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Optimality Conditions and Second-Order Duality for Nondifferentiable Multiobjective Continuous Programming Problems

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ABSTRACT

Fritz John and Karush-Kuhn-Tucker type optimality conditions for a nondifferentiable multiobjective variational problem are derived. As an application of Karush-Kuhn-Tucker type optimality conditions, Mond-weir type second-order nondifferentiable multiobjective dual variational problems is constructed. Various duality results for the pair of Mond-Weir type second-order dual variational problems are proved under second-order pseudoinvexity and second-order quasi-invexity. A pair of Mond-Weir type dual variational problems with natural boundary values is formulated to derive various duality results. Finally, it is pointed out that our results can be considered as dynamic generalizations of their static counterparts existing in the literature.

Keywords: Nondifferentiable Multiobjective Programming; Second-Order Invexity; Second-Order Pseudoinvexity; Second-Order Quasi-Invexity; Second-Order Duality; Nonlinear Multiobjective Programming

1. Introduction

Second-order duality in mathematical programming has been extensively investigated in the literature. In [1], Chen formulated second order dual for a constrained variational problem and established various duality results under an involved invexity-like assumptions. Subsequently, Husain *et al.* [2], have presented Mond-Weir type second order duality for the problem of [1], by introducing continuous-time version of second-order invexity and generalized second-order invexity. Husain and Masoodi [3] formulated a Wolfe type dual for a nondifferentiable variational problem and proved usual duality theorems under second-order pseudoinvexity condition while Husain and Srivastav [4] presented a Mond- Weir type dual to the problem of [2] to study duality under second-order pseudo-invexity and second-order quasi-invexity.

The purpose of this research is to present multiobjective version of the nondifferentiable variational problems considered in [2,4] and study various duality in terms of efficient solutions. The relationship between these multiobjective variational problems and their static counterparts is established through problems with natural boundary values.

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2. Definitions and Related Pre-Requisites

Let $I=[a,b]$ be a real interval, $\phi: I\times R^n\times R^n\to R$ and $\psi: I\times R^n\times R^n\to R^m$ be twice continuously differentiable functions. In order to consider $\phi(t,x(t),\dot{x}(t))$ where $x: I\to R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x and $\phi_{\dot{x}}$ the first order derivatives of ϕ with respect to $x(t)$ and $\dot{x}(t)$, respectively, that is,

$$\phi_x=\left(\frac{\partial\phi}{\partial x^1},\frac{\partial\phi}{\partial x^2},\cdots,\frac{\partial\phi}{\partial x^n}\right)^T,$$

$$\phi_{\dot{x}}=\left(\frac{\partial\phi}{\partial \dot{x}^1},\frac{\partial\phi}{\partial \dot{x}^2},\cdots,\frac{\partial\phi}{\partial \dot{x}^n}\right)^T$$

Further denote by ϕ_{xx}, and ψ_x the $n\times n$ Hessian and $m\times n$ Jacobian matrices respectively.

The symbols $\phi_{\dot{x}},\phi_{\dot{x}x},\phi_{x\dot{x}}$ and $\psi_{\dot{x}}$ have analogous representations.

Designate by X the space of piecewise smooth functions $x: I\to R^n$ with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is given by

$$u=Dx\Leftrightarrow x(t)=\int_a^t u(s)\,\mathrm{d}s,$$

Thus $\frac{\mathrm{d}}{\mathrm{d}t}=D$ except at discontinuities.

We incorporate the following definitions which are

required for the derivation of the duality results.

Definition 1. (*Second-order Invex*): If there exists a vector function $\eta = \eta(t,x,\bar{x}) \in R^n$ where $\eta : I \times R^n \times R^n \to R^n$ and with $\eta = 0$ at $t = a$ and $t = b$ such that for a scalar function $\phi(t,x,\dot{x})$, the functional $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ where $\phi : I \times R^n \times R^n \to R$ satisfies

$$\int_I \phi(t,x,\dot{x})\mathrm{d}t - \int_I \left\{\phi\left(t,\bar{x},\dot{\bar{x}}\right) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}\mathrm{d}t$$

$$\geq \int_I \left\{\eta^T \phi_x\left(t,\bar{x},\dot{\bar{x}}\right) + (D\eta)^T \phi_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right) + \eta^T G\beta(t)\right\}\mathrm{d}t$$

then $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ is second-order invex with respect to η where $G = \phi_{xx} - 2D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}} - D^3\phi_{\ddot{x}\dot{x}}$ and $\beta \in C(I,R^n)$ the space of n-dimensional continuous vector functions.

Definition 2. (*Second-order Pseudoinvex*): If the functional $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ satisfies

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\}\mathrm{d}t \geq 0$$

$$\Rightarrow \int_I \phi(t,x,\dot{x})\mathrm{d}t \geq \int_I \left\{\phi\left(t,\bar{x},\dot{\bar{x}}\right) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}\mathrm{d}t$$

then $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ is said to be second-order pseudo-invex with respect to η.

Definition 3. (Second-order strict-pseudoinvex): If the functional $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ satisfies

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\}\mathrm{d}t \geq 0$$

$$\Rightarrow \int_I \phi(t,x,\dot{x})\mathrm{d}t > \int_I \left\{\phi\left(t,\bar{x},\dot{\bar{x}}\right) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}\mathrm{d}t$$

then $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ is said to be second-order pseudo-invex with respect to η.

Definition 4. (*Second-order Quasi-invex*): If the functional $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ satisfies

$$\int_I \phi(t,x,\dot{x})\mathrm{d}t \leq \int_I \left\{\phi\left(t,\bar{x},\dot{\bar{x}}\right) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}\mathrm{d}t$$

$$\Rightarrow \int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\}\mathrm{d}t \leq 0$$

then $\int_I \phi(t,x,\dot{x})\mathrm{d}t$ is said to be second-order quasi-invex with respect to η.

Remark 1. If ϕ does not depend explicitly on *t*, then the above definitions reduce to those for static cases.

The following inequality will also be required in the forthcoming analysis of the research:

Lemma: 1 (Schwartz inequality): It states that

$$x(t)^T B(t) z(t) \leq \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \left(z(t)^T B(t) z(t)\right)^{\frac{1}{2}}$$

with equality in (1) if $B(t)x(t) - q(t)z(t) = 0$ for some $q(t) \in R, t \in I$.

Throughout the analysis of this research, the following conventions for the inequalities will be used:

If $\alpha, \beta \in R^n$ with $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \cdots, \alpha^n)$ and $\boldsymbol{\beta} = (\beta^1, \beta^2, \cdots, \beta^n)$, then

$$\alpha \geqq \beta \Leftrightarrow \alpha^i \geqq \beta^i, (i = 1,2,\cdots,n)$$

$$\alpha \geq \beta \Leftrightarrow \alpha \geqq \beta \text{ and } \alpha \neq \beta$$

$$\alpha > \beta \Leftrightarrow \alpha^i > \beta^i, (i = 1,2,\cdots,n).$$

3. Statement of the Problem and Necessary Optimality Conditions

Consider the following nondifferentiable Multiobjective variational problem:

(VCP): Minimize

$$\left(\int_I \left[f^1(t,x(t),\dot{x}(t)) + \left(x(t)^T B^1(t) x(t)\right)^{\frac{1}{2}}\right]\mathrm{d}t, \cdots,\right.$$

$$\left.\int_I \left[f^p(t,x(t),\dot{x}(t)) + \left(x(t)^T B^p(t) x(t)\right)^{\frac{1}{2}}\right]\mathrm{d}t\right)$$

subject to

$$x(a) = \alpha, x(b) = \beta \tag{1}$$

$$g(t,x(t),\dot{x}(t)) \leqq 0, t \in I \tag{2}$$

$$x \in C(I,R^n)$$

where 1) $C(I,R^n)$ denote the space of piecewise smooth functions *x* with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where differentiation operator *D* already defined.

