American Journal of Operations Research, **2012, 2, 527-535**
http://dx.doi.org/10.4236/ajor.2012.24062 Published Online November 2012 (http://www.SciRP.org/journal/ajor)

Optimal Stopping Time for Holding an Asset

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Received September 21, 2012; revised October 22, 2012; accepted November 3, 2012

ABSTRACT

In this paper, we consider the problem to determine the optimal time to sell an asset that its price conforms to the Black-Schole model but its drift is a discrete random variable taking one of two given values and this probability distribution behavior changes chronologically. The result of finding the optimal strategy to sell the asset is the first time asset price falling into deterministic time-dependent boundary. Moreover, the boundary is represented by an increasing and continuous monotone function satisfying a nonlinear integral equation. We also conduct to find the empirical optimization boundary and simulate the asset price process.

Keywords: Optimal Stopping Time; Boundary; Brownian Motion; Black-Schole Model

1. Introduction

In [1], Shiryaev and Peskir have considered the problem:

$$V_* = \inf_{\tau \in M}\left(W_\tau - \max_{0\le t\le 1} W_t\right)^2 \qquad (1.1)$$

where W_τ is standard Brownian process and they found the optimal stopping time as:

$$\tau_* = \inf\left\{0 \le t \le 1 : S_t - W_t \ge z^* \sqrt{1-t}\right\} \qquad (1.2)$$

where z^* is the solution of the equation

$$4\Phi(z) - 2z\varphi(z) - 3 = 0, S_t = \max_{0\le s\le t} W_s,\ \varphi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \mathrm{d}y$.

The simulation for W_τ, boundary $z^*\sqrt{1-t}$ and $S_t - W_t$ are given in **Figures 1-3**.

In [2], Albert Shiryaev, Zouquan Xu and Xun Yu Zhou solve the following problem: There is an investor holding a stock, and he needs to decide when to sell it for the last time with given time to sell T. It is obvious that he wants to sell at a time of highest price on the interval from 0 to T. Assume that the discounted share price X_t complies with the following dynamic equation:

$$\mathrm{d}X_t = (a-r) X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t,\ X_0 = 1$$

on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\right)$ where a is the growth rate of the price and $\sigma > 0$ is the volatility, r is interest rate, W_τ is the standard Brownian motion with $W_0 = 0$ under the measure P. Here, $\{\mathcal{F}_t\}_{t\ge 0}$ is P-increasing filter generated by W_τ. Then, $X_t = \mathrm{e}^{\mu t + \sigma W_t}$

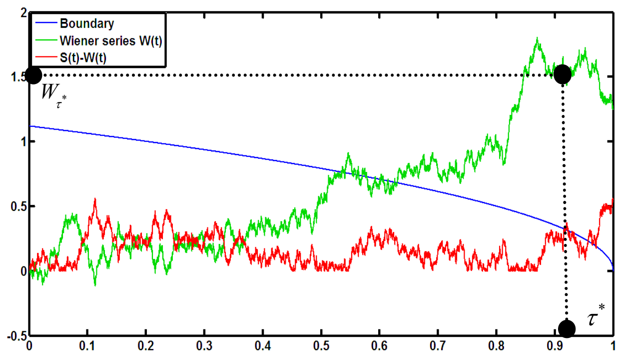

Figure 1. A simulation for stopping time in (1.2). The optimal stopping time τ^* is the first time the boundary line (blue line) lies below the line describes the process $S_t - W_t$ (red line). In this case W_{τ^*} is very large but is not $\max_{0\le t\le 1} W_t$.

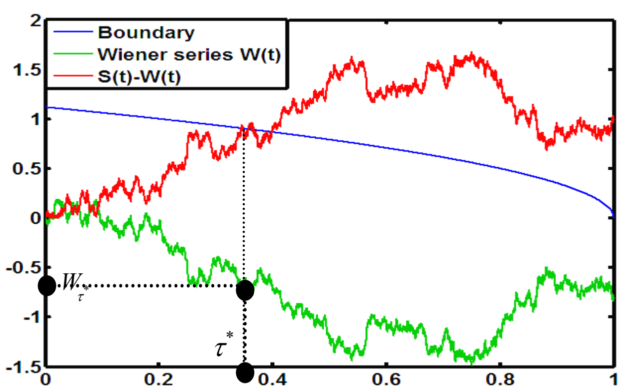

Figure 2. A simulation for the stopping time in (1.2). In this case W_{τ^*} is less than 0 but is not $\max_{0\le t\le 1} W_t$.

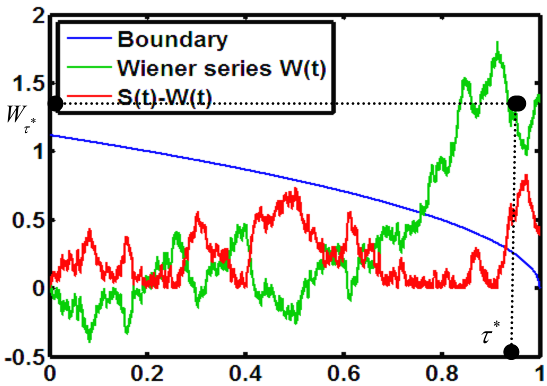

Figure 3. A simulation for the stopping time in (1.2). In this case W_{τ^*} is very large but is not $\max\limits_{0\le t\le 1} W_t$.

where

$\mu = a - r - \frac{1}{2}\sigma^2$. We define: $M_t = \max\limits_{0\le s\le t} X_s, \quad t \ge 0$

and $\alpha = \frac{a-r}{\sigma^2}$.

Then, the following cases:

- If $\alpha > \frac{1}{2}$, $\tau^* = T$ is an unique optimal selling time.
- If $\alpha = \frac{1}{2}, \tau^* = 0$ or $\tau^* = T$ are optimal selling times.
- If $\alpha \le 0$, $\tau^* = 0$ is an unique optimal selling time.

