

Comparison of Different Confidence Intervals of Intensities for an Open Queueing Network with Feedback

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ABSTRACT

In this paper we propose a consistent and asymptotically normal estimator (CAN) of intensities ρ_1, ρ_2 for a queueing network with feedback (in which a job may return to previously visited nodes) with distribution-free inter-arrival and service times. Using this estimator and its estimated variance, some $100(1-\alpha)\%$ asymptotic confidence intervals of intensities are constructed. Also bootstrap approaches such as Standard bootstrap, Bayesian bootstrap, Percentile bootstrap and Bias-corrected and accelerated bootstrap are also applied to develop the confidence intervals of intensities. A comparative analysis is conducted to demonstrate performances of the confidence intervals of intensities for a queueing network with short run data.

Keywords: Coverage Percentage; Relative Coverage; Bayesian Bootstrap; Bias-Corrected and Accelerated Bootstrap; Percentile Bootstrap; Standard Bootstrap

1. Introduction

Consider a queueing network of a computer system with feedback (in which a job may return to previously visited nodes) as shown in **Figure 1**. This queueing network consists of a CPU node and an Input/Output (I/O) node. External jobs arrive at the CPU node according to the rate λ . After service completion at CPU node, the job proceeds to the I/O node with probability p_1 , and with probability p_0 the job departs from the system, where

$p_0 = 1 - p_1$. Jobs leaving the I/O node are always fed back to the CPU node (see **Figure 1**). The service times at each node are with rates μ_1 and μ_2 respectively. The successive service times at both nodes are assumed to be mutually independent and independent of the state of the system. The traffic intensity at the CPU node and I/O node is given by

$$\rho_1 = \frac{\lambda}{p_0\mu_1}, \rho_2 = \frac{p_1\lambda}{p_0\mu_2} \quad (1)$$

respectively. Intensity ρ_1 and ρ_2 can be interpreted as expected number of arrivals per mean service time. The condition for stability of the system is both ρ_1, ρ_2 are less unity.

Basic properties of queueing networks are introduced in Disney [1], Burke [2], Beutler and Melamed [3]

showed that the input process to a service center in a network with feedback is not Poisson in general. It is for this reason that Jackson's result is remarkable. Jackson's [4] theorem states that each node behaves like an independent queue.

The product form solution to open network of Markovian queues with feedback is also discussed in Jackson [4]. Simon and Foley [5], Melamed [6] pointed out that computation of response time distribution is difficult even for Jacksonian networks without feedback. Disney and Kiessel [7] discussed traffic process in queueing networks through Markov renewal approach. Thiruvaiyaru, Basawa and Bhat [8] established Maximum likelihood estimators of the parameters of an open Jackson network are derived, and their joint asymptotic normality. The problem of estimation for tandem queues is discussed as a special case of the Jackson system. These results are valid when the system is not necessarily in equilibrium. Thiruvaiyaru and Basawa [9] considered the problem of estimation for the parameters in a Jackson's type queueing network with the arrival at each node following renewal process and service time distribution being arbitrary. Open queueing networks are useful in studying the behavior of computer communication networks (Kleinrock [10]). More approach to queueing network analysis

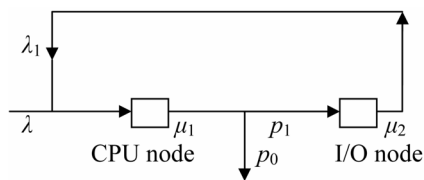


Figure 1. An open queueing network with feedback.

developed by Buzen and Denning [11]. Efron [12-14] the greatest statistician in the field of nonparametric resampling approach, originally developed and proposed the bootstrap, which is a resampling technique that can be effectively applied to estimate the sampling distribution of any statistic. Specifically, one can utilize the bootstrap method to approximate the sampling distribution of a statistic defined by a random sample from a population with unknown probability distribution. And due to the popularity of PC and statistical software, today the bootstrap becomes the most powerful nonparametric estimation procedure. Based upon the bootstrap resampling technique, most statisticians utilize the standard bootstrap (SB), percentile bootstrap (PB), and bias-corrected and accelerated bootstrap (BCaB) approaches to produce confidence intervals for practical problems.

Besides the standard bootstrap (SB) technique, Rubin [15] presented the Bayesian bootstrap (BB) technique of resampling. Miller [16] showed that the SB can be regarded as an extension of the jackknife. The BB is a natural Bayesian analogue of the SB. The BB simulates the posterior distribution of parameters under particular model specifications, whereas the SB simulates the estimated sampling distribution of a statistic estimating the parameters. Both SB and BB can be applied to construct confidence intervals of intensity for a queueing system with distribution-free inter-arrival and service times.

Chu and Ke [17] constructed new confidence intervals of mean response time for an M/G/1 FCFS queueing system. Also, they performed the accuracy of bootstrap confidence intervals through calculating the coverage probability and the average length of confidence intervals. Chu and Ke [18] proposed a consistent and asymptotically normal (CAN) estimator of the mean response time for a G/M/1 queueing system, which is based on the fixed point of empirical Laplace function. Ke and Chu [19] proposed a consistent and asymptotically normal estimator of intensity for a queueing system with distribution-free interarrival and service times. Also, they computed confidence intervals, testing statistical hypothesis of intensity and power function associated with it in this paper. Ke and Chu [20] constructed new confidence intervals of intensity for a queueing system, which are based on different bootstrap methods. They also performed the accuracy of these bootstrap confidence intervals through calculating the coverage probability and the

expected length of confidence intervals. They first proposed bootstrapping technique and concept of relative coverage to queueing system. They studied five estimation approaches of intensity for a queueing system with distribution free inter-arrival and service times for short run. They have introduced a new measure called relative coverage to assess the efficient performances of confidence intervals.

In this paper we propose non parametric interval estimation approach to intensities ρ_1, ρ_2 for an open queueing network with feedback. In Section 2 we prove that the natural estimators $\hat{\rho}_1, \hat{\rho}_2$ of intensities ρ_1 and ρ_2 are strongly consistent and asymptotically normal (CAN). Based on the asymptotical normality of $\hat{\rho}_1, \hat{\rho}_2$, we can construct a CAN confidence interval of ρ_1 and ρ_2 . Next in Section 3 we establish the SB confidence interval of ρ_1, ρ_2 via the standard bootstrap approach. In Section 4 we developed the derivation of the BB confidence interval of ρ_1, ρ_2 in terms of the Bayesian bootstrap approach. The percentile bootstrap (PB) confidence intervals of ρ_1, ρ_2 are developed in Section 5. In Section 6 we developed the bias-corrected and accelerated bootstrap (BCaB) confidence intervals. A numerical simulation study is conducted in Section 7 to demonstrate performances of the interval estimation approaches for an open queueing network with feedback using short run data. All simulation results are shown by appropriate tables for illustrating performances of the five interval estimation approaches. Finally, we make some conclusions in Section 8.

2. Nonparametric Statistical Inference of Intensities

Let X_1 and Y_1 be nonnegative random variables represents the inter-arrival and service time at CPU node. Similarly X_2 and Y_2 be nonnegative random variables represents the inter-arrival and service time at I/O node respectively. Given that a job just completed CPU node burst, it will next request I/O node service with probability p_1 and with probability p_0 , where $p_0 = 1 - p_1$ departs from the system. The random variables at CPU node and I/O node are independent. The intensities are defined as follows:

$$\rho_1 = \frac{p_0 \mu_{Y_1}}{\mu_{X_1}} \quad \text{and} \quad \rho_2 = \frac{p_1 \mu_{Y_2}}{p_1 \mu_{X_2}}, \quad (2)$$

where μ_{X_1}, μ_{X_2} denote the mean inter-arrival times at CPU node and I/O node respectively. Similarly μ_{Y_1}, μ_{Y_2} denote the mean service times at CPU node and I/O node respectively. Equation (2) is equivalent to Equation (1).

2.1. Estimating Intensities

Assume that $X_{11}, X_{12}, \dots, X_{1n}$ is a random sample drawn

from X_1 and $p_0Y_{11}, p_0Y_{12}, \dots, p_0Y_{1n}$ is a random sample drawn from Y_1 . Let (X_{li}, p_0Y_{li}) represents inter-arrival time and service time for the i^{th} customer of CPU node. Similarly assume $p_1X_{21}, p_1X_{22}, \dots, p_1X_{2m}$ is a random sample drawn from X_2 and $p_0Y_{21}, p_0Y_{22}, \dots, p_0Y_{2m}$ is a random sample drawn from Y_2 . Let (p_1X_{2i}, p_0Y_{2i}) represents inter-arrival time and service time for the i^{th} customer of I/O node.

Let $\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2$ be the sample means of X_1, X_2, Y_1, Y_2 respectively.

$$\bar{X}_1 = \frac{1}{n} \sum_{i=1}^n X_{li}, \bar{X}_2 = \frac{1}{m} \sum_{i=1}^m p_1 X_{2i}$$

$$\bar{Y}_1 = \frac{1}{n} \sum_{i=1}^n p_0 Y_{li}, \bar{Y}_2 = \frac{1}{m} \sum_{i=1}^m p_0 Y_{2i}$$

According to the Strong Law of Large Numbers, we know that $\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2$ are strongly consistent estimator of $\mu_{X_1}, \mu_{X_2}, \mu_{Y_1}, \mu_{Y_2}$ respectively. Thus a strongly consistent estimator of intensities are given by

$$\hat{\rho}_1 = \frac{\bar{Y}_1}{\bar{X}_1}, \hat{\rho}_2 = \frac{\bar{Y}_2}{\bar{X}_2} \tag{3}$$

In practical queueing network, the true distributions of X_1, X_2, Y_1, Y_2 are rarely known, so the exact distributions of $\hat{\rho}_1, \hat{\rho}_2$ cannot be derived. But under the assumption that X_1 and Y_1, X_2 and Y_2 being independent, the asymptotic distributions of $\hat{\rho}_1, \hat{\rho}_2$ can be developed by the following procedures.

Firstly, according to the Central Limit Theorem (see [21] p. 234), we have

$$\sqrt{n}(\bar{X}_1 - \mu_{X_1}) \xrightarrow{D} N(0, \sigma_{X_1}^2) \text{ and}$$

$$\sqrt{n}(\bar{Y}_1 - p_0\mu_{Y_1}) \xrightarrow{D} N(0, \sigma_{Y_1}^2) \tag{4}$$

where $\sigma_{X_1}^2$ and $\sigma_{Y_1}^2$ are variances of X_1 and Y_1 , respectively.

Also,

$$\sqrt{m}(\bar{X}_2 - p_1\mu_{X_2}) \xrightarrow{D} N(0, \sigma_{X_2}^2) \text{ and}$$

$$\sqrt{m}(\bar{Y}_2 - p_0\mu_{Y_2}) \xrightarrow{D} N(0, \sigma_{Y_2}^2), \tag{5}$$

where $\sigma_{X_2}^2$ and $\sigma_{Y_2}^2$ are variances of X_2 and Y_2 , respectively, and \xrightarrow{D} denotes convergence in distribution.

Next note that

$$\sqrt{n}(\hat{\rho}_1 - \rho_1)$$

$$= \sqrt{n} \left(\frac{\bar{Y}_1}{\bar{X}_1} - \frac{p_0\mu_{Y_1}}{\mu_{X_1}} \right)$$

$$= \frac{\sqrt{n} \left[\mu_{X_1}(\bar{Y}_1 - p_0\mu_{Y_1}) - p_0\mu_{Y_1}(\bar{X}_1 - \mu_{X_1}) \right]}{\mu_{X_1}\bar{X}_1}$$

Also,

$$\sqrt{m}(\hat{\rho}_2 - \rho_2)$$

$$= \sqrt{m} \left(\frac{\bar{Y}_2}{\bar{X}_2} - \frac{p_0\mu_{Y_2}}{p_1\mu_{X_2}} \right) \tag{6}$$

$$= \frac{\sqrt{m} \left[p_1\mu_{X_2}(\bar{Y}_2 - p_0\mu_{Y_2}) - p_0\mu_{Y_2}(\bar{X}_2 - p_1\mu_{X_2}) \right]}{p_1\mu_{X_2}\bar{X}_2}$$

Therefore by the Slutsky's theorem [see [21] p. 227], we get

$$\sqrt{n}(\hat{\rho}_1 - \rho_1) \xrightarrow{D} N(0, \sigma_1^2),$$

where $\sigma_1^2 = \frac{\mu_{X_1}^2 \sigma_{Y_1}^2 + p_0^2 \mu_{Y_1}^2 \sigma_{X_1}^2}{\mu_{X_1}^4}$.

And

$$\sqrt{m}(\hat{\rho}_2 - \rho_2) \xrightarrow{D} N(0, \sigma_2^2) \tag{7}$$

where $\sigma_2^2 = \frac{p_1^2 \mu_{X_2}^2 \sigma_{Y_2}^2 + p_0^2 \mu_{Y_2}^2 \sigma_{X_2}^2}{p_1^2 \mu_{X_2}^4}$.

