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Decomposition of Mathematical Programming Models for Aircraft Wing Design Facilitating the Use of Dynamic Programming Approach

Prashant K. Tarun[1], Herbert W. Corley[2]

[1]Steven L. Craig School of Business, Missouri Western State University, St. Joseph, Missouri, USA
[2]Department of Industrial, Manufacturing, & Systems Engineering, University of Texas at Arlington, Arlington, Texas, USA
Email: ptarun@missouriwestern.edu, corley@uta.edu

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Abstract

Aircraft designers strive to achieve optimal weight-reliability tradeoffs while designing an aircraft. Since aircraft wing skins account for more than fifty percent of their structural weight, aircraft wings must be designed with utmost care and attention in terms of material types and thickness configurations. In particular, the selection of thickness at each location of the aircraft wing skin is the most consequential task for aircraft designers. To accomplish this, we present discrete mathematical programming models to obtain optimal thicknesses either to minimize weight or to maximize reliability. We present theoretical results for the decomposition of these discrete mathematical programming models to reduce computer memory requirements and facilitate the use of dynamic programming for design purposes. In particular, a decomposed version of the weight minimization problem is solved for an aircraft wing with thirty locations (or panels) and fourteen thickness choices for each location to yield an optimal minimum weight design.

Keywords

Aircraft Wing Design, Maximum Reliability Design, Minimum Weight Design, Dynamic Programming, Optimization, Decomposition

1. Introduction

This paper presents an approach for designing aircraft wings, extending the research work presented by Tarun and Corley [1], who presented a discrete dynamic programming approach for obtaining optimal aircraft wing designs for the following possible design criteria: 1) minimizing an aircraft wing's weight while satisfying reliability requirements by selecting thicknesses for wing box components, 2) maximizing an aircraft wing's reliability within weight limitations by choosing thicknesses for various wing locations, or 3) determining trade-off designs between criteria (1) and (2). A numerical example was presented in [1] to illustrate their approach.

The discrete dynamic programming approach of [1], however, requires a large amount of computer RAM to carry out the computations associated with each stage and a large amount of storage capacity to store all information for each stage when the dynamic programming model involves a large number of stages and states. This paper addresses these issues by decomposing an optimization problem into subproblems prior to applying dynamic programming, thereby reducing computer RAM and storage capacity requirements. As in [1], we focus only on the design of aircraft wings here.

The aircraft wing structure must be suitably designed to withstand loads and aid the aerodynamic capabilities of the aircraft. The design of aircraft wings typically involves identifying the basic shape of the wings first and then designing a sufficiently strong and light wing structure [2]. The components that make up the aircraft wing are skins, spars, and ribs, with skins accounting for fifty to seventy percent of its structural weight. Therefore, it is imperative to design skins with utmost care and attention to detail, taking into consideration the high compressive strength requirement for the upper skin and the high tensile strength requirement for the lower skin. To achieve higher strength, lower weight, and better durability, wing skins are made of composite material and comprise many different layups and thickness configurations at different locations. These locations experience high stress as a result of lift and drag forces. They are typically identified by stress analysis. Generally, each location represents an area over which the internal stress and material strength are approximately constant, and there can be many locations with a choice of thicknesses for each.

Aircraft wing design must balance weight and reliability requirements through the selection of thicknesses at various locations. An increase in thickness at a location will result in an increase in weight of that location. On the other hand, a decrease in thickness at a location under a constant load (accompanied by a decrease in weight at that location) will result in an increase in internal stress and a consequent decrease in the reliability of that location. Therefore, the thickness choice at each location is paramount to aircraft wing design in achieving an optimal balance between weight and reliability requirements. Specifically, the aircraft designers need to consider the following objectives to choose optimal thickness at each wing location: 1) minimize weight within a reliability requirement and 2) maximize reliability within a weight requirement. Consequently, we consider the following problems:

(Problem 1) minimization of aircraft wing weight for a given reliability by selecting a thickness for each location.

(Problem 2) maximization of aircraft wing reliability for a given weight by selecting a thickness for each location.

In addition to [1], related research includes the following. Luo and Grandhi [3] present a methodology to reduce the failure probability in aircraft as a result of the uncertainty and randomness of the input information being used in structural optimization. Pettit and Grandhi [4] minimize the weight of a representative aircraft wing subject to reliability requirements. Pettit and Grandhi [5] go on to utilize probabilistic analysis and structural optimization for wing design. Padmanabhan [6] presents a framework for reliability-based optimization (RBO) in multidisciplinary systems, which facilitates concurrent design optimization and results in significant cost savings. Sobieszczanski-Sobieski and Venter [7] optimize wing box structural design with thousands of degrees of freedom and constraints and hundreds of design variables. Elham *et al.* [8] present a strategy for wing design optimization offering several advantages, including a reduction in the number of design variables, parallel optimization of the airfoils in various spanwise positions, and the use of simpler and faster two-dimensional airfoil analysis tools.

In Section 2 we present mathematical programming models for solving the aforementioned Problems 1 and 2. In Section 3 we present key theoretical results supporting the decomposition of mathematical programming models. In Section 4 we present a solution methodology that combines ideas from finite-element modeling and dynamic programming to solve the decomposed problem and a numerical example illustrating a solution to the discrete version of the weight minimization model. We offer concluding remarks in Section 5.

2. Relevant Mathematical Programming Models

To solve Problems 1 and 2, we use the relationships among *reliability*, *thickness*, and *weight* developed in [1] from Lear Fan 2100 Jet data. For example, *weight* = *area* × *thickness* × *density*, while *weight* at a wing location increases linearly as *thickness* increases since the *area* and *density* of a wing are constant. Furthermore, actual thickness at a location equals baseline thickness at that location times thickness ratio, where baseline thickness is the standard thickness obtained by deterministic structural analysis and actual thickness is the measurement of thickness at that location.

For context and clarity, we reproduce here some relevant information from [1]. A wing box comprises upper skin, lower skin and substructure. **Figure 1** represents the wing model for the upper skin. We assume realistically that internal stress at a location decreases with an increase in thickness at that location, and a decrease in internal stress due to an increase in thickness and weight leads to an increase in reliability. **Figure 2** shows the relationship between reliability and thickness, which is based on **Table 1** data for panel 1 of the Lear Fan 2100 Jet. Both axes in **Figure 2** have been formatted appropriately to clearly show the nonlinear relationship between reliability and thickness. As shown in **Figure 2**, reliability increases nonlinearly with an increase in thickness. The reasons for this nonlinear relationship as explained in [1] are: 1) the use of joint probability of nonlinear functions load and resistance for reliability calculations [9] and 2)

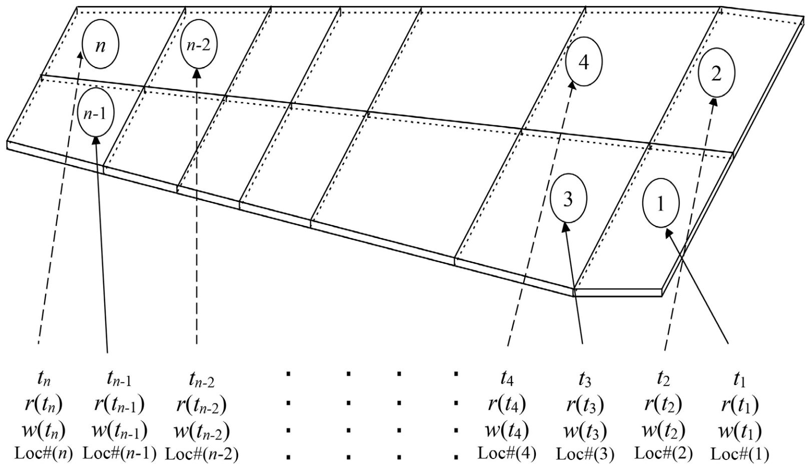

Figure 1. Configuration of a wing skin [1].

