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Minimizing Complementary Pivots in a Simplex-Based Solution Method for a Quadratic Programming Problem

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ABSTRACT

The paper presents an approach for avoiding and minimizing the complementary pivots in a simplex based solution method for a quadratic programming problem. The linearization of the problem is slightly changed so that the simplex or interior point methods can solve with full speed. This is a big advantage as a complementary pivot algorithm will take roughly eight times as longer time to solve a quadratic program than the full speed simplex-method solving a linear problem of the same size. The strategy of the approach is in the assumption that the solution of the quadratic programming problem is near the feasible point closest to the stationary point assuming no constraints.

Keywords: Quadratic Programming; Convex; Karusha-Kuhn-Tucker; Simplex Method

1. Introduction

The quadratic programming (QP) problem is the problem of optimizing a quadratic function over a convex solution space. A variety of methods are available for solving the quadratic programming problem [1-6]. These include interior point, extensions of the simplex, gradient projecttion, conjugate gradient, augmented Lagrangian and active set methods.

Simplex-method: The simplex-method was proposed by George Dantzig in 1947 and it searches the boundary of the feasible solution for the optimal solution [7]. The main problem with this method is that it has an exponent tial complexity and is affected by degeneracy and cycling. This method has undergone so many improvements and has been competitive for solving practical linear programming (LP) problems so far.

Interior point method: This is a method that reaches the optimal solution of a linear programming model by traversing the interior of the feasible region contrary to simplex method [3,8,9]. The interior point approaches were mainly developed for the general nonlinear programming problem but were later abandoned because of their poor performance that time. Karmarkar's breakthrough in 1984 revitalized the study of interior point methods which have been shown to have a polynomial complexity. The interior point algorithm has competed well with the simplex-method on large LPs. The other four methods are not as competitive as these two for LPs. Detailed information on the other methods will not be presented here but readers are encouraged to see Freund [3] or Sun and Yuan [10] for more information.

Application of quadratic programming: Quadratic programming has been successfully applied in areas such as engineering, finance, economics, agriculture, marketing, public policy, water resource management and transportation. The following are some of the so many specific applications of quadratic programming.

- Monopolist's profit maximization
- Portifolio selection
- Inequality constrained least-squares estimation
- Goal programming with quadratic preferences
- Spatial equilibrium analysis for optimal allocation and pricing
- Optimal linear decision rules
- Calculation of current flow in resistors
- Optimization of digital filters
- Image classification in Computer Vision
- Numerical modeling of elastoplastic contact problems
- Optimization of chemical process synthesis

More on applications of quadratic programming can be found in Gupta [11], Horst *et al.* [12] and McCarl *et al.* [5]. With this large number of applications the quadratic programming model certainly requires an efficient solution method. We present an approach for and minimizing the complementary pivots in a simplex based solution method for a quadratic programming problem. The linearization of the problem is slightly changed from the usual Karush Kuhn Tucker (KKT) conditions [13] so that the simplex or interior point methods can solve with full speed. This is a big advantage as a complementary pivot algorithm will take roughly eight times as longer time to solve a quadratic program than the full speed simplex-method solving a linear problem of the same size [4]. The strategy of the approach is in the assumption that the solution of the quadratic programming problem is near the feasible point closest to the stationary point assuming no constraints.

2. The Quadratic Programming Problem

Let a quadratic programming problem be represented by

$$\left.\begin{aligned}&\text{Maximize } f(X) = CX + X^T DX\\&\text{Subject to } AX \le B,\ X > 0\\&\text{where}\\&C = (c_1, c_2, \cdots, c_n), X = (x_1, x_2, \cdots, x_n)^T,\\&D = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \cdots & \cdots & \cdots \\ d_{n1} & \cdots & d_{nm} \end{pmatrix}, A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}\\&\text{and } B = (b_1, b_2, \cdots, b_m)^T.\end{aligned}\right\} \quad (1)$$

The matrix D is assumed symmetric and negative definite. This means that $f(X)$ is strictly concave. The constraints are linear thus a convex solution space is guaranteed. Any maximization quadratic model can be changed into a minimization and vice versa. This model is nonlinear and the idea is to linearize it in such a way that there is minimal complementary pivoting when solving.

3. Linearization of the Quadratic Problem

Let a quadratic programming problem be represented graphically as shown in **Figure 1**.

Where $X^0 = (x_1^0, x_2^0, \cdots, x_n^0)^T$ is the global optimum point.

This point X^0 is not necessarily feasible and these variables are not necessarily nonnegative. In other words they are free variables and Φ is the distance of the required optimal point X from X^0 This distance is positive ($\Phi > 0$)and when $\Phi = 0$ then X^0 is feasible and it is the trivial case. The planes (1), (2), (3), (4), $\cdots$, (m) are the linear constraints of the quadratic programming problem. The distance ℓ is the shortest distance from X^0 to X.