Now, set

$$\hat{\sigma}_1^2 = \frac{\bar{X}_1^2 S_{Y_1}^2 + p_0^2 \bar{Y}_1^2 S_{X_1}^2}{\bar{X}_1^4} \text{ and}$$

$$\hat{\sigma}_2^2 = \frac{p_1^2 \bar{X}_2^2 S_{Y_2}^2 + p_0^2 \bar{Y}_2^2 S_{X_2}^2}{p_1^2 \bar{X}_2^4}, \tag{8}$$

where

$$S_{X_1}^2 = \frac{1}{n} \sum_{i=1}^n (X_{li} - \bar{X}_1)^2, S_{Y_1}^2 = \frac{1}{n} \sum_{i=1}^n (p_0Y_{li} - \bar{Y}_1)^2$$

$$S_{X_2}^2 = \frac{1}{m} \sum_{i=1}^m (p_1X_{2i} - \bar{X}_2)^2 \text{ and } S_{Y_2}^2 = \frac{1}{m} \sum_{i=1}^m (p_0Y_{2i} - \bar{Y}_2)^2$$

Then $\hat{\sigma}_1^2, \hat{\sigma}_2^2$ are strongly consistent estimators of σ_1^2, σ_2^2 respectively. Applying the Slutsky's theorem once again, we deduce that

$$\frac{\sqrt{n}(\hat{\rho}_1 - \rho_1)}{\hat{\sigma}_1} \xrightarrow{D} N(0, 1) \text{ and}$$

$$\frac{\sqrt{m}(\hat{\rho}_2 - \rho_2)}{\hat{\sigma}_2} \xrightarrow{D} N(0, 1) \tag{9}$$

Thus $\hat{\rho}_1, \hat{\rho}_2$ are strongly consistent and asymptotically normal (CAN) estimators with approximate variances $\frac{\hat{\sigma}_1^2}{n}, \frac{\hat{\sigma}_2^2}{m}$ respectively.

2.2. Confidence Intervals

Using the CAN estimators $\hat{\rho}_1, \hat{\rho}_2$ and its associated approximate variances $\frac{\hat{\sigma}_1^2}{n}, \frac{\hat{\sigma}_2^2}{m}$ we construct a confi-

dence intervals of intensities ρ_1 & ρ_2 for a open queueing network with feedback. Let z_α be the upper α^{th} quantile of the standard normal distribution, by the asymptotic distribution of $\frac{\sqrt{n}(\hat{\rho}_1 - \rho_1)}{\hat{\sigma}_1}$ & $\frac{\sqrt{m}(\hat{\rho}_2 - \rho_2)}{\hat{\sigma}_2}$

in expression (9), an approximate $100(1-\alpha)\%$ confidence interval of ρ_1 & ρ_2 are obtained as

$$1-\alpha \approx P\left(-z_{\alpha/2} \leq \frac{\sqrt{n}(\hat{\rho}_1 - \rho_1)}{\hat{\sigma}_1} \leq z_{\alpha/2}\right) = P\left(\hat{\rho}_1 - \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}} \leq \rho_1 \leq \hat{\rho}_1 + \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}}\right)$$

Consequently, an approximate $100(1-\alpha)\%$ confidence interval of ρ_1 is

$$\left(\hat{\rho}_1 - \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}}, \hat{\rho}_1 + \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}}\right). \tag{10}$$

Similarly, an approximate $100(1-\alpha)\%$ confidence interval of ρ_2 is

$$\left(\hat{\rho}_2 - \frac{z_{\alpha/2}\hat{\sigma}_2}{\sqrt{m}}, \hat{\rho}_2 + \frac{z_{\alpha/2}\hat{\sigma}_2}{\sqrt{m}}\right). \tag{11}$$

3. Standard Bootstrap Confidence Intervals of Intensities

Now the bootstrap confidence intervals are developed as follows:

Let $x_{11}, x_{12}, \dots, x_{1n}$ be a random sample of n observations taken from the population X_1 and $p_{0y_{11}}, p_{0y_{12}}, \dots, p_{0y_{1n}}$ be a random sample of n observations taken from the population Y_1 . According to the bootstrap procedure, a simple random sample $x_{11}^*, x_{12}^*, \dots, x_{1n}^*$ can be taken from the empirical distribution function of $x_{11}, x_{12}, \dots, x_{1n}$ called a bootstrap sample from $x_{11}, x_{12}, \dots, x_{1n}$. Also, we can draw a bootstrap sample $p_{0y_{11}}^*, p_{0y_{12}}^*, \dots, p_{0y_{1n}}^*$ from $p_{0y_{11}}, p_{0y_{12}}, \dots, p_{0y_{1n}}$. It follows from Equation (2) that an estimate of intensity ρ_1 can be calculated from bootstrap samples as

$$\hat{\rho}_1^* = \frac{\bar{y}_1^*}{\bar{x}_1^*}, \tag{12}$$

where \bar{x}_1^* and \bar{y}_1^* are the sample means of $x_{11}^*, x_{12}^*, \dots, x_{1n}^*$ and $p_{0y_{11}}^*, p_{0y_{12}}^*, \dots, p_{0y_{1n}}^*$ respectively and $\hat{\rho}_1^*$ is called a bootstrap estimate of ρ_1 . The above resampling process can be repeated N_1 times. The N_1 bootstrap estimates $\hat{\rho}_{11}^*, \hat{\rho}_{12}^*, \dots, \hat{\rho}_{1N_1}^*$ can be computed from the bootstrap resamples. Averaging the N_1 bootstrap estimates we obtain that

$$\hat{\rho}_{N_1} = \frac{1}{N_1} \sum_{i=1}^{N_1} \hat{\rho}_{1i}^* \tag{13}$$

is the bootstrap estimate of ρ_1 . And the standard deviation of $\hat{\rho}_1$ can be estimated by

tion of $\hat{\rho}_1$ can be estimated by

$$sd(\hat{\rho}_{N_1}) = \left\{ \frac{1}{N_1 - 1} \sum_{i=1}^{N_1} (\hat{\rho}_{1i}^* - \hat{\rho}_{N_1})^2 \right\}^{1/2}. \tag{14}$$

Because the central limit theorem implies that the distribution of $\hat{\rho}_1$ is approximately normal. A $100(1-\alpha)\%$ SB confidence interval for ρ_1 is

$$\left(\hat{\rho}_1 - z_{\alpha/2}sd(\hat{\rho}_{N_1}), \hat{\rho}_1 + z_{\alpha/2}sd(\hat{\rho}_{N_1})\right), \tag{15}$$

Similarly $p_{1x_{21}}, p_{1x_{22}}, \dots, p_{1x_{2m}}$ is a random sample of m observation drawn from population X_2 and $p_{0y_{21}}, p_{0y_{22}}, \dots, p_{0y_{2m}}$ is a sample of m observations taken from the population Y_2 . According to the bootstrap procedure, a simple random sample $p_{1x_{21}}^*, p_{1x_{22}}^*, \dots, p_{1x_{2m}}^*$ can be taken from the empirical distribution function of $p_{1x_{21}}, p_{1x_{22}}, \dots, p_{1x_{2m}}$ called a bootstrap sample from $p_{1x_{21}}, p_{1x_{22}}, \dots, p_{1x_{2m}}$. Also, we can draw a bootstrap sample $p_{0y_{21}}^*, p_{0y_{22}}^*, \dots, p_{0y_{2m}}^*$ from $p_{0y_{21}}, p_{0y_{22}}, \dots, p_{0y_{2m}}$. An estimate of intensity ρ_2 can be calculated from bootstrap samples as

$$\hat{\rho}_2^* = \frac{\bar{y}_2^*}{\bar{x}_2^*}, \tag{16}$$

where \bar{x}_2^* and \bar{y}_2^* are the sample means of $p_{1x_{21}}^*, p_{1x_{22}}^*, \dots, p_{1x_{2m}}^*$ and $p_{0y_{21}}^*, p_{0y_{22}}^*, \dots, p_{0y_{2m}}^*$ respectively and $\hat{\rho}_2^*$ is called a bootstrap estimate of ρ_2 . The above resampling process can be repeated M_1 times. The M_1 bootstrap estimates $\hat{\rho}_{21}^*, \hat{\rho}_{22}^*, \dots, \hat{\rho}_{2M_1}^*$ can be computed from the bootstrap resamples. Averaging the M_1 bootstrap estimates we obtain that

$$\hat{\rho}_{M_1} = \frac{1}{M_1} \sum_{i=1}^{M_1} \hat{\rho}_{2i}^* \tag{17}$$

is the bootstrap estimate of ρ_2 . And the standard deviation of $\hat{\rho}_2$ can be estimated by

$$sd(\hat{\rho}_{M_1}) = \left\{ \frac{1}{M_1 - 1} \sum_{i=1}^M (\hat{\rho}_{2i}^* - \hat{\rho}_{M_1})^2 \right\}^{1/2}. \tag{18}$$

Because the central limit theorem implies that the distribution of $\hat{\rho}_2$ is approximately normal. A $100(1-\alpha)\%$ SB confidence interval for ρ_2 is

$$\left(\hat{\rho}_2 - z_{\alpha/2}sd(\hat{\rho}_{M_1}), \hat{\rho}_2 + z_{\alpha/2}sd(\hat{\rho}_{M_1})\right), \tag{19}$$

4. Bayesian Bootstrap Confidence Intervals of Intensities

The Bayesian bootstrap is analogous to the standard bootstrap. Each BB replication generates a posterior probability for each x_i . Specifically, one BB replication is generated by drawing $n-1$ uniform (0,1) random numbers r_1, r_2, \dots, r_{n-1} , ordering them, and calculating the gaps $w_i = r_{(i)} - r_{(i-1)}$, $i = 1, 2, \dots, n$, where $r_{(0)} = 0$ and

$r_{(n)} = 1$. Then $w_{1i} = (w_{11}, w_{12}, \dots, w_{1n})$ is the vector of probabilities attached to the inter-arrival data values $x_{11}, x_{12}, \dots, x_{1n}$ in that BB replication. Considering all BB replications gives the BB distribution of the distribution of X_1 and thus of any parameter of this distribution. Hence for μ_{X_1} (the mean of X_1), in each BB replication we calculate μ_{X_1} as if w_{1i} were the probability that $X_1 = x_i$ that is, we calculate $\bar{x}_1^{**} = \sum_{i=1}^n w_{1i} x_{1i}$. The distribution of the values of \bar{x}_1^{**} overall BB replications is the BB distribution of μ_{X_1} .

Also, generating a vector of probabilities $v_1 = (v_{11}, v_{12}, \dots, v_{1n})$ attached to the service time data values $p_0 y_{11}, p_0 y_{12}, \dots, p_0 y_{1n}$ in a BB replication, and we calculate $\bar{y}_1^{**} = p_0 \sum_{i=1}^n v_{1i} y_{1i}$ for μ_{Y_1} (the mean of Y_1).

Then in terms of equation (2) an estimate of intensity ρ_1 can be calculated from BB replications as

$$\hat{\rho}_1^{**} = \frac{\bar{y}_1^{**}}{\bar{x}_1^{**}}, \tag{20}$$

where $\hat{\rho}_1^{**}$ is called a Bayesian bootstrap estimate of ρ_1 . The above BB process can be repeated N_1 times. The N_1 BB estimates, $\hat{\rho}_{11}^{**}, \hat{\rho}_{12}^{**}, \dots, \hat{\rho}_{1N_1}^{**}$ can be computed from the BB replications. Averaging the N_1 BB estimates, we obtain that

$$\hat{\rho}'_{BB} = \frac{1}{N_1} \sum_{j=1}^{N_1} \hat{\rho}_{1j}^{**}, \tag{21}$$

is the BB estimate of ρ_1 . And the standard deviation of $\hat{\rho}'_{BB}$ can be estimated by

$$sd(\hat{\rho}'_{BB}) = \left\{ \frac{1}{N_1 - 1} \sum_{j=1}^{N_1} (\hat{\rho}_{1j}^{**} - \hat{\rho}'_{BB})^2 \right\}^{\frac{1}{2}}. \tag{22}$$

Applying the asymptotical normality of $\hat{\rho}'_{BB}$, a $100(1-\alpha)\%$ BB confidence interval for ρ_1 is

$$(\hat{\rho}_1 - z_{\alpha/2} sd(\hat{\rho}'_{BB}), \hat{\rho}_1 + z_{\alpha/2} sd(\hat{\rho}'_{BB})). \tag{23}$$

Similarly each BB replication generates a posterior probability for each x_{2i} . Specifically, one BB replication is generated by drawing $m-1$ uniform (0,1) random numbers r_1, r_2, \dots, r_{m-1} , ordering them and calculating the gaps $w_{2i} = r_{(i)} - r_{(i-1)}, i = 1, 2, \dots, m$, where $r_{(0)} = 0$ and $r_{(m)} = 1$. Then $w_{2i} = (w_{21}, w_{22}, \dots, w_{2m})$ is the vector of probabilities attached to the inter-arrival data values $p_1 x_{21}, p_1 x_{22}, \dots, p_1 x_{2m}$ in that BB replication. Considering all BB replications gives the BB distribution of the distribution of X_2 and thus of any parameter of this distribution. Hence for μ_{X_2} (the mean of X_2), in each BB replication we calculate μ_{X_2} as if w_{2i} were the probability that $X_2 = x_{2i}$ that is, we calculate

$$\bar{x}_2^{**} = p_1 \sum_{i=1}^m w_{2i} x_{2i}. \text{ The distribution of the values of } \bar{x}_2^{**}$$

over all BB replications is the BB distribution of μ_{X_2} .