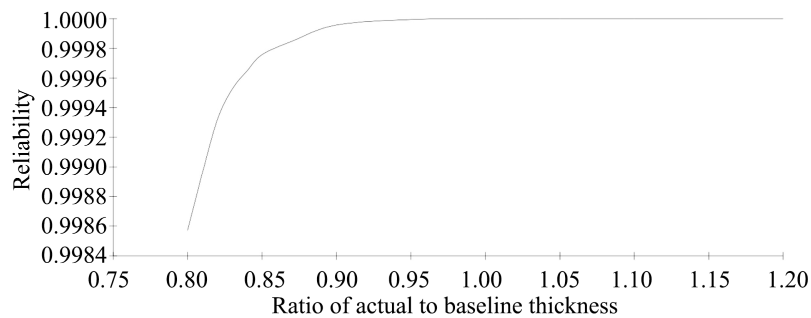

Figure 2. Relationship between reliability and thickness [1].

Table 1. Reliability and weight of panel 1 for 14 different thicknesses [1].

Thickness Number	Ratio of Actual Thickness to Baseline Thickness	Reliability	Weight (pounds)
1	1.20	0.999999995672	5.06
2	1.15	0.999999984471	4.85
3	1.10	0.999999937039	4.64
4	1.05	0.999999759895	4.43
5	1.00	0.999998691860	4.22
6	0.95	0.999992878710	4.01
7	0.90	0.999956177500	3.80
8	0.87	0.999847181000	3.67
9	0.85	0.999757110000	3.59
10	0.84	0.999648429000	3.54
11	0.83	0.999522932000	3.50
12	0.82	0.999318171000	3.46
13	0.81	0.998962640000	3.41
14	0.80	0.998571180000	3.38

the failure occurring due to buckling, a nonlinear function of thickness [10]. Moreover, for each panel, reliability is a strictly increasing function of the panel's thickness.

For the panel $i = 1, 2, 3, \cdots, n$, let t_i denote its thickness, with a resulting reliability $r(t_i)$ and weight $w(t_i)$, which are assumed continuous functions of t_i. Then for a sufficiently large n as determined by a preliminary structural analysis, the total reliability and weight for the upper skin can be estimated by the following expressions:

$$\text{Total Weight} = \sum_{i=1}^{n} w(t_i) \tag{1}$$

$$\text{Total Reliability} = \prod_{i=1}^{n} r(t_i) \tag{2}$$

We now get the following mathematical programming model formulations.

2.1. Weight Minimization

Problem 1 is the optimization problem to minimize the total weight of a wing within a specified minimum reliability level r_0 ($0 < r_0 < 1$) by selecting a thickness for each wing panel. From Equations (1) and (2), the weight minimization model (W) is

$$(W)\ \underset{t_i \ge 0}{\text{minimize}} \sum_{i=1}^{n} w(t_i)$$
$$\text{subject to}$$
$$\prod_{i=1}^{n} r(t_i) \ge r_0,$$

where the t_i are the decision variables and r_0 is the required overall reliability.

2.2. Reliability Maximization

Problem 2 is the optimization problem to maximize the total reliability of a wing for a specified upper weight limit $w_0 > 0$ by selecting a thickness for each wing panel. From Equations (1) and (2), the reliability maximization model (R) is

$$(R)\ \underset{t_i \ge 0}{\text{maximize}} \prod_{i=1}^{n} r(t_i)$$
$$\text{subject to}$$
$$\sum_{i=1}^{n} w(t_i) \le w_0,$$

where the t_i are the decision variables and w_0 is the specified weight limit.

We next present the theoretical results forming the basis for the decomposition of these mathematical programming models.

3. Theoretical Results Supporting the Decomposition of Mathematical Programming Models

For panel location $i = 1, 2, 3, \cdots, n$ in Figure 1, where t_i denotes its thickness, $r(t_i)$ denotes its reliability, and $w(t_i)$ denotes its weight. Then for a sufficiently large *n*, with each panel having a certain number of possible thicknesses, a large amount of computer memory would be required to store information for each panel location. To reduce the required amount of computer memory to make weight minimization problem (*W*) computationally tractable on a classical (as opposed to quantum) computer, (*W*) is decomposed into the following *M* subproblems for suitably chosen *M* and $n_1, n_2, \cdots, n_{M-1}, n_M$ values, where $n_M = n$.

Subproblem (W_1)	Subproblem (W_2)	$\cdots$	Subproblem (W_M)
$\underset{t_i \ge 0}{\text{minimize}} \sum_{i=1}^{n_1} w(t_i)$	$\underset{t_i \ge 0}{\text{minimize}} \sum_{i=n_1+1}^{n_2} w(t_i)$	$\cdots$	$\underset{t_i \ge 0}{\text{minimize}} \sum_{i=n_{M-1}+1}^{n} w(t_i)$
s.t. $\prod_{i=1}^{n_1} r(t_i) \ge r_1$	s.t. $\prod_{i=n_1+1}^{n_2} r(t_i) \ge r_2$	$\cdots$	s.t. $\prod_{i=n_{M-1}+1}^{n} r(t_i) \ge r_M$,

where the reliability level $r_0 = \prod_{j=1}^{M} r_j$ and t_i is the thickness at location *i* = 1, 2, …, *n*.

Similarly, the reliability maximization problem (*R*) is decomposed into the following *M* subproblems for suitably chosen *M* and $n_1, n_2, \cdots, n_{M-1}, n_M$ values, where $n_M = n$.

Subproblem (R_1)	Subproblem (R_2)	$\cdots$	Subproblem (R_M)
$\underset{t_i \ge 0}{\text{maximize}} \prod_{i=1}^{n_1} r(t_i)$	$\underset{t_i \ge 0}{\text{maximize}} \prod_{i=n_1+1}^{n_2} r(t_i)$	$\cdots$	$\underset{t_i \ge 0}{\text{maximize}} \prod_{i=n_{M-1}+1}^{n} r(t_i)$
s.t. $\sum_{i=1}^{n_1} w(t_i) \le w_1$	s.t. $\sum_{i=n_1+1}^{n_2} w(t_i) \le w_2$	$\cdots$	s.t. $\sum_{i=n_{M-1}+1}^{n} w(t_i) \le w_M$,

where the weight limit $w_0 = \sum_{j=1}^{M} w_j$ and t_i is the thickness at location *i* = 1, 2, …, *n*.

The above decompositions for the weight minimization problem (*W*) and reliability maximization problem (*R*) are justified by the following results. Lemma 1 is obvious from the previous discussion.

Lemma 1. Both $w(t_i)$ *and* $r(t_i)$ *are strictly increasing functions of thickness* t_i.

Lemma 2. **(*A*)** *Let* $t_1^*, t_2^*, \cdots, t_n^*$ *solve* (*W*) *for fixed* r_0. *Then* $\sum_{i=1}^{n} w(t_i^*)$ *is minimum if and only if* $\prod_{i=1}^{n} r(t_i^*) = r_0$.

(*B*) *Let* $t_1^*, t_2^*, \cdots, t_n^*$ *solve* (*R*) *for fixed* w_0. *Then* $\prod_{i=1}^{n} r(t_i^*)$ *is maximum if and only if* $\sum_{i=1}^{n} w(t_i^*) = w_0$.