In this paper, we will find the optimal time to sell a stock when the appreciation rate is the random variable taking one of two given values a_l and a_h.

2. The Problem of Finding the Optimal Selling Time

Assume that the asset price process X_t follows a geometric Brownian motion with its drift is a random variable taking one of two given values a_l or a_h, the volatility $\sigma > 0$ is constant, *i.e.*

$$dX_t = a_t X_t dt + \sigma X_t dW_t, \quad t \ge 0 \tag{2.1}$$

where W_t is a standard Brownian motion independent with a_t of the probability space $(\Omega, \mathcal{F}, P)$. Assume $(\Omega, \mathcal{F}, P)$ is a complete probability space with non-decrease σ-field. Suppose a_l and a_h satisfy $a_l < r < a_h$, where r is the interest rate and it is constant and the initial value of assets X_0 is a positive constant.

Investors holding assets need to decide when to sell it for the last time with given time to sell them is T. Knowing that at the initial time distribution of α as

$$P(a = a_h) = \pi_0; P(a = a_\tau) = 1 - \pi_0$$

At time $t > 0$ we put $\pi_t = P\{a = a_h | \mathcal{F}_t^X\}$, where $\{\mathcal{F}_t^X\}, t \in [0, T]$ is the completion of the filtration generated by X.

The problem is finding $\mathcal{F}^X$-stopping time τ, $0 \le \tau \le T$ such that

$$V = \sup_{0\le\tau\le T} E\left[e^{-r\tau} X_\tau\right] \tag{2.2}$$

where supremum is taken in $\mathcal{F}^X$-stopping time τ, $0 \le \tau \le T$.

The price process X_t and posterior probability process π_t satisfying the equations

$$\begin{cases} \dfrac{dX_t}{X_t} = \left(a_l + \pi_t\left(a_h - a_l\right)\right) + dt + \sigma d\bar{W}_t \\ d\pi_t = \dfrac{a_h - a_l}{\sigma}\pi_t\left(1-\pi_t\right) d\bar{W}_t \end{cases} \tag{2.3}$$

where $\left(\bar{W}, \mathcal{F}^X\right)$ is a P-Brownian motion defined by:

$$\bar{W}_t = \int_0^t \frac{dX_u - \left[\left(1-\pi_t\right)a_l - \pi_t a_h\right]du}{\sigma X_u}$$

(see [3], theorem 9.1)

Define the process $\{\tilde{W}_t\}$ by:

$$d\tilde{W}_t = \left(\omega\pi_t - \sigma\right)dt + d\bar{W}_t$$

and a new measure P^* satisfying:

$$\frac{dP^*}{dP} = \exp\left\{-\frac{1}{2}\int_0^T \left(\sigma - \omega\pi_t\right)^2 dt + \int_0^T \left(\sigma - \omega\pi_t\right) d\bar{W}_t\right\}$$

$$= \exp\left\{\frac{1}{2}\int_0^T \left(\sigma - \omega\pi_t\right)^2 dt + \int_0^T \left(\sigma - \omega\pi_t\right) d\tilde{W}_t\right\}$$

where $\omega = \frac{a_h - a_l}{\sigma}$.

According to the Girsanov theorem, $\tilde{W}_t$ is a P^*-Brownian motion. Let $\pi_t = \frac{\Phi_t}{1+\Phi_t}$, we have

$$\frac{d\Phi_t}{\Phi_t} = \omega^2 \pi_t dt + \omega d\bar{W}_t$$

Then, price process X_t and process Φ_t satisfy the equations

$$\begin{pmatrix} \dfrac{dX_t}{X_t} \\ \dfrac{d\Phi_t}{\Phi_t} \end{pmatrix} = \begin{pmatrix} a_l + \sigma \\ \omega\sigma \end{pmatrix} dt + \begin{pmatrix} \sigma \\ \omega \end{pmatrix} d\tilde{W}_t \tag{2.4}$$

So, X and Φ are geometric Brownian motions under measure P^*. Moreover, σ-field generated by $\tilde{W}$ coincides with the one generated by X.

We define the likelihood process

$$\eta_t = \exp\left\{-\frac{1}{2}\int_0^t \left(\sigma - \omega\pi_s\right)^2 ds + \int_0^t \left(\omega\pi_s - \sigma\right) d\tilde{W}_s\right\}$$

We know that $\left(\tilde{W},\mathcal{F}^X\right)$ is a P^*-Brownian motion, so η is an $\mathcal{F}^X$-martingale under measure P^*.

We have:

$$\frac{\mathrm{d}X_t}{X_t}=\left(a_l+\sigma^2\right)\mathrm{d}t+\sigma\mathrm{d}\tilde{W}_t$$

$$\Rightarrow X_t=X_0\exp\left\{\int_0^t\left(a_l+\frac{\sigma^2}{2}\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{W}_t\right)\right\}$$

$$\Rightarrow \eta_t X_t = X_0\exp\left\{\int_0^t\left(a_l-\frac{1}{2}\omega^2\pi_s^2+\sigma\omega_s\right)\mathrm{d}s+\int_0^t\omega\pi_s\mathrm{d}\tilde{W}_s\right\}$$

$$\frac{\mathrm{d}\Phi_t}{\Phi_t}=\omega\sigma\mathrm{d}t+\omega\mathrm{d}\tilde{W}_t$$

$$\Rightarrow \Phi_t=\Phi_0\exp\left\{\int_0^t\left(\omega\sigma-\frac{\omega^2}{2}\right)\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{W}_t\right\}$$