2) $f^i : I \times R^n \times R^n \to R, i \in K = \{1,2,\cdots,p\}$,

$g : I \times R^n \times R^n \to R^m$

are assumed to be continuously differentiable functions, and 3) for each $t \in I, i \in K = \{1,2,\ldots,p\}$, $B^i(t)$ is an $n \times n$ positive semi definite (symmetric) matrix, with $B(.)$ continuous on I.

In this section we will derive Fritz John and Karush-Kuhn-Tucker type necessary optimality conditions for (VCP).

Definition: A point $\bar{x} \in X$ is said to be efficient solution of (VCP) if there exist $x(t) \in X$ such that

$$\int_I \left(\left(f^r\left(t,x,\dot{x}\right)\right) + \left(x(t)^T B^r(t) x(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

$$< \int_I \left(\left(f^r\left(t,\overline{x},\dot{\overline{x}}\right)\right) + \left(\overline{x}(t)^T B^r(t) \overline{x}(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t,$$

for some $r \in K = \{1,2,\cdots,p\}$ and

$$\int_I \left(\left(f^i\left(t,x,\dot{x}\right)\right) + \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

$$\leqq \int_I \left(\left(f^i\left(t,\overline{x},\dot{\overline{x}}\right)\right) + \left(\overline{x}(t)^T B^i(t) \overline{x}(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t,$$

for $r \in K_r = K - \{r\}$.

The following result which is a recast of a result of Chankong and Haimes [5] giving a linkage between an efficient solution of (VCP) and an optimal solution of p-single objective variational problem:

Proposition 1. (Chankong and Haimes [5]): A point $\overline{x}(t) \in X$ is an efficient solution of (VCP) if and only if $\overline{x}(t)$ is an optimal solution of (P_r) for each $r \in K$.

(P_r): Minimize

$$\int_I \left(\left(f^r\left(t,x(t)\right),\dot{x}(t)\right) + \left(x(t)^T B^r(t) x(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

subject to

$$x(a) = \alpha,\ x(b) = \beta,\quad g\left(t,x,\dot{x}\right) \leqq 0,\ t \in I$$

$$\int_I \left(f^i\left(t,x,\dot{x}\right) + \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

$$\leqq \int_I \left(\left(f^i\left(t,\overline{x},\dot{\overline{x}}\right)\right) + \left(\overline{x}(t)^T B^i(t) \overline{x}(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t,\ i \in K_r$$

for obtaining the optimal conditions for (VCP) we will use the optimal conditions obtained by Chandra *et al.* [6] for a single-objective variational problem which does not contain integral inequality constraints of (P_r).

The validity of the following proposition is quite essential in obtaining the optimality conditions for (VCP),

Proposition 2. If $\overline{x}(t) \in X$ is an efficient solution of (VCP), then $\overline{x}(t)$ is an optimal solution of the following problem $(\hat{P}_r)$ for each $r \in K$.

$(\hat{P}_r)$: Minimize

$$\int_I \left(f^r\left(t,x,\dot{x}\right) + \left(x(t)^T B^r(t) x(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

subject to

$$x(a) = \alpha,\ x(b) = \beta,\ g\left(t,x,\dot{x}\right) \leqq 0,\ t \in I$$

$$\int_I \left(f^i\left(t,x,\dot{x}\right) + \left(x(t)^T B^i(t) x(t)\right)^{1/2} \right) \mathrm{d}t$$

$$\leqq \int_I \left(f^i\left(t,\overline{x},\dot{\overline{x}}\right) + \left(\overline{x}(t)^T B^i(t) \overline{x}(t)\right)^{1/2} \right) \mathrm{d}t, i \in K_r, t \in I$$

Proof: Let $\overline{x}(t)$ be an efficient solution of (VCP). Suppose that $\overline{x}(t)$ is not optimal solution of $(\hat{P}_r)$, for some $r \in K$. Then there exists an $x^0(t) \in X$ such that

$$g\left(t,x,\dot{x}\right) \leqq 0,\ t \in I$$

$$\int_I \left(f^i\left(t,x^0,\dot{x}^0\right) + \left(x^0(t)^T B^i(t) x^0(t)\right)^{1/2} \right) \mathrm{d}t$$

$$\leqq \int_I \left(f^i\left(t,\overline{x},\dot{\overline{x}}\right) + \left(\overline{x}(t)^T B^i(t) \overline{x}(t)\right)^{1/2} \right) \mathrm{d}t, i \in K_r, t \in I \tag{3}$$

and

$$\int_I \left(\left(f^r\left(t,x^0,\dot{x}^0\right)\right) + \left(x^0(t)^T B^r(t) x^0(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

$$< \int_I \left(\left(f^r\left(t,\overline{x},\dot{\overline{x}}\right)\right) + \left(\overline{x}(t)^T B^r(t) \overline{x}(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t,$$

The inequality (3) for $i \in K_r$

$$\int_I \left(f^i\left(t,x^0,\dot{x}^0\right) + \left(x^0(t)^T B^i(t) x^0(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t$$

$$\leqq \int_I \left(\left(f^i\left(t,\overline{x},\dot{\overline{x}}\right)\right) + \left(\overline{x}(t)^T B^i(t) \overline{x}(t)\right)^{\frac{1}{2}} \right) \mathrm{d}t \tag{4}$$

The inequalities (3) along with (4) contradicts the fact that $\overline{x}(t)$ is an efficient solution of (VCP).

Hence $\overline{x}(t)$ is an optimal solution of $(\hat{P}_r)$, for some $r \in K$.

Theorem 1. (Fritz John Type necessary optimality condition): Let $\overline{x}(t)$ be an efficient solution of (VCP). Then there exist $\overline{\lambda}^i \in R, i \in K$ and piecewise smooth functions $\overline{y} : \mathrm{I} \to R^m$ and $\overline{z}^i : \mathrm{I} \to R^n, i \in K$, such that

$$\sum_{i=1}^{p} \overline{\lambda}^i \left(f_x^i\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)\right) + B^i(t)\overline{z}^i(t)$$
$$-Df_{\dot{x}}^i\left(\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)\right) + \overline{y}(t)^T g_x\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right) \tag{5}$$
$$-D\overline{y}(t)^T g_{\dot{x}}\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right) = 0, t \in I$$

$$\overline{y}(t)^T g\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right) = 0, t \in I \tag{6}$$

$$\overline{x}(t)^T B^i(t)\overline{z}^i(t) = \left(\overline{x}(t)^T B^i(t) \overline{x}(t)\right)^{\frac{1}{2}}, i \in K \tag{7}$$

$$\overline{z}^i(t)^T B^i(t)\overline{z}^i(t) \leqq 1, i \in K \tag{8}$$

$$\left(\overline{\lambda}, \overline{y}(t)\right) \geq 0, t \in I \tag{9}$$

Proof: Since $\overline{x}(t)$ is an efficient solution of (VCP), by Proposition 2, $\overline{x}(t)$ is an efficient solution of $(\hat{P}_r)$

for each $r \in K$. and hence in particular $\left(\hat{P}_1\right)$. So by the results of [6] there exist $\bar{\lambda}^i \in R, i \in K$ and piecewise smooth functions $\bar{y}: \mathrm{I} \to R^m$ and $\bar{z}^i: \mathrm{I} \to R^n, i \in K$ such that