Proof. (A) Let $t_1^*, t_2^*, \cdots, t_n^*$ solve (*W*). Then $\sum_{i=1}^{n} w(t_i^*) < \sum_{i=1}^{n} w(t_i)$ and $\prod_{i=1}^{n} r(t_i^*) \ge r_0$. Suppose $\prod_{i=1}^{n} r(t_i^*) > r_0$ ($\prod_{i=1}^{n} r(t_i^*) \ne r_0$). Then $\exists\, \hat{t}_1, \hat{t}_2, \cdots, \hat{t}_n$ that is feasible to (*W*) such that $\prod_{i=1}^{n} r(\hat{t}_i) = r_0$, then $\prod_{i=1}^{n} r(\hat{t}_i) = r_0 < \prod_{i=1}^{n} r(t_i^*)$. From Lemma 1, $\sum_{i=1}^{n} w(\hat{t}_i) < \sum_{i=1}^{n} w(t_i^*)$. However, this contradicts $\sum_{i=1}^{n} w(t_i^*) < \sum_{i=1}^{n} w(t_i)$. Therefore, $\prod_{i=1}^{n} r(t_i^*) = r_0$.

Next let $t_1^*, t_2^*, \cdots, t_n^*$ be a feasible solution to (*W*) such that $\prod_{i=1}^{n} r(t_i^*) = r_0$. Then $\prod_{i=1}^{n} r(t_i) \geq \prod_{i=1}^{n} r(t_i^*) = r_0$. From Lemma 1, $\sum_{i=1}^{n} w(t_i) \geq \sum_{i=1}^{n} w(t_i^*)$. Therefore, $\sum_{i=1}^{n} w(t_i^*)$ is the minimum value. ∎

(B) Let $t_1^*, t_2^*, \cdots, t_n^*$ solve (*R*). Then $\prod_{i=1}^{n} r(t_i^*) > \prod_{i=1}^{n} r(t_i)$ and $\sum_{i=1}^{n} w(t_i^*) \leq w_0$. Suppose $\sum_{i=1}^{n} w(t_i^*) < w_0$ ($\sum_{i=1}^{n} w(t_i^*) \neq w_0$). Then $\exists\, \hat{t}_1, \hat{t}_2, \cdots, \hat{t}_n$ that is feasible to (*R*) such that $\sum_{i=1}^{n} w(\hat{t}_i) = w_0$, then $\sum_{i=1}^{n} w(\hat{t}_i) = w_0 > \sum_{i=1}^{n} w(t_i^*)$. From Lemma 1, $\prod_{i=1}^{n} r(\hat{t}_i) > \prod_{i=1}^{n} r(t_i^*)$. However, this contradicts $\prod_{i=1}^{n} r(t_i^*) > \prod_{i=1}^{n} r(t_i)$. Therefore, $\sum_{i=1}^{n} w(t_i^*) = w_0$.

Next let $t_1^*, t_2^*, \cdots, t_n^*$ be a feasible solution to (*R*) such that $\sum_{i=1}^{n} w(t_i^*) = w_0$. Then $\sum_{i=1}^{n} w(t_i) \leq \sum_{i=1}^{n} w(t_i^*) = w_0$. From Lemma 1, $\prod_{i=1}^{n} r(t_i) \leq \prod_{i=1}^{n} r(t_i^*)$. Therefore, $\prod_{i=1}^{n} r(t_i^*)$ is the maximum value. ∎

The results from Lemma 2 are next applied to the above subproblems W_1, W_2, …, W_M and R_1, R_2, …, R_M of (*W*) and (*R*), respectively.

Lemma 3. **(*A*)** *Let* $t_1^*, t_2^*, \cdots, t_{n_1}^*, t_{n_1+1}^*, t_{n_1+2}^*, \cdots, t_{n_2}^*, \cdots, t_n^*$ *be optimal for* (*W*) *and*

$$\prod_{i=1}^{n_1} r(t_i^*) = r_1^*, \prod_{i=n_1+1}^{n_2} r(t_i^*) = r_2^*, \cdots, \prod_{i=n_{M-1}+1}^{n} r(t_i^*) = r_M^*,$$

where $r_1^* \times r_2^* \times \cdots \times r_M^* = r_0$ *and* $r_m = r_m^*$ *for* W_m, $m = 1, 2, \cdots, M$. Then

$$\min_{t_i \geq 0} \sum_{i=1}^{n_1} w(t_i(r_1)) + \min_{t_i \geq 0} \sum_{i=n_1+1}^{n_2} w(t_i(r_2)) + \cdots + \min_{t_i \geq 0} \sum_{i=n_{M-1}+1}^{n} w(t_i(r_M)) = \sum_{i=1}^{n} w(t_i^*)$$

subject to

$$r_1 \times r_2 \times \cdots \times r_M = r_0,$$

where $t_i(r_j)$ *are the decision variables at location i for subproblem* $j = 1, 2, \cdots, M$.

(*B*) *Let* $t_1^*, t_2^*, \cdots, t_{n_1}^*, t_{n_1+1}^*, t_{n_1+2}^*, \cdots, t_{n_2}^*, \cdots, t_n^*$ *be optimal for* (*R*) *and* $\sum_{i=1}^{n_1} w(t_i^*) = w_1^*$, $\sum_{i=n_1+1}^{n_2} w(t_i^*) = w_2^*, \cdots, \sum_{i=n_{M-1}+1}^{n} w(t_i^*) = w_M^*$, *where* $w_1^* + w_2^* + \cdots + w_M^* = w_0$ *and*

$w_m = w_m^*$ *for* R_m, $m = 1, 2, \cdots, M$. *Then*

$$\max_{t_i \ge 0} \prod_{i=1}^{n_1} r\left(t_i\left(w_1\right)\right) \times \max_{t_i \ge 0} \prod_{i=n_1+1}^{n_2} r\left(t_i\left(w_2\right)\right) \times \cdots \times \max_{t_i \ge 0} \prod_{i=n_{M-1}+1}^{n} r\left(t_i\left(w_M\right)\right) = \prod_{i=1}^{n} r\left(t_i^*\right)$$

subject to

$$w_1 + w_2 + \cdots + w_M = w_0,$$

where $t_i\left(w_j\right)$ *are the decision variables at location i for subproblem* $j = 1, 2, \cdots, M$.

Proof. (A) The contrapositive is proved. Obviously,

$$\min_{t_i \ge 0} \sum_{i=1}^{n_1} w\left(t_i\left(r_1\right)\right) + \min_{t_i \ge 0} \sum_{i=n_1+1}^{n_2} w\left(t_i\left(r_2\right)\right) + \cdots + \min_{t_i \ge 0} \sum_{i=n_{M-1}+1}^{n} w\left(t_i\left(r_M\right)\right) \ge \sum_{i=1}^{n} w\left(t_i^*\right),$$

subject to their respective constraints. Now suppose

$$\min_{t_i \ge 0} \sum_{i=1}^{n_1} w\left(t_i\left(r_1\right)\right) + \min_{t_i \ge 0} \sum_{i=n_1+1}^{n_2} w\left(t_i\left(r_2\right)\right) + \cdots + \min_{t_i \ge 0} \sum_{i=n_{M-1}+1}^{n} w\left(t_i\left(r_M\right)\right) > \sum_{i=1}^{n} w\left(t_i^*\right),$$

subject to their respective constraints. Let $\hat{t}_1, \cdots, \hat{t}_{n_1}, \hat{t}_{n_1+1}, \cdots, \hat{t}_{n_2}, \hat{t}_{n_2+1}, \cdots, \hat{t}_n$ solve $W_1, W_2, \cdots, W_M$, respectively. From Lemma 2,

$$\prod_{i=1}^{n_1} r\left(\hat{t}_i\right) = r_1^*, \prod_{i=n_1+1}^{n_2} r\left(\hat{t}_i\right) = r_2^*, \cdots, \prod_{i=n_{M-1}+1}^{n} r\left(\hat{t}_i\right) = r_M^* \text{ and } r_1^* \times r_2^* \times \cdots \times r_M^* = r_0.$$