Denote that E^* is an expectation operator with respect to measure P^* and let $\tau\le T$ is an $\mathcal{F}^X$-stopping time. Then, by the property $\mathcal{F}^X$-martingale under measure P^* of η (see [4], theorem 11), we have:

$$\begin{aligned}&E\left[\mathrm{e}^{-\tau r}\cdot X_\tau\right]=E^*\left[\mathrm{e}^{-\tau r}\cdot\eta_T X_\tau\right]\\&=E^*\left\{\mathrm{e}^{-\tau r}\cdot\eta_\tau X_\tau\right\}=\frac{X_0}{1+\Phi_0}E^*\left[\mathrm{e}^{(a_l-r)\tau}\left(1+\Phi_\tau\right)\right]\end{aligned}\tag{2.5}$$

Lemma 1. X_t can be written as:

$$X_t=X_0\mathrm{e}^{\varepsilon t}\cdot\left(\frac{\Phi_t}{\Phi_0}\right)^\alpha\tag{2.6}$$

where: $\alpha=\frac{\sigma}{\omega}=\frac{\sigma^2}{a_h-a_l}$ and $\varepsilon=\frac{a_h+a_l-\sigma^2}{2}$

Proof.

We have:

$$\begin{aligned}X_t&=X_0\exp\left\{\int_0^t\left(a_l+\frac{\sigma^2}{2}\right)\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{\mathrm{W}}_s\right\}\\&=X_0\mathrm{e}^{t\left(a_l+a_h-\sigma^2\right)/2}\cdot\exp\left\{\int_0^t\left(\frac{a_l-a_h}{2}+\sigma^2\right)\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{\mathrm{W}}_s\right\}\\&=X_0\mathrm{e}^{\varepsilon t}\cdot\exp\left\{\int_0^t\left(\frac{a_l-a_h}{2}+\sigma^2\right)\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{\mathrm{W}}_s\right\}\end{aligned}$$

and

$$\begin{aligned}\Phi_t&=\Phi_0\exp\left\{\int_0^t\left(\omega\sigma-\frac{\omega^2}{2}\right)\mathrm{d}s+\int_0^t\omega\mathrm{d}\tilde{\mathrm{W}}_s\right\}\\&=\Phi_0\exp\left\{\frac{\omega}{2}\left[\int_0^t\left(\sigma^2-\frac{\omega\sigma}{2}\right)\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{\mathrm{W}}_s\right]\right\}\Rightarrow\left(\frac{\Phi_t}{\Phi_0}\right)^{\frac{\sigma}{\omega}}\\&=\exp\left\{\int_0^t\left(\frac{a_l-a_h}{2}+\sigma^2\right)\mathrm{d}s+\int_0^t\sigma\mathrm{d}\tilde{\mathrm{W}}_s\right\}.\end{aligned}$$

We consider an optimal stopping problem as:

$$\Gamma(t,y)=\sup_{0\le\tau\le T-t}E^*\left[\mathrm{e}^{(a_l-r)\tau}\left(1+Y_\tau\right)\right]\tag{2.7}$$

where:

$$Y_u=y\exp\left\{\left(\sigma\omega-\frac{\omega^2}{2}\right)u+\omega\tilde{W}_u\right\},u\ge 0$$

where supremum is taken in $\mathcal{F}^X$-stopping time $\tau, 0\le\tau\le T-t$ with respect to filtration generated by $\tilde{W}$. It can be seen that the optimal stopping time in (2.7) can be turned to the optimal stopping time in the problem (2.2).

Now, we study the optimal stopping problem (2.7). We will prove that existing an increasing and continuous monotone function:

$$b:[0,T]\to[0,\infty)$$

such that the stopping time

$$\tau_{t,y}^*:=\inf\left\{u\in[0,T-t]:Y_u\le b(t+u)\right\}\wedge(T-t)$$

is an optimal stopping time for the problem (2.7).

Y_u satisfies the equation:

$$\frac{\mathrm{d}Y_u}{Y_u}=\left(a_h-a_l\right)\mathrm{d}u+\omega\mathrm{d}W_u,\quad u\ge 0$$

On the other hand, we can write $Y_u=yZ_u$, where

$$Z_u=\exp\left\{\left(\sigma u-\frac{\omega^2}{2}\right)u+u\mathrm{W}_u\right\}.$$

With this notation, we have:

$$\Gamma(t,y)=\sup_{0\le\tau\le T-t}E^*\left[\mathrm{e}^{(a_l-r)\tau}\left(1+yZ\tau\right)\right]$$

Give $\tau=0$ in (2.7), we have

$$\Gamma(t,y)\ge G(y):=E^*\left(1+Y_0\right)=E^*(1+y)=1+y.$$

Define the continuation region *C*:

$$C=\left\{(t,y)\in[0,T]\times(0,\infty):\Gamma(t,y)>G(y)\right\}$$

and the stopping region *D*:

$$D=\left\{(t,y)\in[0,T]\times(0,\infty):\Gamma(t,y)=G(y)\right\}$$

According to general theory about optimal stopping problems, the stopping time

$$\tau_D=\inf\left\{0\le s\le T-t:\left(t+s,y_s\right)\in D\right\}$$

is optimal stopping time problem (2.7). Thus, determining the optimal stopping time is sufficient to defining the stopping region *D*.

Theorem 1. There exists a right continuous and non-decreasing function

$$b:[0,T)\to\left[0,\frac{r-a_l}{a_h-r}\right]$$

such that

$$C=\{(t,y)\in[0,T)\times(0,\infty):y>b(t)\}$$

Furthermore, supremum in (7) is achieved with the stopping time

$$\tau_D=\inf\{0\le u\le T-t:Y_u\le b(t+u)\}$$

Proof.