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f_x^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\right) + B^i(t)\bar{z}^i(t)$$

$$-Df_{\dot{x}}^i\left(\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\right) + \bar{y}(t)^T g_x\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)$$

$$-D\bar{y}(t)^T g_{\dot{x}}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0,\ t \in I$$

$$\bar{y}(t)^T g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0, t \in I$$

$$\bar{x}(t)^T B^i(t)\bar{z}^i(t) = \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, i \in K, t \in I$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1, i \in K, t \in I$$

$$\left(\bar{\lambda}^1, \bar{\lambda}^2, \cdots, \bar{\lambda}^k, y(t)\right) \geq 0, t \in I.$$

The above conditions yield the relations (5) to (9).

Theorem 2 (Kuhn-Tucker type necessary optimality condition):

Let $\bar{x}(t)$ be an efficient solution of (VCP) and let for each $r \in k$, the conditions of $\left(\hat{P}_r\right)$ satisfy Slaters or Robinson condition [6] at $\bar{x}(t)$. Then there exist $\bar{\lambda} \epsilon R^p$ and piecewise smooth functions $\bar{y}: I \to R^m$ and $\bar{z}^i: I \to R^n,\ i \in K$ such that

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f_x^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\right) + B^i(t)\bar{z}^i(t)$$

$$-Df_{\dot{x}}^i\left(\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\right) + \bar{y}(t)^T g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)$$

$$-D\left(\bar{y}(t)^T g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\right) = 0, t \in I \quad (10)$$

$$\bar{y}(t)^T g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0, t \in I \quad (11)$$

$$\bar{x}(t)^T B^i(t)\bar{z}^i(t) = \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, \quad (12)$$

$$i = 1, 2, \cdots, p, t \in I$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1, t \in I \quad (13)$$

$$\bar{y}(t) \geqq 0, t \in I \quad (14)$$

$$\bar{\lambda} > 0 \quad (15)$$

Proof: Since $\bar{x}(t)$ is an efficient solution of (VCP) by Proposition 2, $\bar{x}(t)$ is an optimal solution of $\left(\bar{P}_r\right)$ for each $r \in k$. Since for each $r \in K$, the contradicts of $\left(\bar{P}_r\right)$, satisfy Slaters or Robinson conditions [6] at $\bar{x}(t)$, by Kuhn-Tucker necessary condition of [6], for each $r \in K = \{1, 2, \ldots, p\}$, there exist $\bar{v}_r^i \in R, \left(i \in K_r\right)$ and piecewise smooth function $\mu_r^j(t) \in R, i \in K$ such that

$$f_x^r\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + B^r(t)\bar{z}^r(t) - Df_{\dot{x}}^r\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)$$

$$+\sum_{i \in K_r} \bar{v}_r^i \left(f_x^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + B^i(t)\bar{z}^i(t) - Df_{\dot{x}}^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\right)$$

$$+\sum_{j=1}^{m} \begin{pmatrix} \mu_r^j(t)^T g_x^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \\ -Dg_{\dot{x}}^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \end{pmatrix} = 0, t \in I$$

$$\sum_{j=1}^{m} \mu_r^j(t)^T g^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0, t \in I$$

$$\bar{x}(t)^T B^i(t)\bar{z}^i(t) = \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, i \in K, t \in I$$

$$\mu_r^j(t) \geqq 0, t \in I, j \in \{1, 2, \cdots, m\}$$

$$v_r^i \geqq 0, i \in K_r$$

Summing over $r \in K$ we obtain

$$\sum_{i=1}^{p} \left(\bar{v}_1^i + \bar{v}_2^i + \ldots + \bar{v}_p^i\right) \begin{pmatrix} f_x^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + B^i(t)\bar{z}^i(t) \\ -Df_{\dot{x}}^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \end{pmatrix} +$$

$$\sum_{j=1}^{m} \begin{bmatrix} \left(\mu_1^j(t) + \mu_2^j(t) + \ldots + \mu_p^j(t)\right) g_x^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \\ -D\left(\mu_1^j(t) + \mu_2^j(t) + \ldots + \mu_p^j(t)\right) g_{\dot{x}}^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \end{bmatrix} = 0,$$

$$t \in I$$

$$\sum \left(\mu_1^j(t) + \mu_2^j(t) + \cdots + \mu_p^j(t)\right) g^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0,$$

$$t \in I$$

where $v_r^i = 1$ for each $i \in K$.

These can be written as,

$$\sum_{i=1}^{p} \bar{v}^i \begin{pmatrix} f_x^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + B^i(t)\bar{z}^i(t) \\ -Df_{\dot{x}}^i\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \end{pmatrix}$$

$$+\sum_{j=1}^{m} \begin{bmatrix} \mu^j(t) g_x^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) - \\ D\mu^j(t) g_{\dot{x}}^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \end{bmatrix} = 0,$$

$$\sum_{j=1}^{m} \bar{\mu}^j(t) g^j\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0, t \in I$$

where $\bar{v}^i = 1 + \sum_{r \in K_r} \bar{v}_r^i > 0$

and $\mu^j(t) = \sum_{j=1}^{m} \mu_r^j(t) \geqq 0, t \in I, j \in \{1, 2, \cdots, m\}$

Setting

$$\bar{\lambda}^i = \frac{\bar{v}^i}{\sum_{i=1}^{p} \bar{v}^i}, i \in K \text{ and } \bar{y}^i(t) = \frac{\bar{\mu}^j(t)}{\sum \bar{v}^i}, j \in \{1, 2, \cdots, m\}$$

we get

$$\sum_{i=1}^{p}\bar{\lambda}^i\left(f_x^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)+B^i(t)\bar{z}^i(t)\right.$$
$$\left.-Df_{\dot{x}}^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)\right)$$
$$+\sum_{i=1}^{m}\left(\bar{y}^i(t)g_x^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)\right.$$
$$\left.-D\bar{y}^i(t)g_{\dot{x}}^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)\right)=0,t\in I$$
$$\sum_{j=1}^{m}\bar{y}^j(t)g^j\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)=0,t\in I$$

That is

$$\sum_{i=1}^{p}\bar{\lambda}^i\left(f_x^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)+B^i(t)\bar{z}^i(t)\right.$$
$$\left.-Df_{\dot{x}}^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)\right)$$
$$+\bar{y}(t)^T g\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)$$
$$-D\left(\bar{y}(t)^T g\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)\right)=0,t\in I$$
$$\bar{y}(t)^T g\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)=0,t\in I$$
$$\bar{x}(t)^T B^i(t)\bar{z}^i(t)=\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}},i\in K$$
$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t)\leqq 1,i\in K$$
$$\bar{y}(t)=\left(\bar{y}^1(t),\bar{y}^2(t),\cdots,\bar{y}^m(t)\right)\geqq 0,t\in I$$
$$\bar{\lambda}=\left(\bar{\lambda}^1,\bar{\lambda}^2,\cdots,\bar{\lambda}^p\right)>0.$$