Since $\sum_{i=1}^{n_1} w\left(\hat{t}_i\left(r_1\right)\right) + \sum_{i=n_1+1}^{n_2} w\left(\hat{t}_i\left(r_2\right)\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(\hat{t}_i\left(r_M\right)\right) > \sum_{i=1}^{n} w\left(t_i^*\right)$, at least one sum above on the left-hand side, say, $\sum_{i=1}^{n_1} w\left(\hat{t}_i\left(r_1\right)\right) > \sum_{i=1}^{n_1} w\left(t_i^*\right)$. Then $\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_{n_1}$ does not form an optimal solution to W_1. Therefore,

$$\min_{t_i \ge 0} \sum_{i=1}^{n_1} w\left(t_i\left(r_1\right)\right) + \min_{t_i \ge 0} \sum_{i=n_1+1}^{n_2} w\left(t_i\left(r_2\right)\right) + \cdots + \min_{t_i \ge 0} \sum_{i=n_{M-1}+1}^{n} w\left(t_i\left(r_M\right)\right) = \sum_{i=1}^{n} w\left(t_i^*\right),$$

subject to their respective constraints. ∎

(B) The contrapositive is proved. Obviously,

$$\max_{t_i \ge 0} \prod_{i=1}^{n_1} r\left(t_i\left(w_1\right)\right) \times \max_{t_i \ge 0} \prod_{i=n_1+1}^{n_2} r\left(t_i\left(w_2\right)\right) \times \cdots \times \max_{t_i \ge 0} \prod_{i=n_{M-1}+1}^{n} r\left(t_i\left(w_M\right)\right) \le \prod_{i=1}^{n} r\left(t_i^*\right),$$

subject to their respective constraints. Now suppose

$$\max_{t_i \ge 0} \prod_{i=1}^{n_1} r\left(t_i\left(w_1\right)\right) \times \max_{t_i \ge 0} \prod_{i=n_1+1}^{n_2} r\left(t_i\left(w_2\right)\right) \times \cdots \times \max_{t_i \ge 0} \prod_{i=n_{M-1}+1}^{n} r\left(t_i\left(w_M\right)\right) < \prod_{i=1}^{n} r\left(t_i^*\right),$$

subject to their respective constraints. Let $\hat{t}_1, \cdots, \hat{t}_{n_1}, \hat{t}_{n_1+1}, \cdots, \hat{t}_{n_2}, \hat{t}_{n_2+1}, \cdots, \hat{t}_n$ solve $R_1, R_2, \cdots, R_M$, respectively. From Lemma 2, $\sum_{i=1}^{n_1} w\left(\hat{t}_i\right) = w_1^*$, $\sum_{i=n_1+1}^{n_2} w\left(\hat{t}_i\right) = w_2^*$, $\cdots$, $\sum_{i=n_{M-1}+1}^{n} w\left(\hat{t}_i\right) = w_M^*$ and $w_1^* + \cdots + w_M^* = w_0$.

Since $\prod_{i=1}^{n_1} r\left(\hat{t}_i\left(w_1\right)\right) \times \prod_{i=n_1+1}^{n_2} r\left(\hat{t}_i\left(w_2\right)\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(\hat{t}_i\left(w_M\right)\right) < \prod_{i=1}^{n} r\left(t_i^*\right)$, at least one product above on the left-hand side, say, $\prod_{i=1}^{n_1} r\left(\hat{t}_i\left(w_1\right)\right) < \prod_{i=1}^{n_1} r\left(t_i^*\right)$. Then $\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_{n_1}$ does not form an optimal solution to R_1. Therefore,

$$\max_{t_i \geq 0} \prod_{i=1}^{n_1} r\left(t_i\left(w_1\right)\right) \times \max_{t_i \geq 0} \prod_{i=n_1+1}^{n_2} r\left(t_i\left(w_2\right)\right) \times \cdots \times \max_{t_i \geq 0} \prod_{i=n_{M-1}+1}^{n} r\left(t_i\left(w_M\right)\right) = \prod_{i=1}^{n} r\left(t_i^*\right),$$

subject to their respective constraints. □

Theorem 1. **(*A*)** *In the weight minimization problem* (*W*), *let the M tuples of varying lengths*

$$\left(t_1^*\left(r_1^*\right), \cdots, t_{n_1}^*\left(r_1^*\right)\right), \left(t_{n_1+1}^*\left(r_2^*\right), \cdots, t_{n_2}^*\left(r_2^*\right)\right), \cdots, \left(t_{n_{M-1}+1}^*\left(r_M^*\right), \cdots, t_n^*\left(r_M^*\right)\right)$$

be optimal to problems $W_1, W_2, \cdots, W_M$ *for* $r_1, \cdots, r_M$, *respectively. Then the optimal solution to*

$$\begin{gathered} \underset{r_1, r_2, \cdots, r_M \geq 0}{\text{minimize}} \left[\sum_{i=1}^{n_1} w\left(t_i^*\left(r_1\right)\right) + \sum_{i=n_1+1}^{n_2} w\left(t_i^*\left(r_2\right)\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(t_i^*\left(r_M\right)\right)\right] \\ \text{subject to} \\ r_1 \times r_2 \times \cdots \times r_M = r_0 \end{gathered} \tag{3}$$

solves the weight minimization problem (*W*).

(*B*) *In the reliability maximization problem* (*R*), *let the M tuples of varying lengths*

$$\left(t_1^*\left(w_1^*\right), \cdots, t_{n_1}^*\left(w_1^*\right)\right), \left(t_{n_1+1}^*\left(w_2^*\right), \cdots, t_{n_2}^*\left(w_2^*\right)\right), \cdots, \left(t_{n_{M-1}+1}^*\left(w_M^*\right), \cdots, t_n^*\left(w_M^*\right)\right)$$

be optimal to problems $R_1, R_2, \cdots, R_M$ *for* $w_1, \cdots, w_M$, *respectively. Then the optimal solution to*

$$\begin{gathered} \underset{w_1, w_2, \cdots, w_M \geq 0}{\text{maximize}} \left[\prod_{i=1}^{n_1} r\left(t_i^*\left(w_1\right)\right) \times \prod_{i=n_1+1}^{n_2} r\left(t_i^*\left(w_2\right)\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(t_i^*\left(w_M\right)\right)\right] \\ \text{subject to} \\ w_1 + w_2 + \cdots + w_M = w_0 \end{gathered} \tag{4}$$

solves the reliability maximization problem (*R*).

Proof. (A) The contrapositive is proved. Let $r_1^*, r_2^*, \cdots, r_M^*$ be optimal for problem (3), so $r_1^* \times r_2^* \times \cdots \times r_M^* = r_0$. Then the *M* tuples of varying lengths

$$\left(t_1^*\left(r_1^*\right), \cdots, t_{n_1}^*\left(r_1^*\right)\right), \left(t_{n_1+1}^*\left(r_2^*\right), \cdots, t_{n_2}^*\left(r_2^*\right)\right), \cdots, \left(t_{n_{M-1}+1}^*\left(r_M^*\right), \cdots, t_n^*\left(r_M^*\right)\right)$$

solve $W_1, W_2, \cdots, W_M$, respectively. Let $\hat{t}_1, \cdots, \hat{t}_{n_1}, \hat{t}_{n_1+1}, \cdots, \hat{t}_{n_2}, \cdots, \hat{t}_{n_{M-1}+1}, \cdots, \hat{t}_n$ be optimal for (*W*), and let $\hat{r}_1, \hat{r}_2, \cdots, \hat{r}_M$ be defined by

$$\prod_{i=1}^{n_1} r\left(\hat{t}_i\right) = \hat{r}_1, \quad \prod_{i=n_1+1}^{n_2} r\left(\hat{t}_i\right) = \hat{r}_2, \quad \cdots, \quad \prod_{i=n_{M-1}+1}^{n} r\left(\hat{t}_i\right) = \hat{r}_M,$$

with $\hat{r}_1 \times \hat{r}_2 \times \cdots \times \hat{r}_M = r_0$. Thus $\hat{r}_1, \hat{r}_2, \cdots, \hat{r}_M$ is feasible to problem (3) along with $\hat{t}_1, \cdots, \hat{t}_{n_1}, \hat{t}_{n_1+1}, \cdots, \hat{t}_{n_2}, \cdots, \hat{t}_{n_{M-1}+1}, \cdots, \hat{t}_n$.