We know that

$$C=\{(t,y):\Gamma(t,y)>G(y)\}$$
$$=\{(t,y):\Gamma(t,y)>1+y\}$$

with every fix $t\in[0,T)$ and $y'>y>0$, assume that $(t,y)\in C$, there exists a stopping time τ such that:

$$E^*\left[e^{(a_l-r)\tau}(1+yZ_\tau)\right]>1+y$$

So:

$$\Gamma(t,y')\ge E^*\left[e^{(a_l-r)\tau}(1+y'Z_\tau)\right]$$
$$=E^*\left[e^{(a_l-r)\tau}(1+yZ_\tau)\right]$$
$$+(y'-y)E^*\left[e^{(al-r)\tau}Z_\tau\right]$$
$$>1+y+(y'-y)E^*\left[e^{(a_l-r)\tau}Z_\tau\right]$$

And process $H_t=e^{(a_l-r)t}Z_t$ is a submartingale, so we have:

$$\Gamma(t,y')>1+y+(y'-y)E^*\left[e^{(a_l-r)\cdot 0}Z_0\right]$$
$$=1+y+y'-y=1+y'$$

Therefore, $(t,y')\in C$. This proves that remaining a function $b:[0,T)\to[0,\infty)$

such that: $C=\{(t,y):t\in[0,T):y>b(t)\}$

We have $e^{(a_l-r)t}(1+Y_t)$ to be a submartingale for $t\le\inf\left\{s:Y_s\le\frac{r-a_l}{a_h-r}\right\}$, so all points in region $\left\{(t,z):z>\frac{r-a_l}{a_h-r}\right\}$ belong to the continuation region. Therefore, $b(t)\le\frac{r-a_l}{a_h-r}$. The monotonicity of b follows from monotonicity of function $t\to\Gamma(t,y)$.

The right continuity of $b(t)$ follows from the continuation region C is an open set.

Theorem 2. Assume that b is the function described above whose existence is proved in Lemma 1. Define stopping time:

$$\tau^*=\inf\left\{t:X_t\le\frac{X_0}{\Phi_0^\alpha}e^{\varepsilon t}\cdot b^\alpha(t)\right\}\wedge T \tag{2.8}$$

Then, τ^* attains the supremum in (2.2).

Proof. Deduce directly from Theorem 1 and Lemma 1 by replacing Y_t by Φ_t.

Theorem 3. The optimal stopping boundary b(t) satisfies the integral equation (see Equation (2.9) below):

where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}dy.$

Proof.

Fix $t\in[0,T]$ and $Y_0=y\in(0,\infty)$. Then,

$$e^{(a_l-r)(T-t)}E^*\left[G(Y_{T-t})\right]$$
$$=e^{(a_l-r)(T-t)}E^*\left[\Gamma(T,Y_{T-t})\right]$$

where $G(y)=1+y$.

But:

$$e^{(a_l-r)(T-t)}E^*\left[G(Y_{T-t})\right]$$
$$=E^*\left[e^{(a_l-r)(T-t)}\right]+E^*\left[e^{(a_l-r)(T-t)}Y_{T-t}\right]$$
$$=e^{(a_l-r)(T-t)}+y\cdot e^{(a_h-r)(T-t)}$$

Consider:

$$e^{(a_l-r)(T-t)}E^*\left[\Gamma(T,Y_{T-t})\right]$$
$$=e^{(a_l-r)u}y(a_h-r)E^*\left[Z_uI(Y_u\le b(t+u))\right]$$
$$=e^{(a_l-r)u}(a_l-r)E^*\left[I(Y_u\le b(t+u))\right]$$

We have:

$$1+b(t)=e^{(a_l-r)(T-t)}+b(t)\cdot e^{(a_h-r)(T-t)}-\int_0^{T-t}\left\{(a_l-r)e^{(a_l-r)u}\Phi\left(\frac{1}{\omega\sqrt{u}}\left[\ln\frac{(t+u)}{b(t)}-\omega\sigma u+\frac{\omega^2u}{2}\right]\right)\right.$$
$$\left.+b(t)(a_h-r)e^{(a_h-r)u}\Phi\left(\frac{1}{\omega\sqrt{u}}\left[\ln\frac{b(t+u)}{b(t)}-\omega\sigma u-\frac{\omega^2u}{2}\right]\right)\right\}du \tag{2.9}$$

$$E_{t,y}^{*}\left[Z_u I\left(Y_u \le b(t+u)\right)\right]$$
$$= E_{t,y}^{*}\left[\exp\left(\sigma\omega - \frac{\omega^2}{2}\right)u + \omega W_u\right]$$
$$\times I\left(\frac{W_u}{\sqrt{u}} \le \frac{\ln\left(\frac{b(t+u)}{y}\right) + \left(\frac{\omega^2}{2} - \sigma\omega\right)u}{\omega\sqrt{u}}\right)$$
$$= e^{\left(\sigma\omega - \frac{\omega^2}{2}\right)u} \cdot \int_{-\infty}^{d} e^{\omega z} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2u}} dz = e^{\left(\sigma\omega - \frac{\omega^2}{2}\right)u}$$
$$\cdot \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(z-\omega u)^2}{2u}} \cdot e^{\frac{\omega^2 u}{2}} dz = e^{\sigma\omega u} \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(z-\omega u)^2}{2u}}$$

where $d = \dfrac{\ln\dfrac{b(t+u)}{y} + \left(\dfrac{\omega^2}{2} - \sigma\omega\right)u}{\omega}$.

Let $x = \dfrac{z-\omega u}{\sqrt{u}}$ and

$$d_1 = \frac{\dfrac{\ln\dfrac{b(t+u)}{y} + \left(\dfrac{\omega^2}{2} - \sigma\omega\right)u}{\omega} - \omega u}{\sqrt{u}}$$
$$= \frac{\ln\dfrac{b(t+u)}{y} - \dfrac{\omega^2}{2}u - \sigma\omega u}{\omega\sqrt{u}}$$

then $E_{t,y}^{*}\left[Z_u I\left(Y_u \le b(t+u)\right)\right] = e^{\sigma\omega u} N(d_1)$

In a similar way, we have

$$E_{t,y}^{*}\left[I\left(Y_u \le b(t+u)\right)\right] = N(d_2)$$

where $d_2 = d_1 + \omega\sqrt{u}$.