4. Mond-Weir Type Second Order Duality

In this section, we present the following Mond-Weir type second-order dual to (VCP) and validate duality results:

(M-WD): Maximize

$$\left(\int_I\left(f^1\left(t,x,\dot{x}\right)+x(t)^T B^1(t)z(t)-\frac{1}{2}\beta(t)^T H^1\beta(t)\right)dt,\cdots,\right.$$
$$\left.\int_I\left(f^p\left(t,x,\dot{x}\right)+x(t)^T B^p(t)z(t)-\frac{1}{2}\beta(t)^T H^p\beta(t)\right)dt\right)$$

subject to

$$x(a)=\alpha,x(b)=\beta \tag{16}$$

$$\sum_{i=1}^{n}\lambda^i\left(f_x^i+B^i(t)z^i(t)-Df_{\dot{x}}^i+H^i\beta(t)\right)$$
$$+y(t)^T g_x-Dy(t)^T g_{\dot{x}}+G\beta(t)=0,t\in I \tag{17}$$

$$\int_I\left(y(t)^T g(t,x,\dot{x})-\frac{1}{2}\beta(t)^T G\beta(t)\right)dt\geqq 0 \tag{18}$$

$$\lambda>0,y(t)\geqq 0,\ z^i(t)^T B^i(t)z^i(t)\leqq 1,$$
$$i\in K,t\in I \tag{19}$$

where

$$G=\left(y^T g_x\right)_x-2D\left(y^T g_x\right)_{\dot{x}}$$
$$+D^2\left(y^T g_{\dot{x}}\right)_{\dot{x}}-D^3\left(y^T g_{\dot{x}}\right)_{\ddot{x}}$$

and

$$H^i=f_{xx}^i-2Df_{x\dot{x}}^i+D^2f_{\dot{x}\dot{x}}^i-D^3f_{\dot{x}\ddot{x}}^i,t\in I,$$
$$i\in K=\{1,2,\cdots,p\}.$$

We denote by C_P and C_D the sets of feasible solutions to (VCP) and (M-WD) respectively.

Theorem 3. (Weak Duality): Assume that

(A_1) $\bar{x}(t)\in C_P$ and

$$\left(x(t),y(t),z^1(t),\cdots,z^p(t),\beta(t)\right)\in C_D.$$

(A_2) $\sum\lambda^i\int_I\left[f^i\left(t,.,.\right)+(.)^T B^i z^i(t)\right]dt$ is second-order pseudoinvex.

(A_3) $\int_I y(t)^T g(t,.,.)dt$ is second-order quasi-invex.

Then

$$\int_I\left[f^r\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)+\left(\bar{x}(t)^T B^r(t)\bar{x}(t)\right)^{\frac{1}{2}}\right]dt$$
$$<\int_I\left[f^r\left(t,x,\dot{x}\right)+x(t)^T B^r(t)z^r(t)\right. \tag{20}$$
$$\left.-\frac{1}{2}\beta(t)^T H^r\beta(t)\right]dt,\text{ for some } r\in K$$

and

$$\int_I\left[f^i\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)+\left(\bar{x}(t)^T B^r(t)\bar{x}(t)\right)^{\frac{1}{2}}\right]dt$$
$$\leqq\int_I\left[f^i\left(t,x,\dot{x}\right)+x(t)^T B^i(t)z^i(t)\right. \tag{21}$$
$$\left.-\frac{1}{2}\beta(t)^T H^i\beta(t)\right]dt,i\in K_r$$

cannot hold.

Proof. Suppose to the contrary, that (20) and (21) hold.

Since $\lambda>0$, we have

$$\sum_{i=1}^{p}\int_I \lambda^i\left[f^i\left(t,\bar{x},\dot{\bar{x}}\right)+\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right]\mathrm{d}t$$
$$\leqq \sum_{i=1}^{p}\int_I \lambda^i\Big[f^i(t,x,\dot{x})+x(t)^T B^i(t)z^i(t) -\frac{1}{2}\beta(t)^T H^i\beta(t)\Big]\mathrm{d}t$$

Since $\bar{x}(t)^T B^i(t)z^i(t)\leqq\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, t\in I,$

we have

$$\sum_{i=1}^{p}\int_I \lambda^i\left[f^i\left(t,\bar{x},\bar{x}(t)\right)+\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right]\mathrm{d}t$$
$$< \sum_{i=1}^{p}\int_I \lambda^i\Big[f^i(t,x,\dot{x})+x(t)^T B^i(t)z^i(t) -\frac{1}{2}\beta(t)^T H^i\beta(t)\Big]\mathrm{d}t$$

Now, by the constraints (2), (18) and (19), we have

$$\int_I y(t)^T g(t,x,\dot{x})\mathrm{d}t \leqq \int_I\left\{y(t)^T g(t,x,\dot{x})-\frac{1}{2}\beta(t)^T G\beta(t)\right\}\mathrm{d}t$$

This by (A_3), yields

$$\int_I\Big[\eta^T\left(y(t)^T g_x(t,x,\dot{x})\right) +(D\eta)^T\left(y(t)^T g_{\dot{x}}(t,x,\dot{x})\right)+\eta^T G\beta(t)\Big]\mathrm{d}t\leqq 0$$

By integration by parts, we have

$$0\geqq\int_I\Big[\eta^T\left(y(t)^T g_x\right)-\eta^T D\left(y(t)^T g_{\dot{x}}\right)+\eta^T G\beta(t)\Big]\mathrm{d}t +\eta\left(y(t)^T g_{\dot{x}}\right)\Big|_{t=a}^{t=b}$$

$$0\geqq\int_I\Big[\eta^T\left(y(t)^T g_x\right)-\eta^T D\left(y(t)^T g_{\dot{x}}\right)+\eta^T G\beta(t)\Big]\mathrm{d}t$$

This, by using (4) gives

$$\sum_{i=1}^{p}\lambda^i\int_I\eta^T\left(f_x^i+B^i z^i(t)-Df_{\dot{x}}^i+H^i\beta(t)\right)\mathrm{d}t\geqq 0 \quad (22)$$

$$0\leqq\sum_{i=1}^{p}\lambda^i\int_I\Big[\eta^T\left(f_x^i+B^i(t)z^i(t)\right)-\eta^T Df_{\dot{x}}^i+\eta^T H^i\beta(t)\Big]\mathrm{d}t$$

$$0=\sum_{i=1}^{p}\lambda^i\int_I\begin{bmatrix}\eta^T\left(f_x^i+B^i(t)z^i(t)\right)+(D\eta)^T f_{\dot{x}}^i\\ +\eta^T H^i\beta(t)\end{bmatrix}\mathrm{d}t -\sum_{i=1}^{p}\eta^T\lambda^i f_{\dot{x}}^i\Big|_{t=b}^{t=a}$$

$$0\leqq\sum_{i=1}^{p}\lambda^i\int_I\Big[\eta^T\left(f_x^i+B^i(t)z^i(t)\right) +(D\eta)^T f_{\dot{x}}^i+\eta^T H^i\beta(t)\Big]\mathrm{d}t$$

By hypothesis (A_1), it implies

$$\sum_{i=1}^{p}\lambda^i\int_I\left(f^i\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^T B^i(t)z^i(t)\right)\mathrm{d}t$$
$$\geqq\sum_{i=1}^{p}\lambda^i\int_I\Big(f^i(t,x,\dot{x})+x(t)^T B^i(t)z^i(t) -\frac{1}{2}\beta(t)^T H^i\beta(t)\Big)\mathrm{d}t$$

Using $\bar{x}(t)^T B^i(t)z^i(t)\leqq\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, t\in I$ in the above, we have

$$\sum_{i=1}^{p}\lambda^i\int_I\left(f^i\left(t,\bar{x},\dot{\bar{x}}\right)+\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right)\mathrm{d}t$$
$$\geqq\sum_{i=1}^{p}\lambda^i\int_I\Big(f^i(t,x,\dot{x})+x(t)^T B^i(t)z^i(t) -\frac{1}{2}\beta(t)^T H^i\beta(t)\Big)\mathrm{d}t$$

This contradicts (20) and (21). Hence the result.