Therefore,

$$\sum_{i=1}^{n_1} w\left(t_i^*\left(\hat{r}_1\right)\right) + \sum_{i=n_1+1}^{n_2} w\left(t_i^*\left(\hat{r}_2\right)\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(t_i^*\left(\hat{r}_M\right)\right) \geq \sum_{i=1}^{n} w\left(\hat{t}_i\right),$$

where $\sum_{i=1}^{n} w(\hat{t}_i) = \left[\sum_{i=1}^{n_1} w(\hat{t}_i) + \sum_{i=n_1+1}^{n_2} w(\hat{t}_i) + \cdots + \sum_{i=n_{M-1}+1}^{n} w(\hat{t}_i)\right]$.

Now suppose

$$\sum_{i=1}^{n_1} w\left(t_i^*\left(\hat{r}_1\right)\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(t_i^*\left(\hat{r}_M\right)\right) > \sum_{i=1}^{n_1} w\left(\hat{t}_i\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(\hat{t}_i\right).$$

Then by definition, $\hat{t}_1,\cdots,\hat{t}_{n_1},\hat{t}_{n_1+1},\cdots,\hat{t}_{n_2},\cdots,\hat{t}_{n_{M-1}+1},\cdots,\hat{t}_n$ is not optimal to (*W*). Consequently,

$$\sum_{i=1}^{n_1} w\left(t_i^*\left(r_1^*\right)\right) + \sum_{i=n_1+1}^{n_2} w\left(t_i^*\left(r_2^*\right)\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(t_i^*\left(r_M^*\right)\right) = \sum_{i=1}^{n} w\left(\hat{t}_i\right). \square$$

(B) The contrapositive is proved. Let $w_1^*, w_2^*, \cdots, w_M^*$ be optimal to problem (4), so $w_1^* + w_2^* + \cdots + w_M^* = w_0$. Then the *M* tuples of varying lengths

$$\left(t_1^*\left(w_1^*\right),\cdots,t_{n_1}^*\left(w_1^*\right)\right),\left(t_{n_1+1}^*\left(w_2^*\right),\cdots,t_{n_2}^*\left(w_2^*\right)\right),\cdots,\left(t_{n_{M-1}+1}^*\left(w_M^*\right),\cdots,t_n^*\left(w_M^*\right)\right)$$

solve $R_1, R_2, \cdots, R_M$, respectively. Let $\hat{t}_1,\cdots,\hat{t}_{n_1},\hat{t}_{n_1+1},\cdots,\hat{t}_{n_2},\cdots,\hat{t}_{n_{M-1}+1},\cdots,\hat{t}_n$ be optimal for (*R*), and let $\hat{w}_1, \hat{w}_2, \cdots, \hat{w}_M$ be defined by

$$\sum_{i=1}^{n_1} w\left(\hat{t}_i\right) = \hat{w}_1, \quad \sum_{i=n_1+1}^{n_2} w\left(\hat{t}_i\right) = \hat{w}_2, \quad \cdots, \quad \sum_{i=n_{M-1}+1}^{n} w\left(\hat{t}_i\right) = \hat{w}_M,$$

with $\hat{w}_1 + \hat{w}_2 + \cdots + \hat{w}_M = w_0$. Thus $\hat{w}_1, \hat{w}_2, \cdots, \hat{w}_M$ is feasible to problem (4) along with $\hat{t}_1,\cdots,\hat{t}_{n_1},\hat{t}_{n_1+1},\cdots,\hat{t}_{n_2},\cdots,\hat{t}_{n_{M-1}+1},\cdots,\hat{t}_n$. Therefore,

$$\prod_{i=1}^{n_1} r\left(t_i^*\left(\hat{w}_1\right)\right) \times \prod_{i=n_1+1}^{n_2} r\left(t_i^*\left(\hat{w}_2\right)\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(t_i^*\left(\hat{w}_M\right)\right) \le \prod_{i=1}^{n} r\left(\hat{t}_i\right),$$

where $\prod_{i=1}^{n} r\left(\hat{t}_i\right) = \prod_{i=1}^{n_1} r\left(\hat{t}_i\right) \times \prod_{i=n_1+1}^{n_2} r\left(\hat{t}_i\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(\hat{t}_i\right)$.

Now suppose

$$\prod_{i=1}^{n_1} r\left(t_i^*\left(\hat{w}_1\right)\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(t_i^*\left(\hat{w}_M\right)\right) < \prod_{i=1}^{n_1} r\left(\hat{t}_i\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(\hat{t}_i\right).$$

Then by definition, $\hat{t}_1,\cdots,\hat{t}_{n_1},\hat{t}_{n_1+1},\cdots,\hat{t}_{n_2},\cdots,\hat{t}_{n_{M-1}+1},\cdots,\hat{t}_n$ is not optimal to (*R*). Consequently, $\prod_{i=1}^{n_1} r\left(t_i^*\left(w_1^*\right)\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(t_i^*\left(w_M^*\right)\right) = \prod_{i=1}^{n} r\left(\hat{t}_i\right)$. $\square$

From Theorem 1, it now follows that the weight minimization problem (*W*) can be rewritten as

$$\begin{aligned} &\underset{t_i \ge 0}{\text{minimize}} \left[\sum_{i=1}^{n_1} w\left(t_i\right) + \sum_{i=n_1+1}^{n_2} w\left(t_i\right) + \cdots + \sum_{i=n_{M-1}+1}^{n} w\left(t_i\right)\right] \\ &\text{subject to} \\ &\left[\prod_{i=1}^{n_1} r\left(t_i\right) \times \prod_{i=n_1+1}^{n_2} r\left(t_i\right) \times \cdots \times \prod_{i=n_{M-1}+1}^{n} r\left(t_i\right)\right] \ge r_0, \end{aligned} \tag{5}$$

where r_0 is the required reliability and t_i is the thickness at location *i* = 1, 2, …, *n*.

Similarly, the reliability maximization problem (*R*) can be rewritten as

$$\begin{gathered}\underset{t_i \geq 0}{\text{maximize}}\left[\prod_{i=1}^{n_1} r(t_i)\times\prod_{i=n_1+1}^{n_2} r(t_i)\times\cdots\times\prod_{i=n_{M-1}+1}^{n} r(t_i)\right] \\ \text{subject to} \\ \left[\sum_{i=1}^{n_1} w(t_i)+\sum_{i=n_1+1}^{n_2} w(t_i)+\cdots+\sum_{i=n_{M-1}+1}^{n} w(t_i))\right]\leq w_0,\end{gathered} \tag{6}$$

where w_0 is the weight limit and t_i is the thickness at location $i = 1, 2, \ldots, n$.

In the next section, we use (5) to solve the discrete version of the weight minimization problem (W) consisting of M subproblems as shown in (5) for suitably chosen M and $n_1, n_2, \cdots, n_{M-1}, n_M$, where $n_M = n$. We apply dynamic programming in an example.