Put $y = b(t)$ we attain (2.9).

3. The Numerical Solution of the Integral Equation and Simulation Results

Below we follow [5], devided $[0,T]$ by the points $t_k = kh$, $k = 0,1,\cdots,n$ which $h = T/n$, then Equation (2.8) can be discretized as:

$$1 + b(ih) = e^{(a_l - r)(n-i)h} + b(ih)e^{(a_h - r)(n-i)h} - h\sum_{k=1}^{n-i}\left[(a_l - r)e^{(a_l - r)kh}\Phi\left(\frac{1}{\omega\sqrt{kh}}\left(\ln\frac{b(ih+kh)}{b(ih)} - \omega\sigma kh + \frac{\omega^2 kh}{2}\right)\right)\right]$$
$$- h\sum_{k=1}^{n-i}\left[b(ih)(a_h - r)e^{(a_h - r)kh}\Phi\left(\frac{1}{\omega\sqrt{kh}}\left(\ln\frac{b(ih+kh)}{b(ih)} - \omega\sigma kh - \frac{\omega^2 kh}{2}\right)\right)\right]$$

For $i = n-1$, we have equation

$$1 + b(t_{n-1}) = e^{(a_l - r)h} + b(t_{n-1})e^{(a_h - r)h} - h(a_l - r)e^{(a_l - r)h}\Phi\left(\frac{1}{\omega\sqrt{h}}\left(\ln\frac{b(T)}{b(t_{n-1})} - \omega\sigma h + \frac{\omega^2 h}{2}\right)\right)$$
$$- hb(t_{n-1})(a_h - r)e^{(a_h - r)h}\Phi\left(\frac{1}{\omega\sqrt{h}}\left(\ln\frac{b(T)}{b(t_{n-1})} - \omega\sigma h - \frac{\omega^2 h}{2}\right)\right)$$

Due to $b(T) \approx \dfrac{r - a_l}{a_h - r}$, from the above equation we determine $b(t_{n-1})$, continue to $i = n-2$, we obtain the following equation for determining $b(t_{n-2})$:

$$1 + b(t_{n-2}) = e^{(a_l - r)2h} + b(t_{n-2})e^{(a_h - r)2h} - h(a_l - r)e^{(a_l - r)2h}\Phi\left(\frac{1}{\omega\sqrt{2h}}\left(\ln\frac{b(T)}{b(t_{n-2})} - \omega\sigma 2h + \frac{\omega^2 2h}{2}\right)\right)$$
$$- hb(t_{n-1})(a_h - r)e^{(a_h - r)2h}\Phi\left(\frac{1}{\omega\sqrt{2h}}\left(\ln\frac{b(T)}{b(t_{n-2})} - \omega\sigma 2h - \frac{\omega^2 2h}{2}\right)\right)$$
$$- h(a_l - r)e^{(a_l - r)h}\Phi\left(\frac{1}{\omega\sqrt{h}}\left(\ln\frac{b(t_{n-1})}{b(t_{n-2})} - \omega\sigma h + \frac{\omega^2 h}{2}\right)\right)$$
$$- hb(t_{n-1})(a_h - r)e^{(a_h - r)h}\Phi\left(\frac{1}{\omega\sqrt{h}}\left(\ln\frac{b(t_{n-1})}{b(t_{n-2})} - \omega\sigma h - \frac{\omega^2 h}{2}\right)\right)$$

Just do so until $i = 0$, it has been determined $b(t_0)$. Thus, we obtain a sequence of values $b(t_0), b(t_1), \cdots, b(t_n)$ of $b(t)$ and approximate the optimal boundaries for the asset liquidation process.

We have the approximate solution of Equation (2.9) by a computer program written in Matlab software, then set the boundaries for the process $X(t)$ is $B(t) = \frac{X_0}{\Phi_0^{\alpha}} \mathrm{e}^{\varepsilon t} \cdot b^{\alpha}(t)$. Then the optimal time to sell is the first time the line describes the process $B(t)$ lies below the line describes the process $X(t)$. These figures (**Figures 4-19**) and tables below (**Tables 1-3**) illustrates the stopping time in (2.8) and the solutions of Equation (2.9) in some cases.

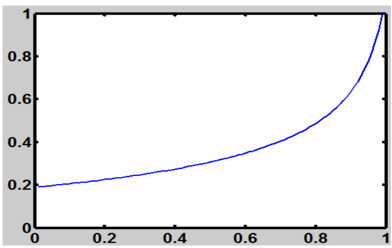

Figure 4. The line describes $b(t)$ with $a_l = 0.1$; $a_h = 0.2$; $r = 0.15$; $\sigma = 0.2$.

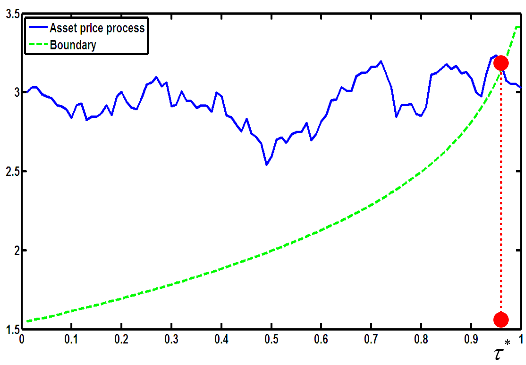

Figure 5. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.1$; $a_h = 0.2$; $r = 0.15$; $\sigma = 0.2$; $\pi_0 = 0.5$.