Also, generating a vector of probabilities $v_2 = (v_{21}, v_{22}, \dots, v_{2m})$ attached to the service time data values $p_0 y_{21}, p_0 y_{22}, \dots, p_0 y_{2m}$ in a BB replication, and we calculate $\bar{y}_2^{**} = p_0 \sum_{i=1}^m v_{2i} y_{2i}$ for μ_{Y_2} (the mean of Y_2).

Then in terms of Equation (2) an estimate of intensity ρ_2 can be calculated from BB replications as

$$\hat{\rho}_2^{**} = \frac{\bar{y}_2^{**}}{\bar{x}_2^{**}}, \tag{24}$$

where $\hat{\rho}_2^{**}$ is called a Bayesian bootstrap estimate of ρ_2 . The above BB process can be repeated M_1 times. The M_1 BB estimates, $\hat{\rho}_{21}^{**}, \hat{\rho}_{22}^{**}, \dots, \hat{\rho}_{2M_1}^{**}$ can be computed from the BB replications. Averaging the M_1 BB estimates, we obtain that

$$\hat{\rho}''_{BB} = \frac{1}{M_1} \sum_{j=1}^{M_1} \hat{\rho}_{2j}^{**}, \tag{25}$$

is the BB estimate of ρ_2 . And the standard deviation of $\hat{\rho}''_{BB}$ can be estimated by

$$sd(\hat{\rho}''_{BB}) = \left\{ \frac{1}{M_1 - 1} \sum_{j=1}^{M_1} (\hat{\rho}_{2j}^{**} - \hat{\rho}''_{BB})^2 \right\}. \tag{26}$$

Applying the asymptotical normality of $\hat{\rho}''_{BB}$, a $100(1-\alpha)\%$ BB confidence interval for ρ_2 is

$$(\hat{\rho}_2 - z_{\alpha/2} sd(\hat{\rho}''_{BB}), \hat{\rho}_2 + z_{\alpha/2} sd(\hat{\rho}''_{BB})) \tag{27}$$

5. Percentile Bootstrap Confidence Intervals of Intensities

Let $\hat{\rho}_{11}^*, \hat{\rho}_{12}^*, \dots, \hat{\rho}_{1N_1}^*$ and $\hat{\rho}_{21}^*, \hat{\rho}_{22}^*, \dots, \hat{\rho}_{2M_1}^*$ call the bootstrap distribution of $\hat{\rho}_1, \hat{\rho}_2$ respectively. Let $\hat{\rho}_1^*(1), \hat{\rho}_1^*(2), \dots, \hat{\rho}_1^*(N_1)$ and $\hat{\rho}_2^*(1), \hat{\rho}_2^*(2), \dots, \hat{\rho}_2^*(M_1)$ be order statistics of $\hat{\rho}_{11}^*, \hat{\rho}_{12}^*, \dots, \hat{\rho}_{1N_1}^*$ and $\hat{\rho}_{21}^*, \hat{\rho}_{22}^*, \dots, \hat{\rho}_{2M_1}^*$ respectively. Then utilizing the $100(\alpha/2)$ th and $100(1-\alpha/2)$ th percentage points of the bootstrap distribution, a $100(1-\alpha)\%$ PB confidence interval for ρ_1, ρ_2 are obtained as

$$\left(\hat{\rho}_1^* \left(\left[N_1 \left(\frac{\alpha}{2} \right) \right] \right), \hat{\rho}_1^* \left(\left[N_1 \left(1 - \frac{\alpha}{2} \right) \right] \right) \right), \tag{28}$$

$$\left(\hat{\rho}_2^* \left(\left[M_1 \left(\frac{\alpha}{2} \right) \right] \right), \hat{\rho}_2^* \left(\left[M_1 \left(1 - \frac{\alpha}{2} \right) \right] \right) \right), \tag{29}$$

where $[x]$ denotes the greatest integer less than or equal to x .

6. Bias-Corrected and Accelerated Bootstrap Confidence Intervals of Intensities

The bootstrap distribution $\hat{\rho}_{11}^*, \hat{\rho}_{12}^*, \dots, \hat{\rho}_{1N_1}^*$ and $\hat{\rho}_{21}^*$,

$\hat{\rho}_{22}^*, \dots, \hat{\rho}_{2M_1}^*$ may be biased. This method is designed to correct this potential bias of the bootstrap designed. Set

$$p' = \sum_{j=1}^{N_1} \frac{I(\hat{\rho}_{1j}^* < \hat{\rho}_1)}{N_1} \quad \text{and} \quad p'' = \sum_{j=1}^{M_1} \frac{I(\hat{\rho}_{2j}^* < \hat{\rho}_2)}{M_1},$$

where $I(\cdot)$ is the indicator function. Define $\hat{z}_0 = \phi^{-1}(p')$ and $\hat{z}_1 = \phi^{-1}(p'')$, where ϕ^{-1} denotes the inverse function of the standard normal distribution ϕ . Except for correcting the potential bias of the bootstrap distribution, we can accelerate convergence of bootstrap distribution. Let $\tilde{X}_1(i)$ and $p_0\tilde{Y}_1(i)$ denote the original samples with the i^{th} observation x_{1i} and p_0y_{1i} deleted, also let $\hat{\rho}_{1i}$ be the estimator of ρ_1 calculated by using $\tilde{X}_1(i)$ and $p_0\tilde{Y}_1(i)$. Define $\tilde{\rho}_1 = \frac{1}{n} \sum_{i=1}^n \hat{\rho}_{1i}$, Similarly $p_1\tilde{X}_2(i)$ and $p_0\tilde{Y}_2(i)$ denote the original samples with the i^{th} observation p_1x_{2i} and p_0y_{2i} deleted, also $\hat{\rho}_{2i}$ be the estimator of ρ_2 calculated by using $p_1\tilde{X}_2(i)$ and $p_0\tilde{Y}_2(i)$.

Define $\tilde{\rho}_2 = \frac{1}{m} \sum_{i=1}^m \hat{\rho}_{2i}$

And

$$\hat{a}_1 = \frac{\sum_{i=1}^n (\tilde{\rho}_1 - \hat{\rho}_{1i})^3}{\left\{ 6 \left(\sum_{i=1}^n (\tilde{\rho}_1 - \hat{\rho}_{1i})^2 \right)^{\left(\frac{3}{2}\right)} \right\}},$$

$$\hat{a}_2 = \frac{\sum_{i=1}^m (\tilde{\rho}_2 - \hat{\rho}_{2i})^3}{\left\{ 6 \left(\sum_{i=1}^m (\tilde{\rho}_2 - \hat{\rho}_{2i})^2 \right)^{\left(\frac{3}{2}\right)} \right\}} \tag{30}$$

where $\hat{z}_0, \hat{z}_1, \hat{a}_1$, and \hat{a}_2 are named bias-correction and acceleration respectively.

Thus a $100(1-\alpha)\%$ Bias-corrected and accelerated bootstrap (BCaB) Confidence Interval of intensities ρ_1, ρ_2 are constructed by

$$\left(\hat{\rho}_1^*([N_1\alpha_1]), \hat{\rho}_1^*([N_1\alpha_2]) \right) \tag{31}$$

$$\left(\hat{\rho}_2^*([M_1\alpha_3]), \hat{\rho}_2^*([M_1\alpha_4]) \right) \tag{32}$$

where

$$\alpha_1 = \phi \left\{ \hat{z}_0 + \frac{(\hat{z}_0 - z_{\alpha/2})}{1 - \hat{a}_1(\hat{z}_0 - z_{\alpha/2})} \right\}$$

$$\alpha_2 = \phi \left\{ \hat{z}_0 + \frac{(\hat{z}_0 + z_{\alpha/2})}{1 - \hat{a}_1(\hat{z}_0 + z_{\alpha/2})} \right\}$$

$$\alpha_3 = \phi \left\{ \hat{z}_1 + \frac{(\hat{z}_1 - z_{\alpha/2})}{1 - \hat{a}_2(\hat{z}_1 - z_{\alpha/2})} \right\}$$

$$\alpha_4 = \phi \left\{ \hat{z}_1 + \frac{(\hat{z}_1 + z_{\alpha/2})}{1 - \hat{a}_2(\hat{z}_1 + z_{\alpha/2})} \right\}$$

7. Simulation Study

To evaluate performances of the different interval estimation approaches mentioned above for an open queueing network with feedback using short run data, a numerical simulation study was undertaken. Most of the statisticians assess performances of interval estimations in terms of coverage percentages or average lengths of confidence intervals. However, through simulation study in the research work, we find that larger coverage percentages of confidence interval may often be due to wider standard deviation of interval estimation methods. Moreover, narrower confidence intervals may often lead to smaller coverage percentages. Hence, both coverage percentage and average length are not efficient for appraising interval estimation methods. In order to overcome above two shortcomings, we propose a measure called relative coverage to evaluate performances of interval estimation methods where,

$$\text{Relative coverage} = \frac{\text{Coverage percentage}}{\text{Average length}}.$$

The larger of the relative coverage implies the better performance of the corresponding confidence interval. In order to reach this goal, we set a continuous distribution with mean $1/\lambda$ on inter-arrival time X_1 and X_2 . Also set continuous distribution with mean $1/\mu_1$ on the service time Y_1 at CPU node and continuous distribution with mean $1/\mu_2$ on the service time Y_2 at I/O node. The levels of p_0 considered in the simulation study are 0.1 to 0.9 where as levels of p_1 are 0.9 to 0.1, where p_0 is the probability that the job departs from the system and p_1 is the probability that after service completion at CPU node, the job proceeds to the I/O node. This means with probability $p_0 = 0.1$ the job departs from the system and with probability $p_1 = 0.9$, after service completion at CPU node, the job proceeds to the I/O node and so on. Also we have considered the values of ρ_1 and ρ_2 such that $\rho_1 < 1$ and $\rho_2 < 1$ for simulation study. Note that in **Table 1** wherever $\rho_1 \geq 1$ and $\rho_2 \geq 1$ such values of ρ_1 and ρ_2 are not considered for simulation study. The intensity parameters ρ_1 and ρ_2 are calculated using Equation (1). The different values of $\lambda, \mu_1, \mu_2, p_0$ and p_1 are considered for simulation study as shown in **Table 1**.

For different levels of ρ_1 , random samples of inter-arrival times $X_{11}, X_{12}, \dots, X_{1n}$ and service times $p_0Y_{11},$

Table 1. Different levels of intensity parameters considered in the simulation study.