4. Weight Minimization Using Dynamic Programming on the Decomposed Version

We established in Section 3 that the weight minimization problem (W) and the set of decomposed subproblems (W_i, i = 1, 2… M) will yield the same optimal solution. Tarun and Corley [1] used dynamic programming to solve the weight minimization problem (W). In this section, we use dynamic programming to solve the decomposed version (5) of the weight minimization problem (W). The advantage to problem (5) is that the computer memory requirement will be smaller than that for the original problem (W). The decomposition developed in Section 3 will be used to solve the weight minimization problem (W) where each of the major aircraft wing box components *upper skin*, *lower skin*, and *substructure* has ten different panels, totaling thirty panels across the wing box structure illustrated in **Figures 3-5** [1].

The problem is discretized since letting the t_i be continuous presents mathematical complications. Thus, for each panel, fourteen different thicknesses will be considered that adequately represent a panel for design purposes. Structural reliability of the wing box for each thickness in each panel was obtained by

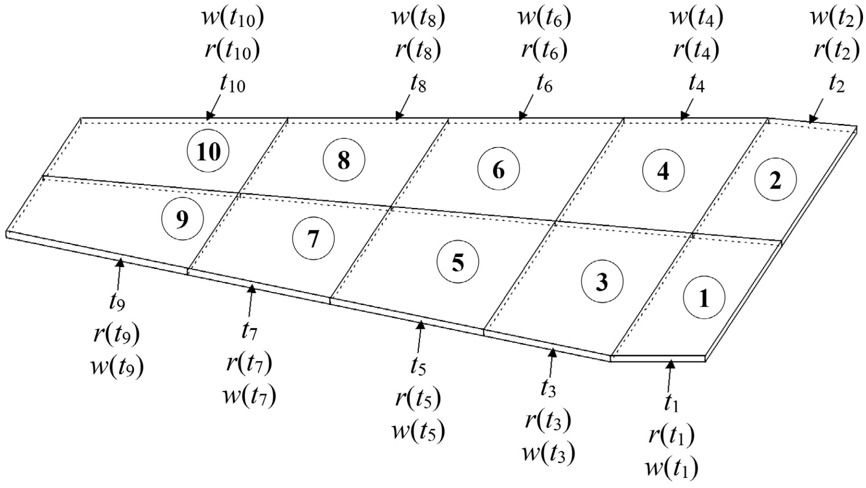

Figure 3. Upper skin [1].

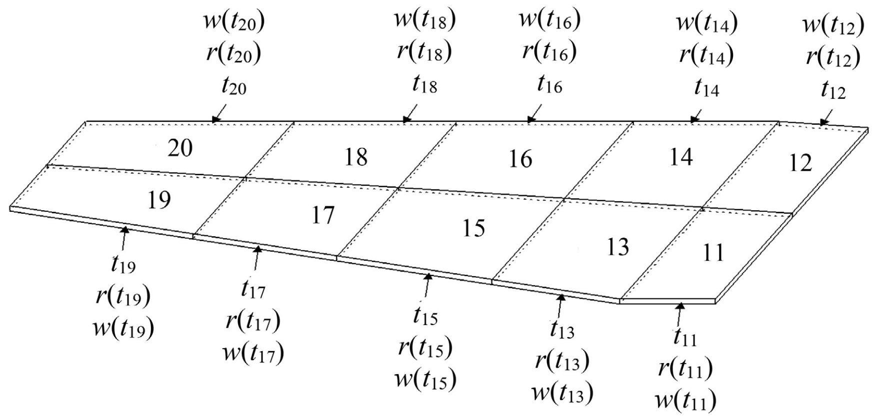

Figure 4. Lower skin [1].

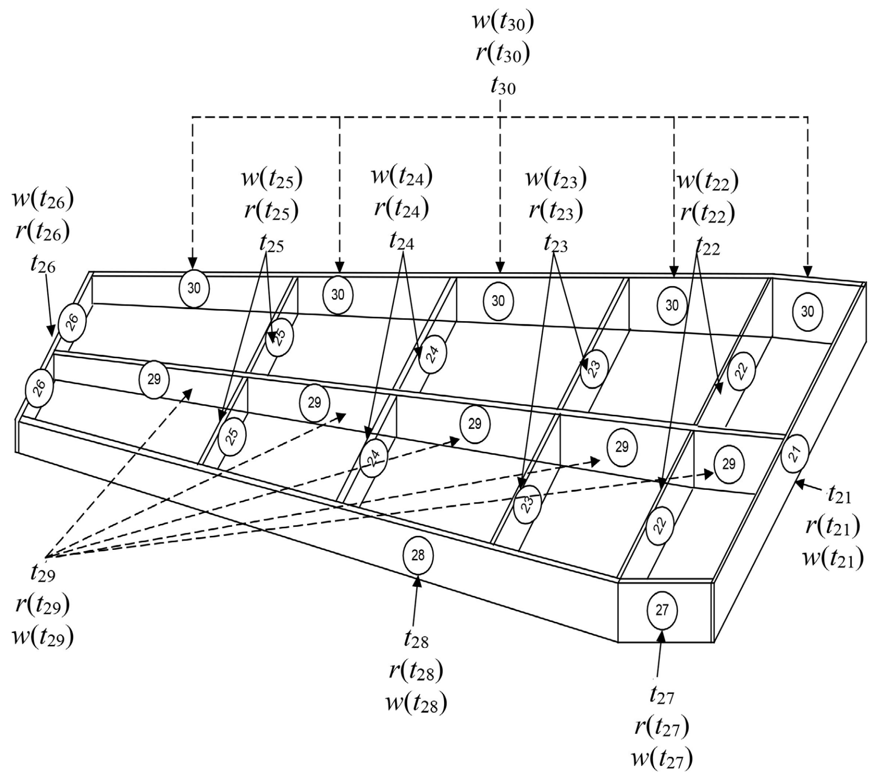

Figure 5. Substructure [1].

incorporating material strength, operational damage, manufacturing defects, moisture absorption, and gust into the probabilistic design program of Northrop Grumman Commercial Aircraft Division (NGCAD). The resulting reliability and weight associated with different thicknesses for panel 1 with baseline thickness of 0.2 inch are shown in **Table 1**, where reliability $r(t_i)$ is the probability in (0, 1) that panel *i* will not fail.

The weight minimization problem (W) represents a generalized resource allocation problem with a single constraint, which can be solved using the optimization technique called dynamic programming [11]. Dynamic programming has been extensively applied to a variety of areas, including inventory analysis, allocation problems, discrete control theory, and chemical engineering.

Tarun and Corley [1] used dynamic programming to obtain optimal solution to the weight minimization problem (W) for 30 panel locations and $r_0 = 0.99999$, using 14 thicknesses, reliabilities, and weights illustrated in **Table 1** for panel 1. This weight minimization problem ($\tilde{W}$) is shown below for context and subsequent discussion:

$$(\tilde{W}) \ \underset{t_i \geq 0}{\text{minimize}} \ \sum_{i=1}^{30} w(t_i)$$

$$\text{subject to}$$

$$\prod_{i=1}^{30} r(t_i) \geq r_0,$$

where t_i are the decision variables and $r_0 = 0.99999$. For the weight minimization problem ($\tilde{W}$), the stages i, state variables s_i, decision variables t_i, return functions $g(s_i, t_i)$, state transformations, and recursive equations associated with the dynamic programming process are defined in **Table 2**.