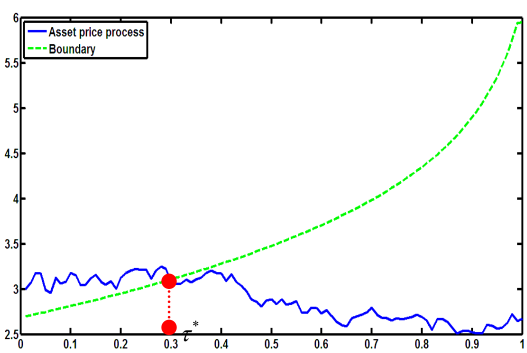

Figure 6. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.1$; $a_h = 0.2$; $r = 0.15$; $\sigma = 0.2$; $\pi_0 = 0.2$. In this case $\pi_0 = 0.2$ is small so is τ^*.

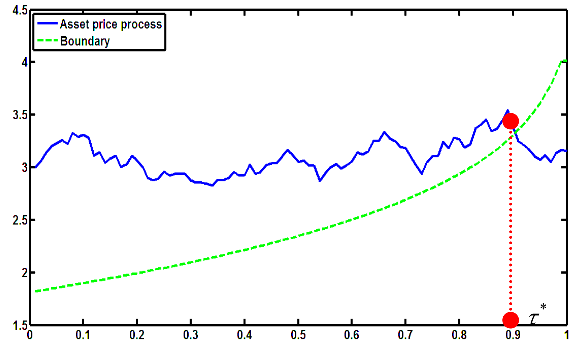

Figure 7. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.1$; $a_h = 0.2$; $r = 0.15$; $\sigma = 0.2$; $\pi_0 = 0.4$.

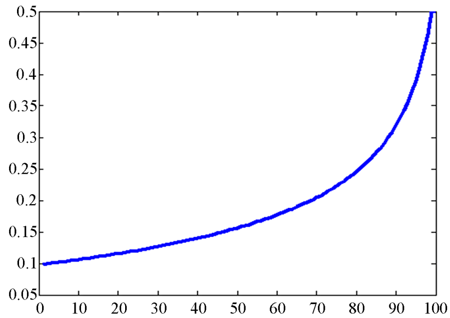

Figure 8. The line describes $b(t)$ with $a_l = 0.09$; $a_h = 0.15$; $r = 0.11$; $\sigma = 0.1$.

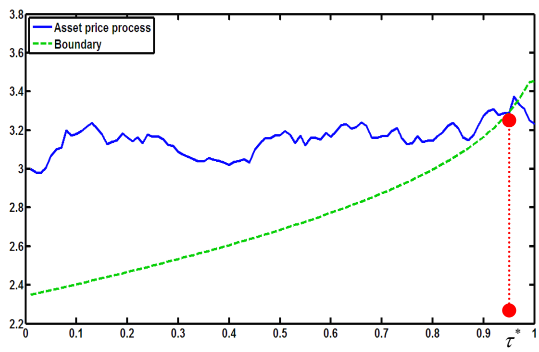

Figure 9. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.11$; $\sigma = 0.1$; $\pi_0 = 0.5$.

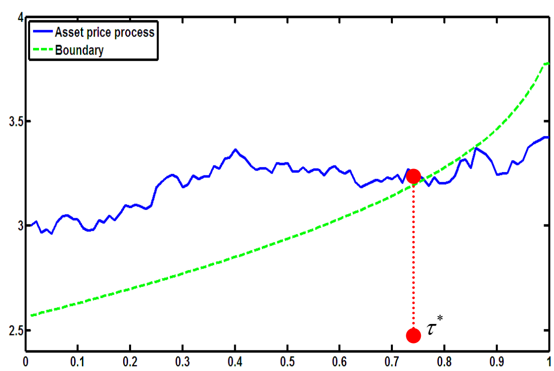

Figure 10. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.11$; $\sigma = 0.1$; $\pi_0 = 0.4$.

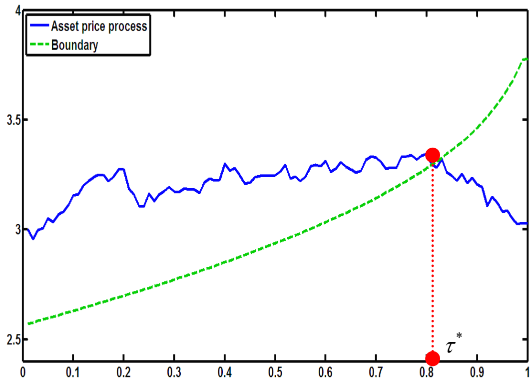

Figure 11. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.11$; $\sigma = 0.1$; $\pi_0 = 0.3$.

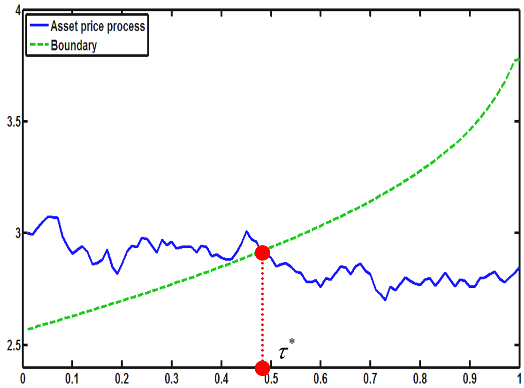

Figure 12. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.11$; $\sigma = 0.1$; $\pi_0 = 0.2$.

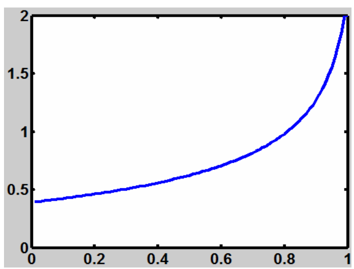

Figure 13. The line describes $b(t)$ with $a_l = 0.09$; $a_h = 0.15$; $r = 0.13$; $\sigma = 0.1$.