Theorem 4 (Strong duality): Let $\bar{x}(t)$ be normal and is an efficient solution of (VP). Then there exist $\lambda\in R^p$, a piecewise smooth function $y:I\to R^m$ such that $\left(\bar{x}(t),\bar{y}(t),\bar{\lambda},z^1(t),\cdots,z^p(t),\bar{\beta}(t)=0\right)$ is feasible for (M-WD) and the two objective functions are equal. Furthermore, if the hypotheses of Theorem 3 hold for all feasible solutions of (VCP) and (M-WD) ,then $\left(\bar{x}(t),\bar{y}(t),\bar{\lambda},z^1(t),\cdots,z^p(t),\bar{\beta}(t)\right)$ is an efficient solution of (M-WD).

Proof: Since $\bar{x}(t)$ is normal and an efficient solution of (VP), by Proposition 2, there exist $\lambda\in R^p$ and piecewise smooth $y:I\to R^m$ and $w^i:I\to R^m, i=1,2,\cdots,n$ satisfying

$$\sum_{i=1}^{p}\lambda^i\left(f_x^i(t,x,\dot{x})+B^i(t)z^i(t)-Df_{\dot{x}}^i\left(t,\bar{x},\dot{\bar{x}}\right) +\left(y(t)^T g_x\right)-D\left(y(t)^T g_{\dot{x}}\right)\right)=0, t\in I \quad (23)$$

$$y(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)=0, t\in I \quad (24)$$

$$\bar{x}(t)^T B^i(t)z^i(t)=\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}, t\in I \quad (25)$$

$$z^i(t)^T B^i(t)z^i(t)\leqq 1, t\in I \quad (26)$$

$$\lambda\geqq 0, y(t)\geqq 0, t\in I. \quad (27)$$

From (24) along with $\beta(t)=0, t\in I$, we have

$$\int_I\left(y(t)^T g(t,x,\dot{x})-\frac{1}{2}\beta(t)^T G\beta(t)\right)\mathrm{d}t=0$$

Hence

$$\left(\overline{x}(t),\overline{y}(t),\overline{\lambda},z^1(t),\cdots,z^p(t),\overline{\beta}(t)=0\right)$$

satisfies the constraints of (M-WD) and

$$\int_I\left(f^i(t,x,\dot{x})+\left(\overline{x}(t)^T B^i(t)\overline{x}(t)\right)^{\frac{1}{2}}\right)\mathrm{d}t$$

$$=\int_I\left(f^i(t,x,\dot{x})+\overline{x}(t)^T B^i(t)z^i(t)-\frac{1}{2}\beta(t)^T H^i\beta(t)\right)\mathrm{d}t$$

That is, the two objective functionals have the same value.

Suppose that $\left(x(t),z^1(t),\cdots,z^p(t),y(t),\beta(t)=0\right)$ is not the efficient solution of (M-WD). Then there exists $\left(\hat{x}(t),\hat{y}(t),\hat{z}^1(t),\cdots,\hat{z}^p(t),\hat{\beta}(t)\right)\in C_D$, such that

$$\begin{bmatrix}\int_I\left(f^1(t,\hat{x},\dot{\hat{x}})+\hat{x}(t)^T B^1(t)\hat{z}^1(t)-\frac{1}{2}\hat{\beta}(t)^T H^1\hat{\beta}(t)\right)\mathrm{d}t,\ldots,\\ \int_I\left(f^p(t,\hat{x},\dot{\hat{x}})+\hat{x}(t)^T B^p(t)\hat{z}^p(t)-\frac{1}{2}\hat{\beta}(t)^T H^p\hat{\beta}(t)\right)\mathrm{d}t\end{bmatrix}\overset{\geqq}{=}$$

$$\begin{bmatrix}\int_I\left(f^1(t,\overline{x},\dot{\overline{x}})+\hat{x}(t)^T B^1(t)\hat{z}^1(t)-\frac{1}{2}\overline{\beta}(t)^T H^1\overline{\beta}(t)\right)\mathrm{d}t,\ldots,\\ \int_I\left(f^p(t,\overline{x},\dot{\overline{x}})+\hat{x}(t)^T B^p(t)\hat{z}^p(t)-\frac{1}{2}\overline{\beta}(t)^T H^p\overline{\beta}(t)\right)\mathrm{d}t\end{bmatrix}$$

As $\hat{\beta}(t)=0$, we have

$$\begin{bmatrix}\int_I\left(f^1(t,\hat{x},\dot{\hat{x}})+\hat{x}(t)^T B^1(t)\hat{z}^1(t)-\frac{1}{2}\hat{\beta}(t)^T H^1\hat{\beta}(t)\right)\mathrm{d}t,\ldots,\\ \int_I\left(f^p(t,\hat{x},\dot{\hat{x}})+\hat{x}(t)^T B^p(t)\hat{z}^p(t)-\frac{1}{2}\hat{\beta}(t)^T H^p\hat{\beta}(t)\right)\mathrm{d}t\end{bmatrix}$$

$$\overset{\geqq}{=}\begin{bmatrix}\int_I\left(f^1(t,\overline{x},\dot{\overline{x}})+\left(\overline{x}(t)^T B^1(t)\overline{x}(t)^T\right)^{\frac{1}{2}}\right)\mathrm{d}t,\ldots,\\ \int_I\left(f^p(t,\overline{x},\dot{\overline{x}})+\left(\overline{x}(t)^T B^1(t)\overline{x}(t)^T\right)^{\frac{1}{2}}\right)\mathrm{d}t\end{bmatrix}$$

This contradicts the conclusion of Theorem 3. Hence $\left(\overline{x}(t),\overline{y}(t),\overline{\lambda},z^1(t),\cdots,z^p(t),\overline{\beta}(t)\right)$ is an efficient solution of (M-WD).