We now describe the steps involved with the use of the dynamic programming process in solving the weight minimization problem ($\tilde{W}$). Dynamic programming involves working backward as follows. At stage 30 we minimize $g(s_{30}, t_{30})$ over the finite preselected values of t_{30} for each possible value of s_{30}. At stages $i = 29, 28, \cdots, 1$ in that order, for each possible value of s_i we solve $f_i(s_i) = \min\left[g(s_i, t_i) + f_{i+1}(s_{i+1})\right], i = 1, 2, \cdots, 29$ over the fourteen preselected values for t_i, where $s_{i+1} = s_i / r(t_i)$ from the stage transformations. There are only a finite possible number of values of each s_i at stage i because there are only fourteen possible t_i at each stage. However, finding and solving the optimal t_i for each s_i is memory intensive. Finally, at stage 1, $s_1 = r_0$. At that point, we proceed forward. For $s_1 = r_0$, the optimal t_1 is obtained by minimizing $\left[g(s_1, t_1) + f_2(s_1 / r(t_1))\right]$ over the fourteen values of t_1, where each $g(s_1, t_1)$ is computed and each $f_2(s_1 / r(t_1))$ is known from previously obtaining $f_2(s_2)$ for all s_2. With the optimal t_1 known, $s_2 = s_1 / r(t_1)$ is computed and the optimal t_2 is determined at stage 2. Then the optimal $t_3, t_4, \cdots, t_{30}$ are similarly obtained.

Figure 6 illustrates the dynamic programming process. The top of **Figure 6** indicates working backward for all possible s_i, while the bottom indicates working forward to obtain the actual values of the optimal t_i and the minimum total weight.

4.1. Problem Decomposition and Use of Dynamic Programming Approach

We now decompose the weight minimization problem ($\tilde{W}$) into six subproblems (W_1), (W_2), (W_3), (W_4), (W_5), and (W_6) for wing panel locations 1 to 5, 6 to 10, 11 to 15, 16 to 20, 21 to 25, and 26 to 30, respectively, as shown in **Table 3**.

Consequently, the weight minimization problem ($\tilde{W}$) can be rewritten as

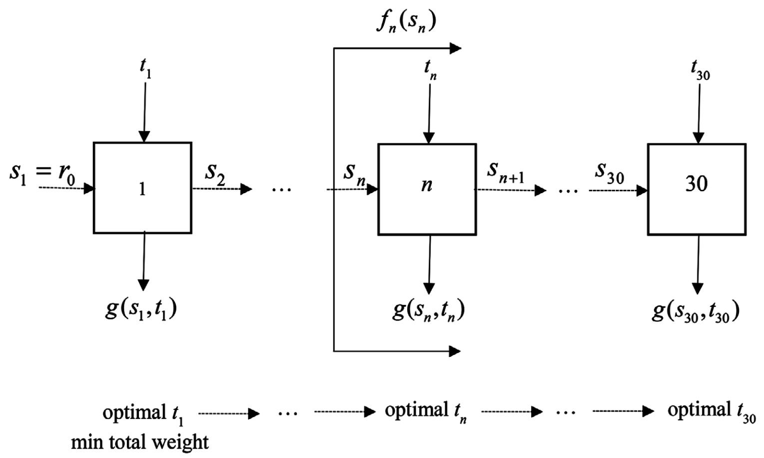

Figure 6. Flow chart for solving problem ($\tilde{W}$) using dynamic programming [1].

Table 2. Definitions for dynamic programming formulation of problem ($\tilde{W}$) [1].

stage number = i	There are 30 stages (panel locations).
decision variables = t_i	Thickness for panel i
state variable at stage $i = s_i$	Overall reliability ($\geq r_0$) required for the remaining stages i, $i+1, \ldots, 30$ presenting restrictions on future decisions
return function at stage $i = g(s_i, t_i)$	Weight of panel i for a fixed remaining reliability s_i and thickness for this s_i
state transformations	$s_{i+1} = \frac{s_i}{r(t_i)}, i = 1, 2, \cdots, 29$
recursive equations	$f_i(s_i) = \min_{t_i}\left[g(s_i, t_i) + f_{i+1}\left(\frac{s_i}{r(t_i)}\right)\right], i = 1, 2, \cdots, 29$ $f_{30}(s_{30}) = \min_{t_{30}} g(s_{30}, t_{30})$

Table 3. Subproblems for the weight minimization problem ($\tilde{W}$).

Subproblems	Covered Locations
(W_1)	locations 1 to 5
(W_2)	locations 6 to 10
(W_3)	locations 11 to 15
(W_4)	locations 16 to 20
(W_5)	locations 21 to 25
(W_6)	locations 26 to 30

$$(E_1)\ \underset{t_i \geq 0}{\text{minimize}} \left[\sum_{i=1}^{5} w(t_i) + \sum_{i=6}^{10} w(t_i) + \sum_{i=11}^{15} w(t_i) + \sum_{i=16}^{20} w(t_i) + \sum_{i=21}^{25} w(t_i) + \sum_{i=26}^{30} w(t_i)\right]$$

subject to

$$\left[\prod_{i=1}^{5} r(t_i) \times \prod_{i=6}^{10} r(t_i) \times \prod_{i=11}^{15} r(t_i) \times \prod_{i=16}^{20} r(t_i) \times \prod_{i=21}^{25} r(t_i) \times \prod_{i=26}^{30} r(t_i)\right] \geq r_0,$$

where t_i are the decision variables and $r_0 = 0.99999$.

The results in Section 3 indicate that the solution to the weight minimization problem ($\tilde{W}$) presented in Tarun and Corley [1] will be the same as the solution to the optimization problem (E_1) comprising the decomposed subproblems (W_1), (W_2), (W_3), (W_4), (W_5), and (W_6) shown in **Table 3**. The decomposed version of the weight minimization problem (E_1) facilitates the application of dynamic programming as the computer memory usage requirement for each subproblem would be much smaller than the computer memory usage requirement for the original, non-decomposed weight minimization problem ($\tilde{W}$). We now describe the steps involved in solving (E_1) using dynamic programming.

Step 1 (5-stage subproblems). Use dynamic programming to obtain optimal weight-reliability level choices for each subproblem W_j, $j = 1, 2, \ldots, 6$ by varying the reliability level. If an optimal weight is reached by more than two reliability levels, the maximal reliability and this optimal weight will be selected as a choice. The dynamic programming formulation for (W_1) is shown in **Table 4** as an illustration. We now describe the dynamic programming process for (W_1). At stage 5 we minimize $g(s_5, t_5)$ over the finite preselected values of t_5 for each possible value of s_5. At stages $i = 4,3,2,1$ in that order, for each possible value of s_i we solve $f_i(s_i) = \min\left[g(s_i, t_i) + f_{i+1}(s_{i+1})\right], i = 1,2,3,4$ over the fourteen preselected values for t_i, where $s_{i+1} = s_i / r(t_i)$ from the stage transformations. There are only a finite possible number of values of each s_i at stage i because there are only fourteen possible t_i at each stage. Finally, at stage 1, $s_1 = r_1$ At that point, we proceed forward. For $s_1 = r_1$ the optimal t_1 is obtained by minimizing $\left[g(s_1, t_1) + f_2(s_1/r(t_1))\right]$ over the fourteen values of t_1, where each $g(s_1, t_1)$ is computed and each $f_2(s_1/r(t_1))$ is known from previously obtaining $f_2(s_2)$ for all s_2. With the optimal t_1 known, $s_2 = s_1/r(t_1)$ is computed and the optimal t_2 is determined at stage 2. Then the optimal t_3, t_4, t_5 are similarly obtained. The dynamic programming process is similarly applied to subproblems (W_2), (W_3), (W_4), (W_5), and (W_6) as well.