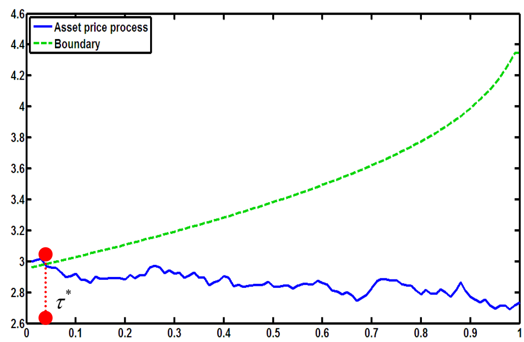

Figure 14. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.13$; $\sigma = 0.1$; $\pi_0 = 0.2$.

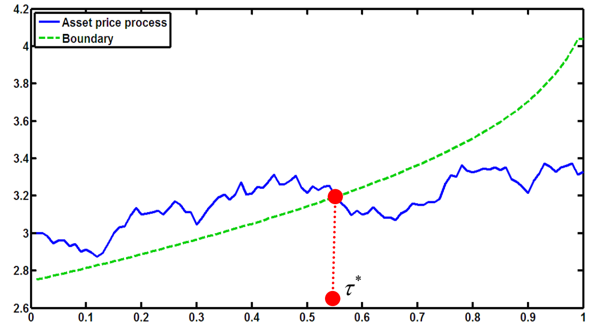

Figure 15. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.13$; $\sigma = 0.1$; $\pi_0 = 0.4$.

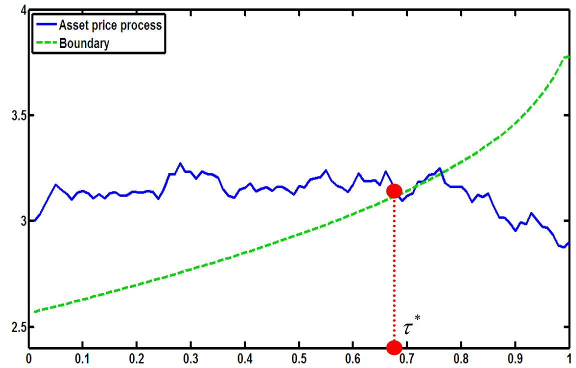

Figure 16. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.13$; $\sigma = 0.1$; $\pi_0 = 0.5$.

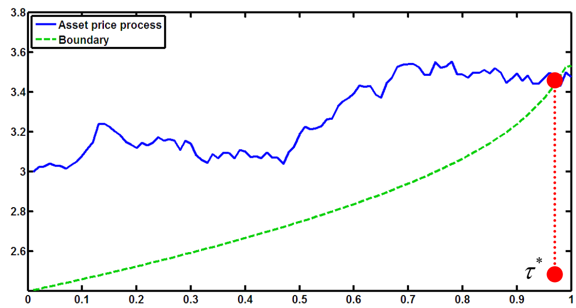

Figure 17. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = 0.09$; $a_h = 0.15$; $r = 0.13$; $\sigma = 0.1$; $\pi_0 = 0.5$.

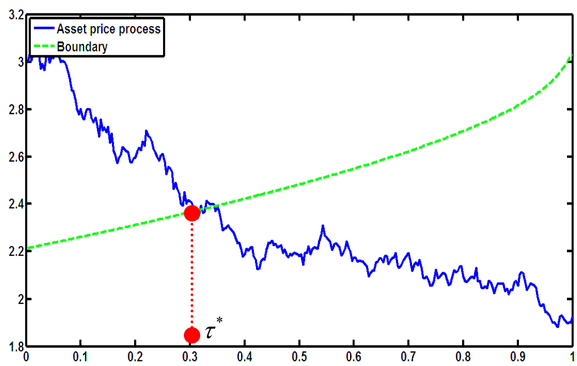

Figure 18. A simulation for the stopping time in (2.8) with parameters $X_0 = 3$; $a_l = -0.3$; $a_h = 0.5$; $r = 0.1$; $\sigma = 0.2$; $\pi_0 = 0.4$.

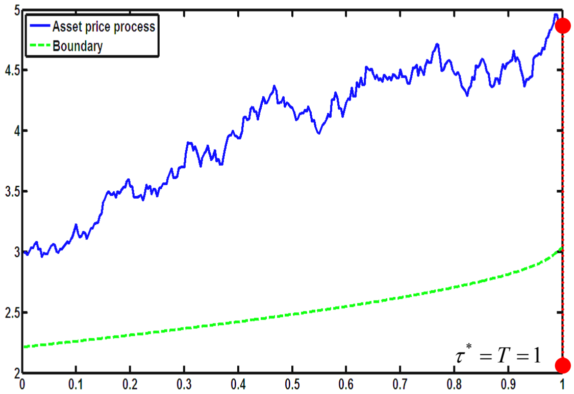

Figure 19. A simulation for the stopping time in (2.9) with parameters $X_0 = 3$; $a_l = -0.3$; $a_h = 0.5$; $r = 0.1$; $\sigma = 0.2$; $\pi_0 = 0.6$.