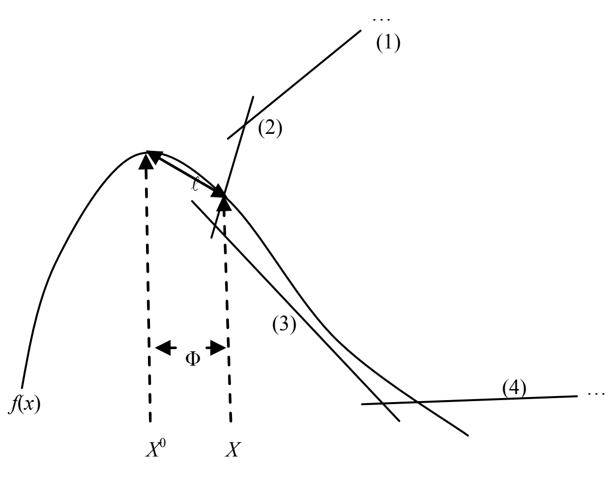

Figure 1. Graphical representation of the quadratic programming problem.

3.1. Finding Point X^0

The point X^0 can easily be obtained from

$$\nabla f(X^0) = 0 \quad (2)$$

It is not necessary to solve for X^0 own its own. The point X^0 is represented by a set of Equation (2) in the linearized model.

3.2. Finding Point X

The point X is feasible and is not known and what are readily available are the limits of the variables. The limits for the variables are given by

$$\left.\begin{aligned}x_1^0 - \varepsilon_1 \le x_1 \le x_1^0 + \varepsilon_1\\x_2^0 - \varepsilon_2 \le x_2 \le x_2^0 + \varepsilon_2\\\cdots\\x_n^0 - \varepsilon_n \le x_n \le x_n^0 + \varepsilon_n\end{aligned}\right\} \quad (3)$$

where $\varepsilon_j \ge 0$.

This can also be represented as

$$X^0 - \Phi \le X \le X^0 + \Phi \quad (4)$$

where

$$\Phi = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)^T \quad (5)$$

3.3. Nearness of Point X to X^0 and the Objective Function

The point X becomes close to optimality when Φ is minimal. In other words we need to find the smallest value of Φ that gives a feasible point.

$$\ell = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + \cdots + (x_n - x_n^0)^2} \quad (6)$$

$$\ell = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2} \quad (7)$$

What minimizes ℓ also minimizes ℓ^2 because $\varepsilon \ge 0$. *i.e.*

$$\ell^2 = \varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2 \quad (8)$$

Theorem 1: What minimizes ℓ^2 also minimizes. $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$.

Proof: Let $\tau_j = \varepsilon_j^2$, then whatever minimizes $\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2$ also minimizes $\tau_1 + \tau_2 + \cdots + \tau_n$ since $\tau_n \ge 0$.

The objective function of the linear model is:

$$\text{Minimize } \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n \quad (9)$$

This is the same as

$$\text{Minimize } \Phi^T I_n \quad (10)$$

where $I_n = (1, 1, \cdots, 1)^T$.

3.4. Linear Model for the Quadratic Function

$$\left.\begin{array}{l}\text{Minimize } \Phi^T I_n \\ \text{Subject to} \\ \nabla f(X^0) = 0 \\ X^0 - \Phi \le X \le X^0 + \Phi \\ AX \le B \\ X \ge 0, \Phi \ge 0 \text{ and } X^0 \text{ are free variables}\end{array}\right\} \quad (11)$$

3.5. Redundant Constraints

In the linear model given above there are redundant constraints and it is necessary to discard them without necessarily losing the optimal point. The set of constraints given in (4) can be split into two inequality sets.

$$X^0 - \Phi \le X \le X^0 + \Phi$$

The first inequality set is given by

$$X^0 - \Phi \le X \quad (12)$$

$$X - X^0 + \Phi \ge 0 \quad (13)$$

Similarly the second inequality set is

$$X \le X^0 + \Phi \quad (14)$$

$$X - X^0 - \Phi \le 0 \quad (15)$$

The set of inequalities in (15) are redundant.

Theorem 2: Whatever feasible solution that satisfies $X - X^0 + \Phi \ge 0$ will also satisfy $X - X^0 - \Phi \le 0$.

Proof

Equality Case 1: If $X - X^0 + \Phi = 0$ then $X - X^0 - \Phi = 0$.

If this is true then the following simultaneous equations are feasible

$$X - X^0 + \Phi = 0 \quad (16)$$

$$X - X^0 - \Phi = 0 \quad (17)$$

This is possible when

$$X = X^0 \quad (18)$$

$$\Phi = 0 \quad (19)$$

This is the trivial case. The equality case holds when $X = X^0$ and $\Phi = 0$.

Inequality Case 2: Suppose $\Phi > 0$ and we need to show that if $X - X^0 + \Phi > 0$ then $X - X^0 - \Phi < 0$.