Table 4. Dynamic programming formulation of subproblem (W_1).

stage number = *i*	There are 5 stages (panel locations 1 to 5).
decision variables = t_i	Thickness for panel *i*
state variable at stage *i* = s_i	Overall reliability ($\geq r_1$) required for the remaining stages *i*, *i* + 1, …, 5 presenting restrictions on future decisions
return function at stage *i* = $g(s_i, t_i)$	Weight of panel *i* for a fixed remaining reliability s_i and thickness for this s_i
state transformations	$s_{i+1} = \dfrac{s_i}{r(t_i)}, i = 1,2,3,4$
recursive equations	$f_i(s_i) = \min_{t_i}\left[g(s_i, t_i) + f_{i+1}\left(\dfrac{s_i}{r(t_i)}\right)\right], i = 1,2,3,4$ $f_5(s_5) = \min_{t_5} g(s_5, t_5)$

Step 2 (2-stage combined subproblems). Use dynamic programming with the optimal weight-reliability level choices for subproblems (W_1) and (W_2) to obtain optimal weight-reliability level choices for the combined subproblem (W_1) & (W_2). The dynamic programming formulation for the combined subproblem (W_1) & (W_2) has 2 stages only, making it computationally more efficient than the problem formulation involving panel locations 1 to 10. Similarly, we obtain the optimal weight-reliability level choices for the combined subproblems (W_3) & (W_4) and (W_5) & (W_6).

Step 3 (3-stage final problem). Use dynamic programming with the optimal weight-reliability level choices for the combined subproblem (W_1) & (W_2), (W_3) & (W_4), and (W_5) & (W_6) to obtain minimal weight for the problem (E_1), reliability associated with the minimal weight, and the thickness choices for each panel location. The dynamic programming formulation involving these three combined subproblems has 3 stages only, which is less computationally intensive than the weight minimization problem ($\tilde{W}$) involving panel locations 1 to 30.

4.2. Dynamic Programming Implementation Results

We now present the results for steps 1, 2, and 3 above. The results for steps 1 and 2 have been condensed to keep this paper to an appropriate length. The complete results for these steps can be provided upon request.

Step 1 Results: In this step, the optimal weight-reliability level choices for each of the subproblems from Table 3 are calculated for various reliability levels. The number of optimal weight-reliability level choices for subproblems (W_1), (W_2), (W_3), (W_4), (W_5), and (W_6) are 84, 212, 174, 288, 276, and 300, respectively, where the reliability level varies from 0.999 to 1. Tables 5-10 show the partial results for subproblems (W_1), (W_2), (W_3), (W_4), (W_5), and (W_6), respectively. The complete results for these subproblems can be provided upon request.

Step 2 Results: In this step, the optimal weight-reliability level choices for the combined subproblems (W_1) & (W_2), (W_3) & (W_4), and (W_5) & (W_6) are calculated. The optimal weight-reliability level choices for subproblems (W_1) and (W_2) from Step 1 are used to compute the optimal weight-reliability level choices for the combined subproblem (W_1) & (W_2). Similarly, the optimal weight-reliability level choices for the combined subproblems (W_3) & (W_4) and (W_5) & (W_6) are obtained. The number of optimal weight-reliability level choices for the combined subproblems (W_1) & (W_2), (W_3) & (W_4), and (W_5) & (W_6) are 290, 255, and 80, respectively. Tables 11-13 show the partial results for the combined subproblems (W_1) & (W_2), (W_3) & (W_4), and (W_5) & (W_6), respectively. The complete results for these subproblems can be provided upon request.

Step 3 Results: In this step, we use Step 2 results for the three combined subproblems (W_1) & (W_2), (W_3) & (W_4), and (W_5) & (W_6) to solve (E_1). This results in the minimal weight of 249.93 pounds for the aircraft wing, the associated reliability of 0.9999905 (greater than $r_0 = 0.99999$), and the thickness choices for panels shown in Table 14. Evidently, these results match the results from Tarun and Corley [1].

Table 5. Optimal weight-reliability level choices for subproblem (W_1) (locations 1 to 5).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99990545	30.7023
2	0.99991093	30.8054
…	…	…
…	…	…
83	0.99999996	39.21233
84	0.99999997	39.62026

Table 6. Optimal weight-reliability level choices for subproblem (W_2) (locations 6 to 10).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99994428	51.7205
2	0.9999456	51.7567
…	…	…
…	…	…
211	0.99999999993	71.1159
212	1	74.3483

Table 7. Optimal weight-reliability level choices for subproblem (W_3) (locations 11 to 15).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99999999983	58.8166
2	0.99999999983	56.3424
…	…	…
…	…	…
173	0.99999999441	39.6861
174	0.99999999423	39.6252

Table 8. Optimal weight-reliability level choices for subproblem (W_4) (locations 16 to 20).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99999999996	60.5623
2	0.999999999958	59.8828
…	…	…
…	…	…
287	0.999999999517	40.4631
288	0.999999999508	40.375

Table 9. Optimal weight-reliability level choices for subproblem (W_5) (locations 21 to 25).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.999999999977	81.8398
2	0.9999999999767	81.3963
…	…	…
…	…	…
275	0.999999999906	55.5899
276	0.99999999986	54.5597

Table 10. Optimal weight-reliability level choices for subproblem (W_6) (locations 26 to 30).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99999999998702	38.1604
2	0.99999999998677	37.7267
…	…	…
…	…	…
299	0.99999999996234	25.4763
300	0.99999999996204	25.4401

Table 11. Optimal weight-reliability level choices for subproblem (W_1) & (W_2).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99999998	114.5874
2	0.999999979	108.3334
…	…	…
…	…	…
289	0.999851	82.459
290	0.999849	82.4228

Table 12. Optimal weight-reliability level choices for subproblem (W_3) & (W_4).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.99999999978	116.6166
2	0.99999999977	114.1424
…	…	…
…	…	…
254	0.99999999391	80.0611
255	0.99999999373	80.0002

Table 13. Optimal weight-reliability level choices for subproblem (W_5) & (W_6).

Choices	Reliability Level	Optimal Weight (pounds)
1	0.999999999964	120.0002
2	0.999999999963	118.0067
…	…	…
…	…	…
79	0.99999999986	81.03
80	0.99999999982	79.9998

Table 14. Choices for panels in the example [1].

Panel	Ratio of Actual Thickness to Baseline Thickness as in Table 1
1	1.05
2	1.05
3	0.95
4	0.90
5	0.95
6	0.87
7	0.87
8	0.87
9	0.85
10	0.87
11	0.80
12	0.80
13	0.80
14	0.80
15	0.80
16	0.80
17	0.80
18	0.80
19	0.80
20	0.80
21	0.80
22	0.80
23	0.80
24	0.80
25	0.80

Continued

26	0.80
27	0.80
28	0.80
29	0.80
30	0.80

Similarly, the reliability maximization problem can be solved to obtain maximal reliability within a weight limit. Further examples are found in [12].

5. Conclusions and Future Work

This paper presented the theoretical results as the basis for the decomposition of discrete mathematical programming models, decomposed the weight minimization and reliability maximization problems for an aircraft wing with 30 panels and 14 thickness choices for each panel, and demonstrated the use of a dynamic programming approach to solve the decomposed version of the weight minimization problem. More importantly, this solution methodology resulted in the minimal weight, maximal reliability, and thickness choices for aircraft wing panels presented in the work by Tarun and Corley [1]. One advantage of this solution methodology is its flexibility in the way it can be expanded to as many stages as needed and the total number of stages decomposed into as many subproblems as needed to reduce the computer memory storage requirements. In other words, with this approach, no matter the number of locations and thicknesses in an aircraft wing, an optimal solution can be obtained. In addition, with our approach, it would be a simple matter to change the specified weight limit or overall reliability requirement to obtain a new design. As a result, sensitivity analysis is not difficult with our approach.

We can envisage at least four major directions for future research. First, the aircraft wing model could be reconfigured. Second, instead of using a discrete version of the model with only a fixed number of thickness values, a continuous version of the model could be developed and possibly solved. Third, the dynamic programming approach could be implemented in a parallel processing environment to solve large, real-world problems. Fourth, optimization problems displaying relationships similar to problems (W) and (R) could be examined.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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