Theorem 5 (Converse duality):

(A_1): Assume that $\left(\overline{x},\overline{\lambda},\overline{y},z^1,\cdots,z^p,\beta\right)$ is an efficient solution of (M-WD)

(A_2): The vectors $\left\{H^i_j,G_j,t\in I,i\in K,j=1,2,\cdots,n\right\}$ are linearly independent where H^i_j the j^{th} row of is H^i and G_j is the j^{th} row of G,

(A_3) $f^i_x\left(t,\overline{x},\dot{\overline{x}}\right)+B^i(t)z^i(t)$

$-Df^i_{\dot{x}}\left(t,\overline{x},\dot{\overline{x}}\right)+H^i\beta(t),t\in I,i\in K$

are linearly independent and

(A_4) for $t\in I$ either

a) $\int_I\beta(t)^T\left(G+\left(y(t)^T g_x\right)_x\right)\beta(t)\mathrm{d}t>0$

and $\int_I\beta(t)^T\left(y^T(t)g_x\right)\overset{\geqq}{=}0$

or b) $\int_I\beta(t)\left(G+\left(y(t)^T g_x\right)_x\right)\beta(t)\mathrm{d}t<0$

and $\int_I\beta(t)^T\left(y^T(t)g_x\right)\overset{\leqq}{=}0$

Then $\overline{x}$ is feasible for (VCP) and the two objective functionals have the same value. Also, if Theorem 3 holds for all feasible solutions of (CP) and (M-WD), the $\overline{x}$ is an efficient solution of (VCP).

Proof: Since $\left(\overline{x},\overline{\lambda},\overline{y},z^1,\cdots,z^p,\beta\right)$ is an efficient solution of (M-WD), there exist $\tau\in R^p$, $\alpha\in R^p$ and $r\in R$, and piecewise smooth $\theta:I\to R^n$, $\varnothing^i:I\to R$, $i=1,2,\cdots,n$ and $\mu:I\to R^n$ such that the following Fritz John optimality conditions (Theorem 1)

$$\sum\tau^i\begin{pmatrix}f^i_x+B^i(t)z^i(t)-Df^i_{\dot{x}}-\frac{1}{2}\left(\beta(t)^T H^i\beta(t)\right)_x\\ +\frac{1}{2}D\left(\beta(t)^T H^i\beta(t)\right)_{\dot{x}}-\frac{1}{2}D^2\left(\beta(t)^T H^i\beta(t)\right)_{\ddot{x}}\\ +\frac{1}{2}D^3\left(\beta(t)^T H^i\beta(t)\right)_{\dddot{x}}-\frac{1}{2}D^4\left(\beta(t)^T H^i\beta(t)\right)_{\ddddot{x}}\end{pmatrix}$$

$$-\sum_{i=1}^p\overline{\lambda}^i\begin{pmatrix}H^i+\left(H^i\beta(t)\right)_x-D\left(H^i\beta(t)\right)_{\dot{x}}\\ +D^2\left(H^i\beta(t)\right)_{\ddot{x}}-D^3\left(H^i\beta(t)\right)_{\dddot{x}}+D^4\left(H^i\beta(t)\right)_{\ddddot{x}}\\ +G+\left(G\beta(t)\right)_x-D\left(G\beta(t)\right)_{\dot{x}}+D^2\left(G\beta(t)\right)_{\ddot{x}}\\ -D^3\left(G\beta(t)\right)_{\dddot{x}}+D^4\left(G\beta(t)\right)_{\ddddot{x}}\end{pmatrix}\theta(t)$$

$$+\gamma\begin{pmatrix}\left(y(t)^T g_x\right)-D\left(y(t)^T g_x\right)-\frac{1}{2}\left(\beta(t)^T H^i\beta(t)\right)_x\\ +\frac{1}{2}D\left(\beta(t)^T H^i\beta(t)\right)_{\dot{x}}-\frac{1}{2}D^2\left(\beta(t)^T H^i\beta(t)\right)_{\ddot{x}}\\ +\frac{1}{2}D^3\left(\beta(t)^T H^i\beta(t)\right)_{\dddot{x}}-\frac{1}{2}D^4\left(\beta(t)^T H^i\beta(t)\right)_{\ddddot{x}}\end{pmatrix}=0,t\in I \tag{28}$$

$$\theta(t)^T\left(f^i_x+B^i(t)z^i(t)+H^i\beta(t)\right)-\alpha^i=0,\quad t\in I,i\in K \tag{29}$$

$$\theta(t)^T\left(g^j_x+g^j_{xx}\beta(t)\right)-\gamma\left(g^j-\frac{1}{2}\beta(t)^T g^j_{xx}\beta(t)\right)-\mu^j(t)=0,\ t\in I,\ j=1,2,\cdots,m \tag{30}$$

$$\sum_{i=1}^p\tau^i H^i\beta(t)+\theta(t)^T\left(\sum_{i=1}^p\lambda^i H^i+G\right)+\gamma G\beta(t)=0, \tag{31}$$

$t\in I,\ j=1,2,\cdots,m$

$$\tau^i x^T(t)B^i(t)+\theta(t)^T B^i(t)-2\phi^i(t)B^i(t)z^i(t)=0, \tag{32}$$

$i\in K.$

$$\gamma\int_I\left(y(t)^T g-\frac{1}{2}\beta(t)^T G\beta(t)\right)\mathrm{d}t=0 \tag{33}$$

$$\mu(t)^T y(t) = 0, t \in I \tag{34}$$

$$\phi^i(t)\left(z^i(t)^T B^i(t) z^i(t) - 1\right) = 0, i = 1,2,\cdots,n \tag{35}$$

$$\alpha^T \lambda = 0 \tag{36}$$

$$\left(\tau, \gamma, \mu(t), \alpha, \phi^1, \cdots, \phi^n\right) \geqq 0 \tag{37}$$

$$\left(\tau, \theta(t), \gamma, \mu(t), \alpha, \phi^1, \phi^2, \cdots, \phi^n\right) \neq 0 \tag{38}$$

From (31), we have

$$\sum_{i=1}^{p} H^i\left(\tau^i \beta(t) + \lambda^i \theta(t)\right) + \left(\theta(t) + \gamma\beta(t)\right) G = 0 \tag{39}$$

This, by the hypothesis (A_2) gives

$$\tau^i \beta(t) + \lambda^i \theta(t) = 0, t \in I \tag{40}$$

and

$$\theta(t) + \gamma\beta(t) = 0, t \in I \tag{41}$$

Using (40), (41) and (17), we have

$$\sum (\tau^i - \gamma\lambda^i)\left(f_x^i + B^i(t) z^i(t) - D f_{\dot{x}}^i + H^i \beta(t)\right)$$

$$-\sum \tau^i \begin{pmatrix} -\frac{1}{2}\left(\beta(t)^T H^i \beta(t)\right)_x + \frac{1}{2} D\left(\beta(t)^T H^i \beta(t)\right)_{\dot{x}} \\ -\frac{1}{2} D^2\left(\beta(t)^T H^i \beta(t)\right)_{\ddot{x}} + \frac{1}{2} D^3\left(\beta(t)^T H^i \beta(t)\right)_{\dddot{x}} \\ -\frac{1}{2} D^4\left(\beta(t)^T H^i \beta(t)\right)_{\ddddot{x}} \end{pmatrix}$$

$$-\sum_{i=1}^{p} \bar{\lambda}^i \begin{pmatrix} \left(H^i \beta(t)\right)_x - D\left(H^i \beta(t)\right)_{\dot{x}} + D^2\left(H^i \beta(t)\right)_{\ddot{x}} \\ -D^3\left(H^i \beta(t)\right)_{\dddot{x}} + D^4\left(H^i \beta(t)\right)_{\ddddot{x}} + G \\ +\left(G\beta(t)\right)_x - D\left(G\beta(t)\right)_{\dot{x}} + D^2\left(G\beta(t)\right)_{\ddot{x}} \\ -D^3\left(G\beta(t)\right)_{\dddot{x}} + D^4\left(G\beta(t)\right)_{\ddddot{x}} \end{pmatrix} \theta(t)$$

$$+\gamma \begin{pmatrix} -\frac{1}{2}\left(\beta(t)^T H^i \beta(t)\right)_x \\ +\frac{1}{2} D\left(\beta(t)^T H^i \beta(t)\right)_{\dot{x}} - \frac{1}{2} D^2\left(\beta(t)^T H^i \beta(t)\right)_{\ddot{x}} \\ +\frac{1}{2} D^3\left(\beta(t)^T H^i \beta(t)\right)_{\dddot{x}} - \frac{1}{2} D^4\left(\beta(t)^T H^i \beta(t)\right)_{\ddddot{x}} \end{pmatrix} = 0,$$

