fan and X. Qi, "Existence Results for Generalized Nonlinear Vector Variational-Like Inequalities with Set-Valued Mapping," *Applied Mathematics Letter*, Vol. 23, No. 1, 2010, pp. 44-47. [doi:10.1016/j.aml.2009.07.023](https://doi.org/10.1016/j.aml.2009.07.023)
- [8] J.-W. Peng and X.-M. Yang, "Generalized Vector Quasi-Variational-Like Inequalities," *Journal of Inequalities and Applications*, Vol. 2006, No. 1, 2006, pp. 1-11.
- [9] S. Husain and S. Gupta, "Existence of Solutions for Generalized Nonlinear Vector Quasi-Variational-Like Inequalities with Set-Valued Mappings," *Filomat*, Vol. 26, No. 5, 2012, pp. 909-916. [doi:10.2298/FIL1205909H](https://doi.org/10.2298/FIL1205909H)
- [10] S. Husain and S. Gupta, "Generalized Nonlinear Vector Quasi-Variational-Like Inequality," *International Journal of Pure and Applied Mathematics*, Vol. 66, No. 2, 2011, pp. 157-169.
- [11] P. Hartman and G. Stampacchia, "On Some Nonlinear Elliptic Differential Function Equations," *Acta Mathematica*, Vol. 115, No. 1, 1966, pp. 271-310. [doi:10.1007/BF02392210](https://doi.org/10.1007/BF02392210)
- [12] S. H. Hou, H. Yu and G. Y. Chen, "On Vector Quasi-Equilibrium Problems with Set-Valued Maps," *Journal of Optimization Theory and Applications*, Vol. 119, No. 3, 2003, pp. 485-498. [doi:10.1023/B:JOTA.0000006686.19635.ad](https://doi.org/10.1023/B:JOTA.0000006686.19635.ad)
- [13] Q. M. Liu, L. Y. Fan and G. H. Wang, "Generalized Vector Quasi-Equilibrium Problems with Set-Valued Mappings," *Applied Mathematics Letters*, Vol. 21, No. 9, 2008, pp. 946-950. [doi:10.1016/j.aml.2007.10.007](https://doi.org/10.1016/j.aml.2007.10.007)
- [14] G. Q. Tian and J. Zhou, "Quasi-Variational Inequalities without the Concavity Assumption," *Journal of Mathematical Analysis and Applications*, Vol. 172, No. 1, 1993, pp. 289-299. [doi:10.1006/jmaa.1993.1025](https://doi.org/10.1006/jmaa.1993.1025)
- [15] J. P. Aubin and I. Ekeland, "Applied Nonlinear Analysis," Wiley-Interscience, New York, 1984.
- [16] C. H. Su and V. M. Sehgal, "Some Fixed Point Theorems for Condensing Multifunctions in Locally Convex Spaces," *Proceedings of the American Mathematical Society*, Vol. 50, No. 1, 1975, pp. 150-154. [doi:10.1090/S0002-9939-1975-0380530-7](https://doi.org/10.1090/S0002-9939-1975-0380530-7)
- [17] X. Q. Yang, "Generalized Convex Functions and Vector Variational Inequalities," *Journal of Optimization Theory and Applications*, Vol. 79, No. 3, 1993, pp. 563-580. [doi:10.1007/BF00940559](https://doi.org/10.1007/BF00940559)
- [18] Q. H. Ansari and J. C. Yao, "A Fixed Point Theorem and Its Applications to the System of variational Inequalities," *Bulletin of the Australian Mathematical Society*, Vol. 59, No. 3, 1999, pp. 433-442. [doi:10.1017/S0004972700033116](https://doi.org/10.1017/S0004972700033116)