| $\lambda = 0.1, \mu_1 = 1, \mu_2 = 1$ | | | | $\lambda = 0.1, \mu_1 = 1, \mu_2 = 2$ | | | | $\lambda = 0.1, \mu_1 = 2, \mu_2 = 1$ | | | |
|---------------------------------------|-------|----------|----------|---------------------------------------|-------|----------|----------|---------------------------------------|-------|----------|----------|
| p_0 | p_1 | ρ_1 | ρ_2 | p_0 | p_1 | ρ_1 | ρ_2 | p_0 | p_1 | ρ_1 | ρ_2 |
| 0.1 | 0.9 | 1 | 0.9 | 0.1 | 0.9 | 1 | 0.45 | 0.1 | 0.9 | 0.5 | 0.9 |
| 0.2 | 0.8 | 0.5 | 0.4 | 0.2 | 0.8 | 0.5 | 0.2 | 0.2 | 0.8 | 0.25 | 0.4 |
| 0.3 | 0.7 | 0.33 | 0.23 | 0.3 | 0.7 | 0.33 | 0.12 | 0.3 | 0.7 | 0.17 | 0.23 |
| 0.4 | 0.6 | 0.25 | 0.15 | 0.4 | 0.6 | 0.25 | 0.08 | 0.4 | 0.6 | 0.13 | 0.15 |
| 0.5 | 0.5 | 0.2 | 0.1 | 0.5 | 0.5 | 0.2 | 0.05 | 0.5 | 0.5 | 0.1 | 0.1 |
| 0.6 | 0.4 | 0.17 | 0.07 | 0.6 | 0.4 | 0.17 | 0.03 | 0.6 | 0.4 | 0.08 | 0.07 |
| 0.7 | 0.3 | 0.14 | 0.04 | 0.7 | 0.3 | 0.14 | 0.02 | 0.7 | 0.3 | 0.07 | 0.04 |
| 0.8 | 0.2 | 0.13 | 0.03 | 0.8 | 0.2 | 0.13 | 0.01 | 0.8 | 0.2 | 0.06 | 0.03 |
| 0.9 | 0.1 | 0.11 | 0.01 | 0.9 | 0.1 | 0.11 | 0.01 | 0.9 | 0.1 | 0.06 | 0.01 |
| $\lambda = 0.5, \mu_1 = 1, \mu_2 = 1$ | | | | $\lambda = 0.5, \mu_1 = 1, \mu_2 = 2$ | | | | $\lambda = 0.5, \mu_1 = 2, \mu_2 = 1$ | | | |
| p_0 | p_1 | ρ_1 | ρ_2 | p_0 | p_1 | ρ_1 | ρ_2 | p_0 | p_1 | ρ_1 | ρ_2 |
| 0.1 | 0.9 | 5 | 4.5 | 0.1 | 0.9 | 5 | 2.25 | 0.1 | 0.9 | 2.5 | 4.5 |
| 0.2 | 0.8 | 2.5 | 2 | 0.2 | 0.8 | 2.5 | 1 | 0.2 | 0.8 | 1.25 | 2 |
| 0.3 | 0.7 | 1.67 | 1.17 | 0.3 | 0.7 | 1.67 | 0.58 | 0.3 | 0.7 | 0.83 | 1.17 |
| 0.4 | 0.6 | 1.25 | 0.75 | 0.4 | 0.6 | 1.25 | 0.38 | 0.4 | 0.6 | 0.63 | 0.75 |
| 0.5 | 0.5 | 1 | 0.5 | 0.5 | 0.5 | 1 | 0.25 | 0.5 | 0.5 | 0.5 | 0.5 |
| 0.6 | 0.4 | 0.83 | 0.33 | 0.6 | 0.4 | 0.83 | 0.17 | 0.6 | 0.4 | 0.42 | 0.33 |
| 0.7 | 0.3 | 0.71 | 0.21 | 0.7 | 0.3 | 0.71 | 0.11 | 0.7 | 0.3 | 0.36 | 0.21 |
| 0.8 | 0.2 | 0.63 | 0.13 | 0.8 | 0.2 | 0.63 | 0.06 | 0.8 | 0.2 | 0.31 | 0.13 |
| 0.9 | 0.1 | 0.56 | 0.06 | 0.9 | 0.1 | 0.56 | 0.03 | 0.9 | 0.1 | 0.28 | 0.06 |
| $\lambda = 0.9, \mu_1 = 1, \mu_2 = 1$ | | | | $\lambda = 0.9, \mu_1 = 1, \mu_2 = 2$ | | | | $\lambda = 0.9, \mu_1 = 2, \mu_2 = 1$ | | | |
| p_0 | p_1 | ρ_1 | ρ_2 | p_0 | p_1 | ρ_1 | ρ_2 | p_0 | p_1 | ρ_1 | ρ_2 |
| 0.1 | 0.9 | 9 | 8.1 | 0.1 | 0.9 | 9 | 4.05 | 0.1 | 0.9 | 4.5 | 8.1 |
| 0.2 | 0.8 | 4.5 | 3.6 | 0.2 | 0.8 | 4.5 | 1.8 | 0.2 | 0.8 | 2.25 | 3.6 |
| 0.3 | 0.7 | 3 | 2.1 | 0.3 | 0.7 | 3 | 1.05 | 0.3 | 0.7 | 1.5 | 2.1 |
| 0.4 | 0.6 | 2.25 | 1.35 | 0.4 | 0.6 | 2.25 | 0.68 | 0.4 | 0.6 | 1.13 | 1.35 |
| 0.5 | 0.5 | 1.8 | 0.9 | 0.5 | 0.5 | 1.8 | 0.45 | 0.5 | 0.5 | 0.9 | 0.9 |
| 0.6 | 0.4 | 1.5 | 0.6 | 0.6 | 0.4 | 1.5 | 0.3 | 0.6 | 0.4 | 0.75 | 0.6 |
| 0.7 | 0.3 | 1.29 | 0.39 | 0.7 | 0.3 | 1.29 | 0.19 | 0.7 | 0.3 | 0.64 | 0.39 |
| 0.8 | 0.2 | 1.13 | 0.23 | 0.8 | 0.2 | 1.13 | 0.11 | 0.8 | 0.2 | 0.56 | 0.23 |
| 0.9 | 0.1 | 1 | 0.1 | 0.9 | 0.1 | 1 | 0.05 | 0.9 | 0.1 | 0.5 | 0.1 |

$p_0Y_{12}, \dots, p_0Y_{1n}$ are drawn from X_1 and Y_1 respectively. Also for each level of ρ_2 random samples of inter-arrival times $p_1X_{21}, p_1X_{22}, \dots, p_1X_{2m}$ and service times $p_0Y_{21}, p_0Y_{22}, \dots, p_0Y_{2m}$ are drawn from X_2 and Y_2 respectively. Next $N = 1000$ bootstrap resamples each of size n and $m = 10, 20, 29$ are drawn from the original samples, as well as $N = 1000$ BB replications are simulated for the original samples. According to Equations (10), (11), (15), (19), (23), (27)-(29), (31) and (32) in respective, we obtain CAN1, CAN2, SB1, SB2, BB1, BB2, PB1, PB2, BCaB1 and BCaB2 confidence intervals of intensities ρ_1 and ρ_2 with confidence level 90%. The above simulation process is replicated $N = 1000$ times and we compute coverage percentages, average lengths and relative coverage of the above mentioned confidence intervals. We utilize a PC Dual Core and apply Matlab[®]7.0.1 to accomplish all simulations.

Here M represents exponential distribution, E_4 a 4-stage Erlang distribution, H_4^{Pe} a 4-stage hyper-expo-

ponential distribution and H_4^{Po} a 4-stage hypo-exponential distribution.

Based on the above mentioned interval estimation approaches, the coverage percentage, average lengths and relative coverage of intensities ρ_1 and ρ_2 are shown in **Tables 3 to 7** for queueing network models (presented in **Table 2**) with short run data, we find that average lengths

Table 2. Different queueing network models simulated for study.

| Queueing Networks | Models Simulated |
|--------------------|--|
| $M/G/1$ to $G/M/1$ | $M/E_4/1$ to $E_4/M/1$ |
| | $M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$ |
| $G/G/1$ to $G/G/1$ | $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$ |
| | $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$ |
| | $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$ |

Table 3. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network. $M/E_4/1$ to $E_4/M/1$.

| Intensity Parameters | Estimation Approach | Coverage Percentage | | | Average Length | | | Relative Coverage | | |
|---|---------------------|---------------------|----------|----------|----------------|----------|----------|-------------------|----------------|----------------|
| | | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ |
| $p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$ | CAN1 | 0.878 | 0.895 | 0.878 | 0.611 | 0.416 | 0.345 | 1.437 | 2.151 | 2.548 |
| | CAN2 | 0.840 | 0.881 | 0.871 | 0.220 | 0.162 | 0.135 | 3.816 | 5.431 | 6.465 |
| | SB1 | 0.916 | 0.910 | 0.900 | 0.748 | 0.455 | 0.365 | 1.225 | 2.002 | 2.465 |
| | SB2 | 0.835 | 0.875 | 0.869 | 0.217 | 0.161 | 0.134 | 3.843 | 5.425 | 6.480 |
| | BB1 | 0.879 | 0.898 | 0.877 | 0.628 | 0.418 | 0.344 | 1.399 | 2.148 | 2.547 |
| | BB2 | 0.817 | 0.864 | 0.856 | 0.205 | 0.157 | 0.131 | 3.978 | 5.517 | 6.514 |
| | PB1 | 0.832 | 0.874 | 0.867 | 0.687 | 0.438 | 0.356 | 1.211 | 1.994 | 2.437 |
| | PB2 | 0.831 | 0.874 | 0.869 | 0.214 | 0.159 | 0.133 | 3.892 | 5.490 | 6.537 |
| | BCaB1 | 0.831 | 0.877 | 0.872 | 0.668 | 0.433 | 0.353 | 1.243 | 2.026 | 2.473 |
| BCaB2 | 0.837 | 0.871 | 0.871 | 0.214 | 0.160 | 0.133 | 3.913 | 5.454 | 6.535 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.11$ and $\rho_2 = 0.01$ | CAN1 | 0.870 | 0.885 | 0.868 | 0.139 | 0.092 | 0.076 | 6.276 | 9.648 | 11.410 |
| | CAN2 | 0.832 | 0.878 | 0.867 | 0.006 | 0.004 | 0.004 | 134.515 | 195.434 | 232.230 |
| | SB1 | 0.910 | 0.912 | 0.887 | 0.171 | 0.100 | 0.081 | 5.310 | 9.116 | 11.013 |
| | SB2 | 0.827 | 0.879 | 0.866 | 0.006 | 0.004 | 0.004 | 135.068 | 196.738 | 232.756 |
| | BB1 | 0.879 | 0.882 | 0.867 | 0.143 | 0.092 | 0.076 | 6.138 | 9.586 | 11.400 |
| | BB2 | 0.807 | 0.871 | 0.858 | 0.006 | 0.004 | 0.004 | 139.854 | 200.876 | 235.214 |
| | PB1 | 0.824 | 0.871 | 0.859 | 0.156 | 0.096 | 0.078 | 5.270 | 9.031 | 10.946 |
| | PB2 | 0.826 | 0.869 | 0.866 | 0.006 | 0.004 | 0.004 | 137.508 | 197.084 | 235.030 |
| | BCaB1 | 0.833 | 0.872 | 0.868 | 0.152 | 0.095 | 0.078 | 5.478 | 9.169 | 11.159 |
| BCaB2 | 0.823 | 0.871 | 0.865 | 0.006 | 0.004 | 0.004 | 136.752 | 196.995 | 234.219 | |

Continued

| | | | | | | | | | | |
|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$ | CAN1 | 0.851 | 0.876 | 0.866 | 1.012 | 0.697 | 0.579 | 0.841 | 1.258 | 1.496 |
| | CAN2 | 0.829 | 0.891 | 0.891 | 0.186 | 0.135 | 0.112 | 4.459 | 6.600 | 7.935 |
| | SB1 | 0.893 | 0.894 | 0.882 | 1.249 | 0.760 | 0.613 | 0.715 | 1.176 | 1.438 |
| | SB2 | 0.826 | 0.890 | 0.887 | 0.184 | 0.134 | 0.112 | 4.492 | 6.631 | 7.925 |
| | BB1 | 0.856 | 0.879 | 0.865 | 1.045 | 0.700 | 0.578 | 0.819 | 1.256 | 1.496 |
| | BB2 | 0.814 | 0.886 | 0.878 | 0.173 | 0.131 | 0.110 | 4.693 | 6.783 | 8.010 |
| | PB1 | 0.827 | 0.866 | 0.848 | 1.135 | 0.732 | 0.598 | 0.729 | 1.184 | 1.418 |
| | PB2 | 0.830 | 0.884 | 0.886 | 0.181 | 0.132 | 0.111 | 4.595 | 6.673 | 7.993 |
| | BCaB1 | 0.827 | 0.866 | 0.848 | 1.108 | 0.722 | 0.593 | 0.747 | 1.199 | 1.431 |
| BCaB2 | 0.824 | 0.883 | 0.881 | 0.181 | 0.133 | 0.111 | 4.553 | 6.636 | 7.934 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$ | CAN1 | 0.882 | 0.891 | 0.879 | 0.682 | 0.470 | 0.384 | 1.293 | 1.895 | 2.286 |
| | CAN2 | 0.859 | 0.861 | 0.875 | 0.031 | 0.022 | 0.019 | 27.501 | 38.453 | 46.779 |
| | SB1 | 0.915 | 0.912 | 0.893 | 0.841 | 0.515 | 0.408 | 1.088 | 1.771 | 2.189 |
| | SB2 | 0.850 | 0.858 | 0.873 | 0.031 | 0.022 | 0.019 | 27.505 | 38.524 | 46.903 |
| | BB1 | 0.889 | 0.892 | 0.878 | 0.704 | 0.472 | 0.385 | 1.262 | 1.889 | 2.283 |
| | BB2 | 0.831 | 0.850 | 0.867 | 0.029 | 0.022 | 0.018 | 28.516 | 39.337 | 47.492 |
| | PB1 | 0.847 | 0.861 | 0.852 | 0.767 | 0.496 | 0.397 | 1.104 | 1.737 | 2.144 |
| | PB2 | 0.850 | 0.848 | 0.877 | 0.030 | 0.022 | 0.018 | 28.019 | 38.473 | 47.602 |
| | BCaB1 | 0.846 | 0.856 | 0.852 | 0.746 | 0.490 | 0.394 | 1.133 | 1.749 | 2.163 |
| BCaB2 | 0.857 | 0.850 | 0.877 | 0.030 | 0.022 | 0.018 | 28.158 | 38.438 | 47.496 | |
| $p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$ | CAN1 | 0.842 | 0.893 | 0.880 | 0.588 | 0.418 | 0.342 | 1.431 | 2.138 | 2.570 |
| | CAN2 | 0.842 | 0.856 | 0.879 | 1.016 | 0.730 | 0.618 | 0.829 | 1.173 | 1.423 |
| | SB1 | 0.894 | 0.909 | 0.899 | 0.716 | 0.456 | 0.362 | 1.249 | 1.993 | 2.481 |
| | SB2 | 0.839 | 0.856 | 0.877 | 1.005 | 0.725 | 0.615 | 0.835 | 1.181 | 1.425 |
| | BB1 | 0.850 | 0.890 | 0.875 | 0.605 | 0.420 | 0.342 | 1.406 | 2.119 | 2.560 |
| | BB2 | 0.830 | 0.847 | 0.874 | 0.948 | 0.705 | 0.603 | 0.875 | 1.201 | 1.449 |
| | PB1 | 0.810 | 0.866 | 0.874 | 0.658 | 0.439 | 0.353 | 1.230 | 1.972 | 2.475 |
| | PB2 | 0.834 | 0.846 | 0.870 | 0.986 | 0.717 | 0.610 | 0.846 | 1.180 | 1.427 |
| | BCaB1 | 0.810 | 0.865 | 0.875 | 0.639 | 0.433 | 0.349 | 1.267 | 1.998 | 2.504 |
| BCaB2 | 0.835 | 0.842 | 0.871 | 0.990 | 0.719 | 0.611 | 0.843 | 1.172 | 1.425 | |