Table 1. The numerical solutions of (2.9) with $a_l = 0.1$; $a_h = 0.2$; $r = 0.15$; $\sigma = 0.2$.

i	b (t_i)	i	b (t_i)	i	b (t_i)	i	b (t_i)	i	b (t_i)
1	0.1909	21	0.2272	41	0.2775	61	0.3541	81	0.4985
2	0.1924	22	0.2294	42	0.2806	62	0.359	82	0.5098
3	0.1941	23	0.2315	43	0.2837	63	0.3642	83	0.5219
4	0.1957	24	0.2337	44	0.2868	64	0.3695	84	0.5347
5	0.1973	25	0.2359	45	0.2901	65	0.3749	85	0.5485
6	0.199	26	0.2382	46	0.2934	66	0.3806	86	0.5632
7	0.2007	27	0.2405	47	0.2968	67	0.3864	87	0.5791
8	0.2024	28	0.2428	48	0.3003	68	0.3925	88	0.5963
9	0.2042	29	0.2452	49	0.3038	69	0.3987	89	0.6151
10	0.2059	30	0.2477	50	0.3074	70	0.4052	90	0.6356
11	0.2077	31	0.2501	51	0.3112	71	0.412	91	0.6582
12	0.2096	32	0.2526	52	0.315	72	0.419	92	0.6833
13	0.2114	33	0.2552	53	0.3189	73	0.4263	93	0.7113
14	0.2133	34	0.2578	54	0.3229	74	0.4339	94	0.743
15	0.2152	35	0.2605	55	0.327	75	0.4419	95	0.7791
16	0.2171	36	0.2632	56	0.3312	76	0.4502	96	0.8207
17	0.2191	37	0.2659	57	0.3355	77	0.4589	97	0.8696
18	0.2211	38	0.2687	58	0.34	78	0.468	98	0.9281
19	0.2231	39	0.2716	59	0.3445	79	0.4777	99	0.9995
20	0.2252	40	0.2745	60	0.3492	80	0.4878	100	1

Table 2. The numerical solutions of (2.8) with $a_l = 0.09$; $a_h = 0.15$; $r = 0.11$; $\sigma = 0.1$.

i	b (t_i)	i	b (t_i)	i	b (t_i)	i	b (t_i)	i	b (t_i)
1	0.0985	21	0.1165	41	0.1414	61	0.1792	81	0.2507
2	0.0993	22	0.1176	42	0.1429	62	0.1817	82	0.2563
3	0.1001	23	0.1186	43	0.1444	63	0.1842	83	0.2623
4	0.1009	24	0.1197	44	0.146	64	0.1868	84	0.2686
5	0.1017	25	0.1208	45	0.1476	65	0.1895	85	0.2754
6	0.1025	26	0.122	46	0.1493	66	0.1923	86	0.2827
7	0.1034	27	0.1231	47	0.1509	67	0.1952	87	0.2906
8	0.1042	28	0.1243	48	0.1526	68	0.1982	88	0.2992
9	0.1051	29	0.1254	49	0.1544	69	0.2013	89	0.3085
10	0.106	30	0.1266	50	0.1562	70	0.2045	90	0.3186
11	0.1069	31	0.1279	51	0.158	71	0.2079	91	0.3299
12	0.1078	32	0.1291	52	0.1599	72	0.2113	92	0.3423
13	0.1087	33	0.1304	53	0.1618	73	0.2149	93	0.3563
14	0.1096	34	0.1317	54	0.1638	74	0.2187	94	0.372
15	0.1106	35	0.133	55	0.1658	75	0.2226	95	0.39
16	0.1115	36	0.1343	56	0.1679	76	0.2268	96	0.4107
17	0.1125	37	0.1357	57	0.1701	77	0.2311	97	0.4351
18	0.1135	38	0.1371	58	0.1723	78	0.2356	98	0.4642
19	0.1145	39	0.1385	59	0.1745	79	0.2403	99	0.4999
20	0.1155	40	0.1399	60	0.1768	80	0.2454	100	0.5

Table 3. The numerical solutions of (2.9) with $a_l = 0.09$; $a_h = 0.15$; $r = 0.13$; $\sigma = 0.1$.

i	b (t_i)	i	b (t_i)	i	b (t_i)	i	b (t_i)	i	b (t_i)
1	0.3944	21	0.4664	41	0.5658	61	0.717	81	1.0027
2	0.3975	22	0.4706	42	0.5718	62	0.7269	82	1.0252
3	0.4007	23	0.4749	43	0.578	63	0.737	83	1.0491
4	0.4039	24	0.4792	44	0.5843	64	0.7475	84	1.0745
5	0.4072	25	0.4836	45	0.5907	65	0.7583	85	1.1018
6	0.4105	26	0.4881	46	0.5972	66	0.7695	86	1.131
7	0.4139	27	0.4927	47	0.6039	67	0.781	87	1.1625
8	0.4173	28	0.4973	48	0.6108	68	0.793	88	1.1967
9	0.4208	29	0.502	49	0.6178	69	0.8054	89	1.2339
10	0.4243	30	0.5068	50	0.6249	70	0.8182	90	1.2746
11	0.4278	31	0.5117	51	0.6323	71	0.8316	91	1.3195
12	0.4314	32	0.5167	52	0.6398	72	0.8454	92	1.3694
13	0.4351	33	0.5217	53	0.6475	73	0.8599	93	1.4252
14	0.4388	34	0.5269	54	0.6554	74	0.8749	94	1.4881
15	0.4426	35	0.5321	55	0.6635	75	0.8907	95	1.56
16	0.4464	36	0.5375	56	0.6719	76	0.9071	96	1.6429
17	0.4503	37	0.5429	57	0.6804	77	0.9244	97	1.7403
18	0.4542	38	0.5485	58	0.6892	78	0.9424	98	1.857
19	0.4582	39	0.5542	59	0.6982	79	0.9615	99	1.9994
20	0.4623	40	0.5599	60	0.7075	80	0.9815	100	2

4. Conclusion

This paper solves the problem to find the optimal stopping time for the holding asset and make a decision when to sell assets with discounted price reaching the greatest expected value. The optimal stopping time is the first time the price of the asset hit the boundary or be at the time T. In next study, we will study the distributions and characteristics of the optimal stopping time.

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