$t \in I$

(42)

Let $\gamma = 0$. Then (41) gives $\theta(t) = 0$ and (40) gives $\tau^i \beta(t) = 0$, $t \in I$

Using $\theta(t) = 0$ and $\tau^i \beta(t) = 0$, (42) implies

$$\sum\left(\tau^i - \gamma\lambda^i\right)\left(f_x^i + B^i(t) z^i(t) - D f_{\dot{x}}^i + H^i \beta(t)\right) = 0$$

This, because of (A_3) yields

$$\tau^i - \gamma\lambda^i = 0 \tag{43}$$

The relation (43) with $\gamma = 0$ gives $\tau^i = 0$

Since $\lambda > 0$, (36) gives $\alpha = 0$. The relation (30) yields $\mu(t) = 0, t \in I$ we have $\phi^i(t) B^i(t) z^i(t) = 0$, $i \in K$ from (32) and $\phi^i(t) z^i(t)^T B^i(t) z^i(t) = \phi^i(t)$, $t \in I$, $i \in K$ from (35). These yield $\phi^i(t) = 0$, $t \in I$, $i \in K$.

Consequently

$$\left(\tau, \theta(t), \gamma, \mu(t), \alpha, \phi^1, \phi^2, \cdots, \phi^n\right) = 0, t \in I$$

contradicting (38).

Hence $\gamma > 0$ and from (43) $\tau^i > 0$.

Multiplying (30) by $y^j(t)$, summing over j, and then using (34) and (41), we have

$$\int_I \beta(t)^T \left(G + \left(y(t)^T g_x\right)_x\right) \beta(t)\,\mathrm{d}t$$

$$+2\int_I \beta(t)^T \left(y(t)^T g_x\right)\mathrm{d}t = 0$$

In view of the hypothesis (A_4), this gives $\beta(t) = 0$, $t \in I$, The relation (30) implies $g^j = -\dfrac{\mu^j(t)}{r} \leqq 0$, $j \in \{1,2,\cdots,m\}$ yielding the feasibility of $\bar{x}(t)$ for (VCP).

The relation (32) with $\theta(t) = 0$ and $\tau^i > 0$ gives

$$x(t)^T B^i(t) = 2\phi^i(t) B^i(t) z^i(t) / \tau^i, t \in I, i \in K \tag{44}$$

This by Schwartz inequality gives

$$x(t)^T B^i(t) z^i(t) = \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}} \left(z^i(t)^T B^i(t) z^i(t)\right)^{\frac{1}{2}} \tag{45}$$

If $\phi^i(t) > 0, i \in K$, then (35) give

$z^i(t)^T B^i(t) z^i(t) = 1$, $i \in K$. and so (45) implies

$$x(t)^T B^i(t) z^i(t) = \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}}, t \in I, i \in K.$$

If $\phi^i(t) = 0$, (44) gives $B^i(t) x(t) = 0$. So we still get

$$x(t)^T B^i(t) z^i(t) = \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}}, t \in I, i \in K$$

Now suppose that $\bar{x}(t)$ is not an efficient of (VCP). Then, there exists $\hat{x}(t)^T \in X$ such that

$$\int_I \left\{ f^r\left(t, \hat{x}, \dot{\hat{x}}\right) + \left(\hat{x}(t)^T B^r(t) \hat{x}(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

$$< \int_I \left\{ f^r\left(t, \bar{x}, \dot{\bar{x}}\right) + \left(\bar{x}(t)^T B^r(t) \bar{x}(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

and

$$\int_I \left\{ f^i\left(t, \hat{x}, \dot{\hat{x}}\right) + \left(\hat{x}(t)^T B^i(t) \hat{x}(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

$$\leqq \int_I \left\{ f^i\left(t, \bar{x}, \dot{\bar{x}}\right) + \left(\bar{x}(t)^T B^i(t) \bar{x}(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

Using $\beta(t)=0, t\in I$ and

$$\left(\overline{x}(t)^T B^i(t)\overline{x}(t)\right)^{\frac{1}{2}} = \overline{x}(t)^T B^i(t) z^i(t), t\in I$$

We have

$$\int_I \left\{ f^r\left(t,\hat{x},\dot{\hat{x}}\right) + \left(\hat{x}(t)^T B^r(t)\hat{x}(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$
$$< \int_I \left\{ f^r\left(t,\overline{x},\dot{\overline{x}}\right) + \overline{x}(t)^T B^r(t)\overline{z}(t) - \frac{1}{2}\beta(t)^T H^r \beta(t) \right\} \mathrm{d}t$$

for some $r\in K$, and

$$\int_I \left\{ f^i\left(t,\hat{x},\dot{\hat{x}}\right) + \left(\hat{x}(t)^T B^i(t)\hat{x}(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$
$$\leqq \int_I \left\{ f^i\left(t,\overline{x},\dot{\overline{x}}\right) + \overline{x}(t)^T B^i(t)\overline{z}(t) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} \mathrm{d}t,\ i\in K_r$$

This contradicts Theorem 3. Hence $\overline{x}(t)$ is an efficient solution for (VCP).

Theorem 6 (Strict converse duality): Assume that

$$\sum_{i=1}^{p} \overline{\lambda}^i \int_I \left(f^i(t,.,.) + (.)^T B^i(t)\overline{z}^i(t) \right) \mathrm{d}t$$

is second-order strictly pseudoinvex, and $\int_I \overline{y}(t)^T g(t,.,.)\,\mathrm{d}t$ is second-order quasi-invex with respect to same η. Assume also that (VCP) has an optimal solution $\overline{x}(t)$ which is normal [6]. If $\left(\overline{\lambda}, \hat{u}(t), \hat{y}(t), \hat{z}^1(t), \cdots, \hat{z}^p(t), \beta(t)\right)$ is an optimal solution of (M-WD), then $\hat{u}(t)$ is an efficient solution of (VCP) with $\overline{x}(t)=\hat{u}(t), t\in I$.