Continued

| | | | | | | | | | | |
|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$ | CAN1 | 0.841 | 0.868 | 0.878 | 0.764 | 0.523 | 0.431 | 1.101 | 1.659 | 2.037 |
| | CAN2 | 0.851 | 0.863 | 0.875 | 0.837 | 0.600 | 0.505 | 1.016 | 1.439 | 1.732 |
| | SB1 | 0.891 | 0.888 | 0.890 | 0.944 | 0.572 | 0.455 | 0.944 | 1.552 | 1.955 |
| | SB2 | 0.850 | 0.859 | 0.875 | 0.828 | 0.596 | 0.503 | 1.026 | 1.440 | 1.739 |
| | BB1 | 0.849 | 0.871 | 0.873 | 0.790 | 0.526 | 0.431 | 1.075 | 1.657 | 2.025 |
| | BB2 | 0.832 | 0.849 | 0.864 | 0.781 | 0.580 | 0.494 | 1.066 | 1.465 | 1.748 |
| | PB1 | 0.810 | 0.849 | 0.863 | 0.864 | 0.551 | 0.444 | 0.937 | 1.540 | 1.943 |
| | PB2 | 0.843 | 0.854 | 0.881 | 0.813 | 0.589 | 0.498 | 1.037 | 1.449 | 1.767 |
| | BCaB1 | 0.817 | 0.853 | 0.862 | 0.840 | 0.544 | 0.441 | 0.973 | 1.569 | 1.957 |
| BCaB2 | 0.841 | 0.854 | 0.869 | 0.814 | 0.592 | 0.499 | 1.033 | 1.443 | 1.740 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$ | CAN1 | 0.853 | 0.887 | 0.896 | 0.334 | 0.231 | 0.193 | 2.554 | 3.841 | 4.639 |
| | CAN2 | 0.831 | 0.873 | 0.862 | 0.063 | 0.045 | 0.037 | 13.279 | 19.278 | 23.249 |
| | SB1 | 0.893 | 0.906 | 0.911 | 0.411 | 0.251 | 0.205 | 2.171 | 3.611 | 4.452 |
| | SB2 | 0.828 | 0.869 | 0.859 | 0.062 | 0.045 | 0.037 | 13.405 | 19.311 | 23.232 |
| | BB1 | 0.856 | 0.890 | 0.897 | 0.345 | 0.232 | 0.193 | 2.484 | 3.844 | 4.645 |
| | BB2 | 0.815 | 0.861 | 0.861 | 0.058 | 0.044 | 0.036 | 13.972 | 19.708 | 23.773 |
| | PB1 | 0.824 | 0.858 | 0.877 | 0.377 | 0.242 | 0.199 | 2.187 | 3.546 | 4.398 |
| | PB2 | 0.838 | 0.872 | 0.861 | 0.061 | 0.044 | 0.037 | 13.825 | 19.620 | 23.521 |
| | BCaB1 | 0.830 | 0.857 | 0.880 | 0.364 | 0.238 | 0.198 | 2.278 | 3.595 | 4.449 |
| BCaB2 | 0.837 | 0.873 | 0.862 | 0.061 | 0.045 | 0.037 | 13.803 | 19.574 | 23.512 | |

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 4. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queuing network. $M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$.

| Intensity Parameters | Estimation Approach | Coverage Percentage | | | Average Length | | | Relative Coverage | | |
|---|---------------------|---------------------|----------|----------|----------------|----------|----------|-------------------|--------------|--------------|
| | | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ |
| $p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$ | CAN1 | 0.875 | 0.876 | 0.903 | 0.620 | 0.432 | 0.359 | 1.412 | 2.026 | 2.514 |
| | CAN2 | 0.825 | 0.870 | 0.881 | 0.234 | 0.169 | 0.140 | 3.530 | 5.155 | 6.275 |
| | SB1 | 0.910 | 0.896 | 0.916 | 0.752 | 0.472 | 0.380 | 1.210 | 1.900 | 2.413 |
| | SB2 | 0.826 | 0.869 | 0.882 | 0.235 | 0.169 | 0.140 | 3.512 | 5.147 | 6.279 |
| | BB1 | 0.876 | 0.875 | 0.899 | 0.635 | 0.435 | 0.359 | 1.379 | 2.011 | 2.502 |
| | BB2 | 0.802 | 0.859 | 0.874 | 0.220 | 0.163 | 0.137 | 3.649 | 5.256 | 6.359 |
| | PB1 | 0.845 | 0.874 | 0.895 | 0.688 | 0.454 | 0.371 | 1.228 | 1.923 | 2.414 |
| | PB2 | 0.836 | 0.864 | 0.880 | 0.229 | 0.167 | 0.139 | 3.645 | 5.185 | 6.331 |
| | BCaB1 | 0.841 | 0.875 | 0.891 | 0.669 | 0.448 | 0.367 | 1.257 | 1.951 | 2.427 |
| BCaB2 | 0.830 | 0.860 | 0.875 | 0.229 | 0.167 | 0.139 | 3.618 | 5.150 | 6.295 | |

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|--|-------|-------|-------|-------|-------|-------|---------|----------------|----------------|----------------|
| $p_0 = 0.9$ $p_1 = 0.1$ and $\rho_2 = 0.01$ | CAN1 | 0.887 | 0.900 | 0.885 | 0.140 | 0.097 | 0.080 | 6.315 | 9.299 | 11.041 |
| | CAN2 | 0.824 | 0.866 | 0.884 | 0.006 | 0.005 | 0.004 | 127.283 | 186.510 | 225.524 |
| | SB1 | 0.911 | 0.917 | 0.904 | 0.171 | 0.105 | 0.085 | 5.313 | 8.711 | 10.681 |
| | SB2 | 0.824 | 0.863 | 0.885 | 0.006 | 0.005 | 0.004 | 127.040 | 185.644 | 225.329 |
| | BB1 | 0.894 | 0.901 | 0.885 | 0.145 | 0.097 | 0.080 | 6.166 | 9.256 | 11.057 |
| | BB2 | 0.805 | 0.855 | 0.879 | 0.006 | 0.005 | 0.004 | 132.218 | 189.933 | 229.041 |
| | PB1 | 0.852 | 0.872 | 0.881 | 0.157 | 0.102 | 0.083 | 5.429 | 8.588 | 10.676 |
| | PB2 | 0.815 | 0.851 | 0.890 | 0.006 | 0.005 | 0.004 | 128.530 | 185.849 | 229.244 |
| | BCaB1 | 0.856 | 0.877 | 0.882 | 0.152 | 0.100 | 0.082 | 5.622 | 8.761 | 10.783 |
| BCaB2 | 0.812 | 0.860 | 0.889 | 0.006 | 0.005 | 0.004 | 127.997 | 187.752 | 228.755 | |
| $p_0 = 0.6$ $p_1 = 0.4$ and $\rho_2 = 0.17$ | CAN1 | 0.871 | 0.883 | 0.876 | 1.082 | 0.730 | 0.599 | 0.805 | 1.209 | 1.463 |
| | CAN2 | 0.834 | 0.873 | 0.897 | 0.199 | 0.143 | 0.117 | 4.186 | 6.097 | 7.688 |
| | SB1 | 0.904 | 0.904 | 0.896 | 1.339 | 0.795 | 0.633 | 0.675 | 1.137 | 1.415 |
| | SB2 | 0.831 | 0.872 | 0.896 | 0.200 | 0.144 | 0.117 | 4.156 | 6.075 | 7.668 |
| | BB1 | 0.879 | 0.884 | 0.878 | 1.118 | 0.734 | 0.599 | 0.786 | 1.205 | 1.467 |
| | BB2 | 0.816 | 0.863 | 0.889 | 0.187 | 0.139 | 0.114 | 4.362 | 6.218 | 7.780 |
| | PB1 | 0.837 | 0.858 | 0.870 | 1.223 | 0.766 | 0.618 | 0.685 | 1.120 | 1.409 |
| | PB2 | 0.824 | 0.869 | 0.895 | 0.195 | 0.142 | 0.116 | 4.219 | 6.140 | 7.746 |
| | BCaB1 | 0.843 | 0.855 | 0.872 | 1.183 | 0.756 | 0.611 | 0.712 | 1.131 | 1.426 |
| BCaB2 | 0.816 | 0.866 | 0.890 | 0.196 | 0.142 | 0.116 | 4.172 | 6.110 | 7.690 | |
| $p_0 = 0.9$ $p_1 = 0.1$ and $\rho_2 = 0.03$ | CAN1 | 0.866 | 0.880 | 0.874 | 0.700 | 0.489 | 0.399 | 1.238 | 1.800 | 2.192 |
| | CAN2 | 0.837 | 0.868 | 0.894 | 0.032 | 0.023 | 0.020 | 26.164 | 37.301 | 44.908 |
| | SB1 | 0.897 | 0.900 | 0.896 | 0.855 | 0.532 | 0.421 | 1.049 | 1.691 | 2.126 |
| | SB2 | 0.828 | 0.871 | 0.898 | 0.032 | 0.023 | 0.020 | 25.819 | 37.366 | 44.974 |
| | BB1 | 0.870 | 0.881 | 0.880 | 0.720 | 0.492 | 0.398 | 1.209 | 1.790 | 2.211 |
| | BB2 | 0.807 | 0.860 | 0.889 | 0.030 | 0.023 | 0.019 | 26.877 | 38.147 | 45.675 |
| | PB1 | 0.844 | 0.868 | 0.864 | 0.784 | 0.513 | 0.411 | 1.076 | 1.693 | 2.103 |
| | PB2 | 0.827 | 0.872 | 0.897 | 0.031 | 0.023 | 0.020 | 26.381 | 37.956 | 45.425 |
| | BCaB1 | 0.845 | 0.868 | 0.865 | 0.763 | 0.506 | 0.407 | 1.107 | 1.716 | 2.123 |
| BCaB2 | 0.824 | 0.870 | 0.892 | 0.031 | 0.023 | 0.020 | 26.212 | 37.782 | 45.098 | |

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|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$ | CAN1 | 0.859 | 0.889 | 0.901 | 0.629 | 0.434 | 0.363 | 1.366 | 2.046 | 2.483 |
| | CAN2 | 0.863 | 0.888 | 0.875 | 1.045 | 0.748 | 0.628 | 0.826 | 1.188 | 1.394 |
| | SB1 | 0.908 | 0.911 | 0.918 | 0.776 | 0.473 | 0.384 | 1.170 | 1.927 | 2.391 |
| | SB2 | 0.860 | 0.887 | 0.880 | 1.049 | 0.749 | 0.628 | 0.820 | 1.184 | 1.400 |
| | BB1 | 0.861 | 0.890 | 0.903 | 0.649 | 0.437 | 0.363 | 1.326 | 2.037 | 2.491 |
| | BB2 | 0.845 | 0.875 | 0.865 | 0.981 | 0.724 | 0.614 | 0.861 | 1.208 | 1.409 |
| | PB1 | 0.831 | 0.866 | 0.895 | 0.706 | 0.456 | 0.375 | 1.178 | 1.899 | 2.390 |
| | PB2 | 0.855 | 0.885 | 0.866 | 1.027 | 0.738 | 0.622 | 0.833 | 1.199 | 1.393 |
| | BCaB1 | 0.839 | 0.865 | 0.892 | 0.687 | 0.451 | 0.371 | 1.221 | 1.919 | 2.404 |
| BCaB2 | 0.851 | 0.886 | 0.868 | 1.026 | 0.739 | 0.622 | 0.829 | 1.198 | 1.395 | |
| $p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$ | CAN1 | 0.881 | 0.899 | 0.900 | 0.778 | 0.551 | 0.451 | 1.133 | 1.632 | 1.995 |
| | CAN2 | 0.848 | 0.879 | 0.862 | 0.884 | 0.635 | 0.529 | 0.959 | 1.385 | 1.629 |
| | SB1 | 0.909 | 0.920 | 0.913 | 0.954 | 0.601 | 0.478 | 0.953 | 1.532 | 1.912 |
| | SB2 | 0.843 | 0.883 | 0.862 | 0.887 | 0.635 | 0.530 | 0.950 | 1.390 | 1.627 |
| | BB1 | 0.881 | 0.900 | 0.903 | 0.804 | 0.554 | 0.451 | 1.096 | 1.624 | 2.001 |
| | BB2 | 0.831 | 0.871 | 0.856 | 0.831 | 0.616 | 0.518 | 1.000 | 1.415 | 1.652 |
| | PB1 | 0.849 | 0.877 | 0.891 | 0.873 | 0.579 | 0.466 | 0.972 | 1.515 | 1.913 |
| | PB2 | 0.833 | 0.884 | 0.867 | 0.868 | 0.626 | 0.523 | 0.960 | 1.412 | 1.657 |
| | BCaB1 | 0.848 | 0.888 | 0.886 | 0.850 | 0.570 | 0.461 | 0.997 | 1.558 | 1.920 |
| BCaB2 | 0.831 | 0.874 | 0.883 | 0.868 | 0.628 | 0.524 | 0.957 | 1.392 | 1.685 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$ | CAN1 | 0.859 | 0.876 | 0.891 | 0.347 | 0.242 | 0.200 | 2.477 | 3.615 | 4.452 |
| | CAN2 | 0.849 | 0.848 | 0.872 | 0.067 | 0.047 | 0.039 | 12.746 | 18.094 | 22.514 |
| | SB1 | 0.894 | 0.898 | 0.902 | 0.422 | 0.264 | 0.212 | 2.117 | 3.402 | 4.263 |
| | SB2 | 0.851 | 0.845 | 0.871 | 0.067 | 0.047 | 0.039 | 12.716 | 18.010 | 22.461 |
| | BB1 | 0.861 | 0.881 | 0.887 | 0.357 | 0.244 | 0.200 | 2.415 | 3.613 | 4.437 |
| | BB2 | 0.831 | 0.845 | 0.867 | 0.063 | 0.045 | 0.038 | 13.279 | 18.605 | 22.872 |
| | PB1 | 0.822 | 0.860 | 0.874 | 0.386 | 0.254 | 0.206 | 2.130 | 3.382 | 4.239 |
| | PB2 | 0.848 | 0.849 | 0.880 | 0.065 | 0.046 | 0.038 | 12.964 | 18.347 | 22.926 |
| | BCaB1 | 0.828 | 0.866 | 0.875 | 0.375 | 0.251 | 0.204 | 2.207 | 3.448 | 4.282 |
| BCaB2 | 0.849 | 0.849 | 0.882 | 0.066 | 0.046 | 0.038 | 12.959 | 18.325 | 22.919 | |

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 5. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queuing network: $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$.

| Intensity Parameters | Estimation Approach | Coverage Percentage | | | Average Length | | | Relative Coverage | | |
|---|---------------------|---------------------|----------|----------|----------------|----------|----------|-------------------|----------------|----------------|
| | | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ |
| $p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$ | CAN1 | 0.881 | 0.873 | 0.885 | 0.402 | 0.286 | 0.236 | 2.193 | 3.051 | 3.754 |
| | CAN2 | 0.875 | 0.888 | 0.897 | 0.161 | 0.113 | 0.094 | 5.441 | 7.837 | 9.518 |
| | SB1 | 0.873 | 0.868 | 0.886 | 0.400 | 0.285 | 0.235 | 2.183 | 3.042 | 3.777 |
| | SB2 | 0.875 | 0.892 | 0.896 | 0.163 | 0.114 | 0.095 | 5.367 | 7.816 | 9.463 |
| | BB1 | 0.851 | 0.861 | 0.879 | 0.375 | 0.277 | 0.230 | 2.268 | 3.111 | 3.817 |
| | BB2 | 0.857 | 0.875 | 0.891 | 0.151 | 0.110 | 0.092 | 5.678 | 7.987 | 9.696 |
| | PB1 | 0.849 | 0.870 | 0.889 | 0.392 | 0.282 | 0.232 | 2.164 | 3.082 | 3.824 |
| | PB2 | 0.853 | 0.879 | 0.881 | 0.159 | 0.113 | 0.094 | 5.352 | 7.796 | 9.393 |
| | BCaB1 | 0.849 | 0.868 | 0.882 | 0.392 | 0.282 | 0.232 | 2.167 | 3.077 | 3.794 |
| BCaB2 | 0.860 | 0.876 | 0.872 | 0.159 | 0.112 | 0.094 | 5.422 | 7.798 | 9.308 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.11$ and $\rho_2 = 0.01$ | CAN1 | 0.871 | 0.878 | 0.893 | 0.090 | 0.063 | 0.053 | 9.722 | 13.893 | 16.935 |
| | CAN2 | 0.856 | 0.891 | 0.888 | 0.004 | 0.003 | 0.003 | 195.200 | 281.820 | 338.563 |
| | SB1 | 0.869 | 0.876 | 0.894 | 0.089 | 0.063 | 0.053 | 9.753 | 13.908 | 17.007 |
| | SB2 | 0.859 | 0.889 | 0.889 | 0.004 | 0.003 | 0.003 | 193.152 | 279.335 | 337.997 |
| | BB1 | 0.849 | 0.862 | 0.886 | 0.084 | 0.061 | 0.051 | 10.148 | 14.137 | 17.206 |
| | BB2 | 0.834 | 0.875 | 0.881 | 0.004 | 0.003 | 0.003 | 201.763 | 286.371 | 344.687 |
| | PB1 | 0.858 | 0.865 | 0.884 | 0.088 | 0.062 | 0.052 | 9.794 | 13.872 | 16.953 |
| | PB2 | 0.848 | 0.891 | 0.878 | 0.004 | 0.003 | 0.003 | 195.057 | 283.504 | 336.941 |
| | BCaB1 | 0.859 | 0.865 | 0.880 | 0.087 | 0.062 | 0.052 | 9.833 | 13.876 | 16.891 |
| BCaB2 | 0.842 | 0.882 | 0.877 | 0.004 | 0.003 | 0.003 | 194.536 | 281.382 | 337.204 | |
| $p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$ | CAN1 | 0.878 | 0.874 | 0.870 | 0.661 | 0.471 | 0.395 | 1.329 | 1.855 | 2.203 |
| | CAN2 | 0.849 | 0.882 | 0.889 | 0.132 | 0.094 | 0.080 | 6.432 | 9.341 | 11.167 |
| | SB1 | 0.872 | 0.869 | 0.867 | 0.658 | 0.470 | 0.393 | 1.325 | 1.851 | 2.204 |
| | SB2 | 0.849 | 0.882 | 0.891 | 0.134 | 0.095 | 0.080 | 6.356 | 9.282 | 11.156 |
| | BB1 | 0.847 | 0.854 | 0.859 | 0.617 | 0.455 | 0.385 | 1.372 | 1.875 | 2.230 |
| | BB2 | 0.822 | 0.864 | 0.879 | 0.124 | 0.091 | 0.078 | 6.635 | 9.473 | 11.335 |
| | PB1 | 0.865 | 0.866 | 0.869 | 0.646 | 0.464 | 0.390 | 1.338 | 1.866 | 2.229 |
| | PB2 | 0.843 | 0.869 | 0.883 | 0.131 | 0.094 | 0.079 | 6.454 | 9.279 | 11.154 |
| | BCaB1 | 0.868 | 0.864 | 0.865 | 0.645 | 0.464 | 0.390 | 1.345 | 1.862 | 2.217 |
| BCaB2 | 0.839 | 0.868 | 0.875 | 0.130 | 0.093 | 0.079 | 6.464 | 9.291 | 11.078 | |

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|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$ | CAN1 | 0.858 | 0.875 | 0.891 | 0.452 | 0.320 | 0.263 | 1.897 | 2.730 | 3.390 |
| | CAN2 | 0.870 | 0.884 | 0.891 | 0.022 | 0.016 | 0.013 | 39.305 | 55.479 | 67.767 |
| | SB1 | 0.858 | 0.874 | 0.890 | 0.450 | 0.320 | 0.262 | 1.906 | 2.731 | 3.397 |
| | SB2 | 0.870 | 0.882 | 0.896 | 0.022 | 0.016 | 0.013 | 38.755 | 54.933 | 67.862 |
| | BB1 | 0.836 | 0.863 | 0.881 | 0.422 | 0.309 | 0.256 | 1.983 | 2.790 | 3.441 |
| | BB2 | 0.836 | 0.879 | 0.881 | 0.021 | 0.015 | 0.013 | 40.177 | 57.074 | 68.727 |
| | PB1 | 0.847 | 0.876 | 0.895 | 0.441 | 0.317 | 0.260 | 1.919 | 2.767 | 3.445 |
| | PB2 | 0.847 | 0.878 | 0.887 | 0.022 | 0.016 | 0.013 | 38.630 | 55.413 | 67.828 |
| | BCaB1 | 0.847 | 0.873 | 0.890 | 0.441 | 0.317 | 0.260 | 1.919 | 2.757 | 3.429 |
| BCaB2 | 0.843 | 0.878 | 0.883 | 0.022 | 0.016 | 0.013 | 38.676 | 55.614 | 67.700 | |
| $p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$ | CAN1 | 0.853 | 0.882 | 0.886 | 0.396 | 0.282 | 0.237 | 2.156 | 3.127 | 3.738 |
| | CAN2 | 0.879 | 0.895 | 0.877 | 0.723 | 0.504 | 0.425 | 1.216 | 1.777 | 2.064 |
| | SB1 | 0.852 | 0.880 | 0.889 | 0.393 | 0.281 | 0.236 | 2.167 | 3.129 | 3.761 |
| | SB2 | 0.882 | 0.897 | 0.875 | 0.736 | 0.506 | 0.426 | 1.199 | 1.771 | 2.055 |
| | BB1 | 0.829 | 0.866 | 0.877 | 0.369 | 0.272 | 0.231 | 2.247 | 3.184 | 3.795 |
| | BB2 | 0.863 | 0.885 | 0.868 | 0.681 | 0.486 | 0.414 | 1.268 | 1.822 | 2.095 |
| | PB1 | 0.850 | 0.869 | 0.887 | 0.386 | 0.278 | 0.234 | 2.203 | 3.124 | 3.785 |
| | PB2 | 0.869 | 0.901 | 0.857 | 0.718 | 0.501 | 0.421 | 1.211 | 1.799 | 2.033 |
| | BCaB1 | 0.843 | 0.869 | 0.883 | 0.384 | 0.278 | 0.234 | 2.194 | 3.126 | 3.767 |
| BCaB2 | 0.869 | 0.898 | 0.859 | 0.713 | 0.499 | 0.421 | 1.219 | 1.799 | 2.043 | |
| $p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$ | CAN1 | 0.858 | 0.890 | 0.891 | 0.503 | 0.358 | 0.295 | 1.707 | 2.485 | 3.018 |
| | CAN2 | 0.871 | 0.857 | 0.891 | 0.595 | 0.427 | 0.354 | 1.463 | 2.005 | 2.517 |
| | SB1 | 0.852 | 0.892 | 0.888 | 0.500 | 0.357 | 0.294 | 1.704 | 2.497 | 3.018 |
| | SB2 | 0.875 | 0.858 | 0.897 | 0.605 | 0.430 | 0.355 | 1.447 | 1.995 | 2.525 |
| | BB1 | 0.825 | 0.880 | 0.883 | 0.469 | 0.346 | 0.288 | 1.759 | 2.546 | 3.066 |
| | BB2 | 0.849 | 0.845 | 0.883 | 0.559 | 0.412 | 0.345 | 1.517 | 2.051 | 2.559 |
| | PB1 | 0.832 | 0.878 | 0.891 | 0.491 | 0.353 | 0.291 | 1.693 | 2.487 | 3.057 |
| | PB2 | 0.848 | 0.854 | 0.884 | 0.591 | 0.425 | 0.352 | 1.436 | 2.010 | 2.514 |
| | BCaB1 | 0.823 | 0.873 | 0.883 | 0.491 | 0.353 | 0.291 | 1.678 | 2.472 | 3.031 |
| BCaB2 | 0.853 | 0.848 | 0.881 | 0.586 | 0.424 | 0.351 | 1.455 | 2.001 | 2.510 | |