Proof: We assume that $\overline{x}(t)\neq\hat{u}(t), t\in I$ and exhibit a contradiction. Since $\overline{x}(t)$ is an efficient solution, it follows from Theorem, that there exist $\lambda^i\in R$, $i\in K$, $\overline{y}: I\to R^m$, $z^i\in R^n$, $i=1,2,\cdots,n$, $t\in I$ and $\overline{\beta}(t)\in R^n$ such that

$$\left(\overline{x}(t), \overline{y}(t), \overline{\beta}(t)=0, \lambda^1, \cdots, \lambda^p, \hat{u}(t), \hat{y}(t), \hat{z}^1(t), \cdots, \hat{z}^p(t)\right)$$

is an efficient solution of (M-WD). Since $\left(\hat{u}(t), \hat{y}(t), \hat{z}^1(t), \cdots, \hat{z}^p(t), \beta(t), \lambda\right)$ is an optimal solution of (M-WD), it follows that

$$\sum_{i=1}^{p} \overline{\lambda}^i \int_I \left(f^i\left(t,\overline{x},\dot{\overline{x}}\right) + \overline{x}(t)^T B^i(t) z^i(t) \right) \mathrm{d}t$$
$$= \sum_{i=1}^{p} \lambda^i \int_I \left\{ f^i\left(t,\hat{u},\dot{\hat{u}}\right) + \hat{u}(t)^T B^i(t) z^i(t) - \frac{1}{2}\beta(t)^T H^i \beta(t) \right\} \mathrm{d}t$$

This, because of second-order strict-pseudoinvexity of

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.) + (\cdot)^T B^i(t) z^i(t) \right) \mathrm{d}t,$$

for $z^i(t)\in R^n$, $i\in P$.

$$\int_I \left[\eta^T \sum_{i=1}^{p} \lambda^i \left(f_x^i + B^i(t) z^i(t) \right) + (D\eta)^T \left(\sum_{i=1}^{p} \lambda^i f_{\dot{x}}^i \right) + \eta^T \sum \lambda^i \left(H^i \beta(t) \right) \right] \mathrm{d}t < 0 \tag{46}$$

Also from the constraint of (VCP) and (M-WD), we have

$$\int_I y(t)^T g\left(t,\overline{x},\dot{\overline{x}}\right)\mathrm{d}t \leqq \left(y(t)^T g\left(t,\hat{u},\dot{\hat{u}}\right) - \frac{1}{2}\beta(t)^T G\beta(t) \right)\mathrm{d}t$$

Because of second-order quasi-invexity of $\int_I y(t)^T g(t,.,.)\,\mathrm{d}t$, this implies

$$\int_I \left[\eta^T \left(y(t)^T g_x \right) + (D\eta)^T \left(y(t)^T g_{\dot{x}} \right) + \eta^T G\beta(t) \right] \mathrm{d}t \leqq 0 \tag{47}$$

Combining (46) and (47), we have

$$0 > \int_I \left[\begin{array}{l} \eta^T \left\{ \sum_{i=1}^{p} \lambda^i \left(\begin{array}{l} f_x^i + B^i(t) z^i(t) + y(t)^T g_x \\ + H^i \beta(t) \end{array} \right) \right\} \\ + (D\eta)^T \left(\sum_{i=1}^{p} \lambda^i f_{\dot{x}}^i + y(t)^T g_{\dot{x}} \right) + \eta^T G\beta(t) \end{array} \right] \mathrm{d}t$$
$$= \int_I \eta^T \left[\begin{array}{l} \sum \lambda^i \left(f_x^i - D f_{\dot{x}}^i + B^i(t) z^i(t) + H^i \beta(t) \right) \\ + y(t)^T g_x - D\left(y(t)^T g_{\dot{x}} \right) + G\beta(t) \end{array} \right] \mathrm{d}t$$
$$+ \eta^T \left(\sum \lambda^i f_{\dot{x}}^i + y(t)^T g_{\dot{x}} \right)\Big|_{t=b}^{t=a},$$

(By integrating by parts)

This, by using $\eta = 0$, at $t = a$ and $t = b$, implies

$$\int_I \eta^T \left[\sum_{i=1}^{p} \lambda^i \left(f_x^i - D f_{\dot{x}}^i + B^i(t) z^i(t) + H^i \beta(t) \right) + y(t)^T g_x - D\left(y(t)^T g_{\dot{x}} \right) + G\beta(t) \right] \mathrm{d}t < 0,$$

contradicting the feasibility of $\left(\hat{u}(t), \lambda, \hat{y}(t), \overline{\beta}(t), \hat{z}^1(t), \cdots, \hat{z}^p(t)\right)$ for (M-WD).

5. Problems with Natural Boundary Values

In this section, we formulate a pair of nondifferentiable Mond-Weir type dual variational problems with natural boundary values rather than fixed end points given bellow

(VCP_0) : Minimize

$$\left(\int_I \left(f^1(t,x,\dot{x})\right)+\left(\overline{x}(t)^T B^1(t)\overline{x}(t)\right)^{\frac{1}{2}} \mathrm{d}t,\cdots,\right.$$

$$\left.\int_I \left(f^p(t,x,\dot{x})\right)+\left(\overline{x}(t)^T B^p(t)\overline{x}(t)\right)^{\frac{1}{2}} \mathrm{d}t\right)$$

Subject to

$$x(a)=0, x(b)=0 \quad g(t,x,\dot{x}) \leqq 0, t \in I$$

(VCD_0) : Maximize

$$\left(\int_I \left(\left(f^1(t,u,\dot{u})+u^T(t)B^1(t)z(t)-\frac{1}{2}\beta(t)^T H^1\beta(t)\right)\mathrm{d}t,\cdots,\right.$$

$$\left.\int_I \left(\left(f^p(t,u,\dot{u})+u^T(t)B^p(t)z(t)-\frac{1}{2}\beta(t)^T H^p\beta(t)\right)\mathrm{d}t\right)\right.$$

Subject to

$$u(a)=0, u(b)=0$$

$$\sum \lambda^i \left(f^i_u + B^i(t)z^i(t)+H^i\beta(t)+y(t)^T g_u\right.$$

$$\left.-Dy(t)^T g_{\dot{u}}+G\beta(t)\right)=0, t \in I$$

$$\int_I \left(\left(y(t)^T g(t,u,\dot{u})\right)-\frac{1}{2}\beta(t)^T G\beta(t)\right)\mathrm{d}t \geqq 0,$$

$$z^i(t)^T B^i(t) z^i(t) \leqq 1,$$

$$\lambda^T f_{\dot{u}}=0 \text{ and } y(t)^T g_{\dot{u}}=0, \text{ at } t=a \text{ and } t=b.$$

$$\lambda>0, y(t) \geqq 0, t \in I.$$

We shall not repeat the proofs of Theorems 3-6 for the above problems, as these follow on the lines of the analysis of the preceding section with slight modifications.

6. Non-Linear Multiobjective Programming Problem

If the time dependency of (VP_0) and $(M-WD_0)$ is ignored, then these problems reduce to the following nondifferentiable second-order nonlinear problems already studied in the literature:

(VP_1): Minimize

$$\left(f^1(x)+\left(x^T B^1 x\right)^{\frac{1}{2}},\cdots,f^p(x)+\left(x^T B^p x\right)^{\frac{1}{2}}\right)$$

subject to

$$g(x) \leqq 0.$$

(VD_1): Maximize

$$\left(f^1(u)+u^T B^1 z^1-\frac{1}{2}\beta^T\nabla^2 f^1\beta,\cdots,f^p(u)\right.$$

$$\left.+u^T B^p z^p-\frac{1}{2}\beta^T\nabla^2 f^p\beta\right)$$

subject to

$$\sum \lambda^i\left(f^i_u+B^i z^i-Df^i+\nabla^2 f^i\beta\right)+y^T g_u+\nabla^2\left(y^T g\right)\beta=0$$

$$y^T g-\frac{1}{2}\beta^T\left(y^T g_{uu}\right)\beta \geqq 0,\ z^{iT}B^i z^i \leqq 1,\ i \in K, \lambda>0, y \geqq 0.$$

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