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|---|-------|-------|-------|-------|-------|-------|-------|---------------|---------------|---------------|
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$ | CAN1 | 0.878 | 0.880 | 0.874 | 0.227 | 0.157 | 0.131 | 3.875 | 5.588 | 6.657 |
| | CAN2 | 0.862 | 0.904 | 0.899 | 0.045 | 0.032 | 0.026 | 19.163 | 28.082 | 34.053 |
| | SB1 | 0.874 | 0.885 | 0.873 | 0.226 | 0.157 | 0.131 | 3.871 | 5.635 | 6.667 |
| | SB2 | 0.864 | 0.903 | 0.897 | 0.046 | 0.032 | 0.027 | 18.942 | 27.925 | 33.823 |
| | BB1 | 0.858 | 0.868 | 0.868 | 0.212 | 0.152 | 0.128 | 4.056 | 5.710 | 6.781 |
| | BB2 | 0.841 | 0.892 | 0.892 | 0.042 | 0.031 | 0.026 | 19.872 | 28.700 | 34.691 |
| | PB1 | 0.864 | 0.869 | 0.870 | 0.222 | 0.155 | 0.130 | 3.898 | 5.595 | 6.704 |
| | PB2 | 0.840 | 0.880 | 0.881 | 0.045 | 0.032 | 0.026 | 18.840 | 27.545 | 33.535 |
| | BCaB1 | 0.862 | 0.874 | 0.866 | 0.221 | 0.155 | 0.130 | 3.909 | 5.637 | 6.681 |
| | BCaB2 | 0.839 | 0.880 | 0.887 | 0.044 | 0.032 | 0.026 | 18.928 | 27.640 | 33.815 |

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 6. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network: $E_4/H_4^{P_0}/1$ to $H_4^{P_0}/E_4/1$.

| Intensity Parameters | Estimation Approach | Coverage Percentage | | | Average Length | | | Relative Coverage | | |
|---|---------------------|---------------------|----------|----------|----------------|----------|----------|-------------------|----------------|---------------|
| | | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ |
| $p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$ | CAN1 | 0.868 | 0.889 | 0.892 | 0.400 | 0.283 | 0.236 | 2.173 | 3.140 | 3.775 |
| | CAN2 | 0.863 | 0.883 | 0.877 | 0.159 | 0.115 | 0.094 | 5.421 | 7.646 | 9.316 |
| | SB1 | 0.865 | 0.888 | 0.895 | 0.397 | 0.282 | 0.236 | 2.179 | 3.149 | 3.796 |
| | SB2 | 0.866 | 0.885 | 0.878 | 0.162 | 0.116 | 0.094 | 5.355 | 7.611 | 9.300 |
| | BB1 | 0.845 | 0.883 | 0.883 | 0.373 | 0.273 | 0.230 | 2.268 | 3.240 | 3.835 |
| | BB2 | 0.843 | 0.871 | 0.871 | 0.150 | 0.112 | 0.092 | 5.632 | 7.805 | 9.498 |
| | PB1 | 0.846 | 0.884 | 0.891 | 0.390 | 0.279 | 0.234 | 2.171 | 3.169 | 3.812 |
| | PB2 | 0.850 | 0.875 | 0.866 | 0.158 | 0.115 | 0.094 | 5.379 | 7.626 | 9.256 |
| | BCaB1 | 0.847 | 0.874 | 0.889 | 0.388 | 0.279 | 0.234 | 2.181 | 3.138 | 3.806 |
| BCaB2 | 0.848 | 0.872 | 0.867 | 0.157 | 0.114 | 0.093 | 5.388 | 7.632 | 9.285 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.11$ and $\rho_2 = 0.01$ | CAN1 | 0.873 | 0.900 | 0.894 | 0.089 | 0.063 | 0.052 | 9.794 | 14.320 | 17.061 |
| | CAN2 | 0.892 | 0.866 | 0.885 | 0.004 | 0.003 | 0.003 | 199.102 | 275.616 | 336.933 |
| | SB1 | 0.874 | 0.899 | 0.891 | 0.089 | 0.063 | 0.052 | 9.846 | 14.339 | 17.045 |
| | SB2 | 0.893 | 0.868 | 0.888 | 0.005 | 0.003 | 0.003 | 196.801 | 274.561 | 336.762 |
| | BB1 | 0.849 | 0.886 | 0.881 | 0.083 | 0.061 | 0.051 | 10.189 | 14.627 | 17.223 |
| | BB2 | 0.873 | 0.859 | 0.874 | 0.004 | 0.003 | 0.003 | 206.607 | 283.242 | 341.123 |
| | PB1 | 0.865 | 0.897 | 0.889 | 0.087 | 0.062 | 0.052 | 9.934 | 14.486 | 17.162 |
| | PB2 | 0.874 | 0.858 | 0.879 | 0.004 | 0.003 | 0.003 | 197.039 | 274.871 | 337.131 |
| | BCaB1 | 0.863 | 0.897 | 0.885 | 0.087 | 0.062 | 0.052 | 9.929 | 14.497 | 17.097 |
| BCaB2 | 0.872 | 0.859 | 0.888 | 0.004 | 0.003 | 0.003 | 197.877 | 275.862 | 341.255 | |

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|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$ | CAN1 | 0.870 | 0.888 | 0.870 | 0.661 | 0.473 | 0.391 | 1.316 | 1.878 | 2.226 |
| | CAN2 | 0.852 | 0.890 | 0.894 | 0.134 | 0.095 | 0.080 | 6.378 | 9.414 | 11.213 |
| | SB1 | 0.867 | 0.884 | 0.871 | 0.658 | 0.471 | 0.390 | 1.318 | 1.875 | 2.236 |
| | SB2 | 0.859 | 0.891 | 0.893 | 0.135 | 0.095 | 0.080 | 6.347 | 9.388 | 11.170 |
| | BB1 | 0.851 | 0.872 | 0.864 | 0.618 | 0.456 | 0.381 | 1.378 | 1.912 | 2.267 |
| | BB2 | 0.835 | 0.875 | 0.887 | 0.126 | 0.091 | 0.078 | 6.651 | 9.588 | 11.401 |
| | PB1 | 0.863 | 0.886 | 0.868 | 0.646 | 0.466 | 0.386 | 1.336 | 1.901 | 2.247 |
| | PB2 | 0.843 | 0.872 | 0.888 | 0.132 | 0.094 | 0.079 | 6.369 | 9.293 | 11.214 |
| | BCaB1 | 0.862 | 0.885 | 0.863 | 0.645 | 0.466 | 0.386 | 1.336 | 1.900 | 2.236 |
| BCaB2 | 0.837 | 0.870 | 0.885 | 0.132 | 0.094 | 0.079 | 6.362 | 9.302 | 11.196 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$ | CAN1 | 0.868 | 0.892 | 0.881 | 0.451 | 0.313 | 0.260 | 1.924 | 2.848 | 3.389 |
| | CAN2 | 0.855 | 0.889 | 0.888 | 0.022 | 0.016 | 0.013 | 38.912 | 56.159 | 67.891 |
| | SB1 | 0.868 | 0.890 | 0.882 | 0.449 | 0.312 | 0.259 | 1.932 | 2.850 | 3.400 |
| | SB2 | 0.855 | 0.886 | 0.889 | 0.022 | 0.016 | 0.013 | 38.389 | 55.660 | 67.803 |
| | BB1 | 0.846 | 0.873 | 0.872 | 0.421 | 0.303 | 0.254 | 2.009 | 2.885 | 3.438 |
| | BB2 | 0.837 | 0.871 | 0.881 | 0.021 | 0.015 | 0.013 | 40.525 | 56.991 | 69.056 |
| | PB1 | 0.852 | 0.885 | 0.871 | 0.441 | 0.309 | 0.257 | 1.931 | 2.868 | 3.388 |
| | PB2 | 0.841 | 0.891 | 0.884 | 0.022 | 0.016 | 0.013 | 38.677 | 56.728 | 68.080 |
| | BCaB1 | 0.853 | 0.878 | 0.872 | 0.440 | 0.308 | 0.257 | 1.940 | 2.846 | 3.390 |
| BCaB2 | 0.839 | 0.892 | 0.886 | 0.022 | 0.016 | 0.013 | 38.827 | 56.980 | 68.421 | |
| $p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$ | CAN1 | 0.856 | 0.889 | 0.896 | 0.393 | 0.283 | 0.239 | 2.178 | 3.137 | 3.754 |
| | CAN2 | 0.866 | 0.895 | 0.888 | 0.727 | 0.508 | 0.427 | 1.191 | 1.763 | 2.081 |
| | SB1 | 0.851 | 0.891 | 0.896 | 0.391 | 0.283 | 0.238 | 2.174 | 3.152 | 3.759 |
| | SB2 | 0.862 | 0.889 | 0.889 | 0.738 | 0.510 | 0.428 | 1.168 | 1.742 | 2.077 |
| | BB1 | 0.829 | 0.878 | 0.886 | 0.367 | 0.274 | 0.232 | 2.259 | 3.206 | 3.816 |
| | BB2 | 0.838 | 0.882 | 0.882 | 0.684 | 0.490 | 0.416 | 1.225 | 1.798 | 2.121 |
| | PB1 | 0.851 | 0.888 | 0.882 | 0.385 | 0.279 | 0.236 | 2.213 | 3.179 | 3.731 |
| | PB2 | 0.836 | 0.876 | 0.874 | 0.722 | 0.504 | 0.424 | 1.158 | 1.739 | 2.064 |
| | BCaB1 | 0.848 | 0.881 | 0.885 | 0.384 | 0.279 | 0.236 | 2.210 | 3.155 | 3.746 |
| BCaB2 | 0.840 | 0.875 | 0.880 | 0.716 | 0.502 | 0.423 | 1.173 | 1.742 | 2.083 | |

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|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$ | CAN1 | 0.864 | 0.901 | 0.904 | 0.510 | 0.362 | 0.294 | 1.694 | 2.486 | 3.076 |
| | CAN2 | 0.878 | 0.889 | 0.896 | 0.607 | 0.424 | 0.351 | 1.447 | 2.096 | 2.551 |
| | SB1 | 0.860 | 0.897 | 0.898 | 0.508 | 0.361 | 0.293 | 1.692 | 2.482 | 3.064 |
| | SB2 | 0.879 | 0.889 | 0.898 | 0.618 | 0.426 | 0.353 | 1.423 | 2.086 | 2.545 |
| | BB1 | 0.844 | 0.884 | 0.894 | 0.477 | 0.350 | 0.286 | 1.769 | 2.526 | 3.121 |
| | BB2 | 0.863 | 0.876 | 0.885 | 0.571 | 0.410 | 0.342 | 1.511 | 2.139 | 2.586 |
| | PB1 | 0.841 | 0.883 | 0.891 | 0.498 | 0.357 | 0.291 | 1.688 | 2.472 | 3.066 |
| | PB2 | 0.857 | 0.882 | 0.885 | 0.602 | 0.421 | 0.350 | 1.424 | 2.096 | 2.530 |
| | BCaB1 | 0.850 | 0.880 | 0.895 | 0.497 | 0.357 | 0.291 | 1.711 | 2.465 | 3.078 |
| BCaB2 | 0.854 | 0.876 | 0.886 | 0.599 | 0.420 | 0.349 | 1.426 | 2.084 | 2.541 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$ | CAN1 | 0.867 | 0.880 | 0.884 | 0.216 | 0.159 | 0.130 | 4.009 | 5.550 | 6.780 |
| | CAN2 | 0.897 | 0.894 | 0.893 | 0.045 | 0.031 | 0.027 | 20.111 | 28.392 | 33.686 |
| | SB1 | 0.860 | 0.879 | 0.885 | 0.215 | 0.158 | 0.130 | 4.005 | 5.556 | 6.806 |
| | SB2 | 0.896 | 0.891 | 0.897 | 0.045 | 0.032 | 0.027 | 19.852 | 28.156 | 33.765 |
| | BB1 | 0.846 | 0.865 | 0.878 | 0.202 | 0.153 | 0.127 | 4.193 | 5.647 | 6.907 |
| | BB2 | 0.875 | 0.880 | 0.880 | 0.042 | 0.030 | 0.026 | 20.922 | 28.966 | 34.057 |
| | PB1 | 0.850 | 0.876 | 0.888 | 0.211 | 0.156 | 0.129 | 4.029 | 5.600 | 6.893 |
| | PB2 | 0.874 | 0.879 | 0.896 | 0.044 | 0.031 | 0.026 | 19.771 | 28.134 | 34.043 |
| | BCaB1 | 0.849 | 0.871 | 0.883 | 0.211 | 0.156 | 0.129 | 4.033 | 5.575 | 6.860 |
| BCaB2 | 0.874 | 0.873 | 0.893 | 0.044 | 0.031 | 0.026 | 19.880 | 28.019 | 34.021 | |

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 7. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queuing network: $H_4^{Pc} / H_4^{Po} / 1$ to $H_4^{Po} / H_4^{Pc} / 1$.

| Intensity Parameters | Estimation Approach | Coverage Percentage | | | Average Length | | | Relative Coverage | | |
|---|---------------------|---------------------|----------|----------|----------------|----------|----------|-------------------|--------------|--------------|
| | | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ | $n = 10$ | $n = 20$ | $n = 29$ |
| $p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$ | CAN1 | 0.858 | 0.858 | 0.901 | 0.430 | 0.306 | 0.256 | 1.993 | 2.799 | 3.518 |
| | CAN2 | 0.864 | 0.870 | 0.895 | 0.174 | 0.122 | 0.102 | 4.977 | 7.119 | 8.803 |
| | SB1 | 0.863 | 0.864 | 0.905 | 0.436 | 0.308 | 0.256 | 1.980 | 2.808 | 3.529 |
| | SB2 | 0.869 | 0.872 | 0.899 | 0.175 | 0.123 | 0.102 | 4.952 | 7.101 | 8.821 |
| | BB1 | 0.839 | 0.845 | 0.897 | 0.405 | 0.296 | 0.250 | 2.071 | 2.855 | 3.592 |
| | BB2 | 0.841 | 0.857 | 0.886 | 0.163 | 0.118 | 0.099 | 5.147 | 7.256 | 8.922 |
| | PB1 | 0.860 | 0.854 | 0.892 | 0.426 | 0.304 | 0.254 | 2.017 | 2.813 | 3.513 |
| | PB2 | 0.850 | 0.869 | 0.882 | 0.171 | 0.121 | 0.101 | 4.960 | 7.176 | 8.738 |
| | BCaB1 | 0.859 | 0.850 | 0.886 | 0.424 | 0.303 | 0.253 | 2.025 | 2.809 | 3.495 |
| BCaB2 | 0.846 | 0.868 | 0.881 | 0.170 | 0.121 | 0.101 | 4.966 | 7.180 | 8.740 | |

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|--|-------|-------|-------|-------|-------|-------|-------|----------------|----------------|----------------|
| | CAN1 | 0.859 | 0.869 | 0.903 | 0.096 | 0.069 | 0.057 | 8.959 | 12.609 | 15.900 |
| | CAN2 | 0.863 | 0.875 | 0.899 | 0.005 | 0.003 | 0.003 | 178.166 | 256.808 | 313.751 |
| | SB1 | 0.861 | 0.872 | 0.902 | 0.097 | 0.069 | 0.057 | 8.882 | 12.572 | 15.840 |
| | SB2 | 0.871 | 0.874 | 0.898 | 0.005 | 0.003 | 0.003 | 177.672 | 255.224 | 312.522 |
| $p_0 = 0.9$ $p_1 = 0.1$ and $\rho_2 = 0.01$ | BB1 | 0.840 | 0.857 | 0.896 | 0.090 | 0.067 | 0.055 | 9.318 | 12.869 | 16.171 |
| | BB2 | 0.844 | 0.865 | 0.895 | 0.005 | 0.003 | 0.003 | 185.562 | 262.597 | 320.580 |
| | PB1 | 0.840 | 0.857 | 0.906 | 0.095 | 0.068 | 0.056 | 8.868 | 12.521 | 16.068 |
| | PB2 | 0.852 | 0.868 | 0.899 | 0.005 | 0.003 | 0.003 | 177.949 | 256.892 | 316.192 |
| | BCaB1 | 0.844 | 0.862 | 0.902 | 0.094 | 0.068 | 0.056 | 8.948 | 12.625 | 16.029 |
| | BCaB2 | 0.850 | 0.861 | 0.899 | 0.005 | 0.003 | 0.003 | 178.207 | 255.533 | 316.061 |
| | CAN1 | 0.857 | 0.877 | 0.892 | 0.727 | 0.512 | 0.430 | 1.178 | 1.712 | 2.076 |
| | CAN2 | 0.879 | 0.892 | 0.904 | 0.143 | 0.102 | 0.084 | 6.127 | 8.703 | 10.757 |
| | SB1 | 0.864 | 0.878 | 0.894 | 0.737 | 0.513 | 0.431 | 1.173 | 1.710 | 2.073 |
| | SB2 | 0.879 | 0.892 | 0.907 | 0.145 | 0.103 | 0.084 | 6.058 | 8.655 | 10.759 |
| $p_0 = 0.6$ $p_1 = 0.4$ and $\rho_2 = 0.17$ | BB1 | 0.842 | 0.865 | 0.886 | 0.684 | 0.495 | 0.419 | 1.230 | 1.747 | 2.115 |
| | BB2 | 0.852 | 0.883 | 0.897 | 0.135 | 0.099 | 0.082 | 6.312 | 8.894 | 10.924 |
| | PB1 | 0.849 | 0.859 | 0.883 | 0.720 | 0.507 | 0.427 | 1.179 | 1.696 | 2.067 |
| | PB2 | 0.855 | 0.885 | 0.911 | 0.142 | 0.102 | 0.083 | 6.031 | 8.703 | 10.910 |
| | BCaB1 | 0.840 | 0.860 | 0.882 | 0.716 | 0.505 | 0.426 | 1.173 | 1.702 | 2.069 |
| | BCaB2 | 0.848 | 0.879 | 0.911 | 0.141 | 0.101 | 0.083 | 6.012 | 8.661 | 10.941 |
| | CAN1 | 0.867 | 0.883 | 0.881 | 0.475 | 0.340 | 0.281 | 1.826 | 2.598 | 3.138 |
| | CAN2 | 0.868 | 0.883 | 0.874 | 0.024 | 0.017 | 0.014 | 36.033 | 52.147 | 62.267 |
| | SB1 | 0.865 | 0.886 | 0.885 | 0.482 | 0.342 | 0.281 | 1.795 | 2.593 | 3.147 |
| | SB2 | 0.869 | 0.887 | 0.871 | 0.024 | 0.017 | 0.014 | 35.563 | 52.065 | 61.943 |
| $p_0 = 0.9$ $p_1 = 0.1$ and $\rho_2 = 0.03$ | BB1 | 0.847 | 0.871 | 0.872 | 0.447 | 0.329 | 0.274 | 1.896 | 2.650 | 3.186 |
| | BB2 | 0.843 | 0.871 | 0.864 | 0.023 | 0.016 | 0.014 | 37.101 | 53.150 | 63.192 |
| | PB1 | 0.863 | 0.870 | 0.884 | 0.471 | 0.337 | 0.278 | 1.833 | 2.579 | 3.176 |
| | PB2 | 0.857 | 0.878 | 0.872 | 0.024 | 0.017 | 0.014 | 35.875 | 52.258 | 62.681 |
| | BCaB1 | 0.856 | 0.873 | 0.876 | 0.468 | 0.336 | 0.278 | 1.831 | 2.598 | 3.153 |
| | BCaB2 | 0.860 | 0.874 | 0.866 | 0.024 | 0.017 | 0.014 | 36.209 | 52.085 | 62.304 |

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|---|-------|-------|-------|-------|-------|-------|--------|---------------|---------------|---------------|
| $p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$ | CAN1 | 0.864 | 0.884 | 0.892 | 0.423 | 0.309 | 0.254 | 2.044 | 2.857 | 3.514 |
| | CAN2 | 0.881 | 0.899 | 0.872 | 0.785 | 0.552 | 0.459 | 1.122 | 1.628 | 1.901 |
| | SB1 | 0.869 | 0.885 | 0.892 | 0.427 | 0.311 | 0.255 | 2.036 | 2.848 | 3.503 |
| | SB2 | 0.884 | 0.897 | 0.879 | 0.793 | 0.555 | 0.459 | 1.115 | 1.617 | 1.913 |
| | BB1 | 0.844 | 0.869 | 0.882 | 0.397 | 0.299 | 0.248 | 2.128 | 2.905 | 3.558 |
| | BB2 | 0.864 | 0.886 | 0.863 | 0.739 | 0.534 | 0.448 | 1.170 | 1.660 | 1.924 |
| | PB1 | 0.878 | 0.864 | 0.891 | 0.417 | 0.307 | 0.252 | 2.104 | 2.817 | 3.536 |
| | PB2 | 0.859 | 0.888 | 0.863 | 0.775 | 0.548 | 0.455 | 1.108 | 1.621 | 1.897 |
| | BCaB1 | 0.870 | 0.863 | 0.887 | 0.414 | 0.306 | 0.251 | 2.100 | 2.822 | 3.527 |
| BCaB2 | 0.852 | 0.887 | 0.865 | 0.770 | 0.546 | 0.455 | 1.106 | 1.624 | 1.903 | |
| $p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$ | CAN1 | 0.861 | 0.878 | 0.894 | 0.541 | 0.382 | 0.319 | 1.592 | 2.297 | 2.803 |
| | CAN2 | 0.861 | 0.872 | 0.898 | 0.639 | 0.460 | 0.385 | 1.347 | 1.896 | 2.334 |
| | SB1 | 0.864 | 0.880 | 0.896 | 0.548 | 0.384 | 0.320 | 1.575 | 2.293 | 2.804 |
| | SB2 | 0.867 | 0.867 | 0.899 | 0.647 | 0.462 | 0.386 | 1.340 | 1.877 | 2.332 |
| | BB1 | 0.842 | 0.870 | 0.888 | 0.509 | 0.369 | 0.311 | 1.656 | 2.359 | 2.853 |
| | BB2 | 0.837 | 0.858 | 0.892 | 0.601 | 0.444 | 0.375 | 1.393 | 1.931 | 2.378 |
| | PB1 | 0.851 | 0.878 | 0.882 | 0.536 | 0.379 | 0.316 | 1.588 | 2.318 | 2.790 |
| | PB2 | 0.853 | 0.861 | 0.893 | 0.633 | 0.455 | 0.382 | 1.347 | 1.890 | 2.340 |
| | BCaB1 | 0.848 | 0.878 | 0.882 | 0.532 | 0.378 | 0.315 | 1.593 | 2.323 | 2.797 |
| BCaB2 | 0.858 | 0.864 | 0.892 | 0.631 | 0.454 | 0.381 | 1.360 | 1.901 | 2.343 | |
| $p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$ | CAN1 | 0.866 | 0.887 | 0.879 | 0.239 | 0.169 | 0.142 | 3.623 | 5.246 | 6.207 |
| | CAN2 | 0.866 | 0.876 | 0.903 | 0.048 | 0.034 | 0.028 | 17.862 | 25.724 | 32.167 |
| | SB1 | 0.870 | 0.885 | 0.877 | 0.242 | 0.170 | 0.142 | 3.598 | 5.210 | 6.180 |
| | SB2 | 0.872 | 0.877 | 0.901 | 0.049 | 0.034 | 0.028 | 17.762 | 25.637 | 32.002 |
| | BB1 | 0.849 | 0.879 | 0.870 | 0.225 | 0.163 | 0.138 | 3.768 | 5.380 | 6.297 |
| | BB2 | 0.852 | 0.871 | 0.891 | 0.046 | 0.033 | 0.027 | 18.679 | 26.465 | 32.544 |
| | PB1 | 0.849 | 0.880 | 0.873 | 0.236 | 0.168 | 0.140 | 3.590 | 5.248 | 6.215 |
| | PB2 | 0.855 | 0.876 | 0.905 | 0.048 | 0.034 | 0.028 | 17.833 | 25.945 | 32.452 |
| | BCaB1 | 0.848 | 0.875 | 0.874 | 0.235 | 0.167 | 0.140 | 3.601 | 5.230 | 6.225 |
| BCaB2 | 0.852 | 0.877 | 0.896 | 0.048 | 0.034 | 0.028 | 17.850 | 26.028 | 32.199 | |

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

are decreasing as p_0 approaches 1 (p_1 approaches 0) but both coverage percentages and relative coverage are increasing as p_0 approaches 1 (p_1 approaches 0).

Also we find that average lengths are decreasing with sample size n but both coverage percentages and relative coverage are increasing with sample size n . From **Tables 3 to 7** one can observe that the coverage percentage can approach to 90% when n increases up to 29.

1) In queueing network models $M/E_4/1$ to $E_4/M/1$ and $M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$, the estimation approach CAN and BB has the greatest relative coverage among the five confidence intervals for ρ_1 and ρ_2 respectively for different values of p_0 and p_1 .

2) In queueing network models $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$, $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$ and $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$, the estimation approach BB has the greatest relative coverage among the five confidence intervals for ρ_1 and ρ_2 for different values of p_0 and p_1 .

3) Average lengths are decreasing as sample size n increases for both ρ_1 and ρ_2 . Also relative coverage increases with n for ρ_1 and ρ_2 .

4) Some poor coverage percentage of above confidence intervals with respect to the nominal level 90% may be due to small sample size n .

8. Conclusion

This paper provides the interval estimations of intensities ρ_1 and ρ_2 for an open queueing network with feedback. Different estimation approaches CAN, SB, BB, PB and BCaB are applied to produce confidence intervals for intensities ρ_1 and ρ_2 . The relative coverage is adopted to understand, compare and assess performance of the resulted confidence intervals. From simulation study it is clear that CAN and BB method has the best performance on interval estimation of intensities ρ_1 and ρ_2 for $M/G/1$ to $M/G/1$ queueing network and BB method has the best performance on interval estimation of intensities ρ_1 and ρ_2 for $G/G/1$ to $G/G/1$ queueing network with short run data. And approach is easily applied to practical queueing network system such as all types of open, closed, mixed queueing networks as well as cyclic, retrial queueing models.

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