If this is true then $X - X^0 + \Phi > 0$ is the same as

$$-\left(X - X^0\right) - \Phi < 0 \quad (20)$$

Let $X - X^0 = \Delta$ then the two inequalities become

$$-\Delta - \Phi < 0 \quad (21)$$

$$\Delta - \Phi < 0 \quad (22)$$

Suppose $\Delta - \Phi < 0$ this implies $\Delta < \Phi$ and this satisfies, $-\Delta - \Phi < 0$, whether $\Delta < 0$ or $\Delta > 0$ This implies that the second set of constraints can be discarded without losing the optimal point as they are redundant. The discarding also reduces the size of the LP significantly.

3.6. Reduced Linear Model

$$\left.\begin{array}{l}\text{Minimize } \Phi^T I_n \\ \text{Subject to} \\ \nabla f(X^0) = 0 \\ X - X^0 + \Phi \ge 0 \\ AX \le B \\ X \ge 0, \Phi \ge 0 \text{ and } X^0 \text{ are free variables.}\end{array}\right\} \quad (23)$$

The solution given by the linear model (23) is not necessarily optimal to the original quadratic problem.

4. Optimality and Linear Model

The solution to (23) becomes optimal if it also satisfies the necessary KKT conditions. In other words there is a need to fuse (23) and KKT conditions.

Reduced linear model KKT conditions

Minimize $\Phi^T I_n$	$DX + A^T \mu^T - Y = -C$
Subject to	$AX + V = B$ (24a)
$\nabla f(X^0) = 0$	Where
$X - X^0 + \Phi \ge 0$	Non-negativity conditions: $Y \ge 0$, $V \ge 0$ and $\mu \ge 0$.
$AX \le B$ (24b)	Complementary slackness: $Y^T X = 0$ and $\mu V = 0$.

Expressions (24a) and (24b) can be represented as just (24).

Linear Model

The linear model becomes

$$\left.\begin{array}{l}\text{Minimize } \Phi^T I_n \\ \text{Subject to} \\ \nabla f(X^0) = 0 \\ X - X^0 + \Phi \ge 0 \\ AX + V = B \\ DX + A^T \mu^T - Y = -C\end{array}\right\} \quad (25)$$

where $X \ge 0$, $\Phi \ge 0$, $Y \ge 0$, $V \ge 0$, $\mu \ge 0$ and X^0 are free variables.

Solving (25) will give a solution that is near optimal.

Complementary slackness

$$Y^T X = \mu V = 0 \quad (26)$$

If the solution given in (25) satisfies the complementary slackness conditions then it is optimal. The complementary slackness conditions cause a restricted basis entry for the simplex-method and this has significant effect on its speed. The idea or strategy in this paper is to find a feasible point closest to X^0 and at the same time satisfying all the KKT conditions except the complementary slackness conditions given in (26). The feasible point is then tested to see if it satisfies KKT or not. If it does then it is optimal else switch to complementary pivots or Wolfe's method [14] until all the necessary conditions are satisfied.

5. Proposed Approach

The steps for the proposed approach are summarized as follows:

Step 1: Solve the linear model:

$$\text{Minimize } \Phi^T I_n$$
$$\text{Subject to}$$
$$\nabla f(X^0) = 0$$
$$X - X^0 + \Phi \geq 0$$
$$AX + V = B$$
$$DX + A^T \mu^T - Y = -C$$

Step 2: If solution in Step 1 satisfies $Y^T X = 0$ and $\mu V = 0$ then it is optimal else go to Step 3.

Step 3: Use the optimal tableau in Step 1, switch to complementary pivots until all the necessary KKT conditions are satisfied.

6. Numerical Illustration

Use the proposed approach to linearize the following quadratic programming problem and then solve for the optimal solution.

$$\left.\begin{aligned} &\text{Maximize } 2x_1 + 2x_2 - x_1^2 - 2x_2^2 + 2x_1x_2 \\ &\text{Subject to} \\ &x_1 + 6x_2 \geq 9 \\ &2x_1 + 7x_2 \leq 12 \\ &x_1, x_2 \geq 0 \end{aligned}\right\} \quad (27)$$

This is a portfolio selection problem where x_j is the number of millions of dollars invested in stock j. In this problem the objective is to obtain a minimum variance portfolio yielding the company's expected desired return. This objective is obtained by solving the company's quadratic problem formulated as (27).

Solution

The linear model is

$$\left.\begin{aligned} &\text{Minimize } \varepsilon_1 + \varepsilon_2 \\ &\text{Subject to} \\ &2x_1^0 - 2x_2^0 = 2 \\ &-2x_1^0 + 4x_2^0 = 2 \\ &x_1 - x_1^0 + \varepsilon_1 \geq 0 \\ &x_2 - x_2^0 + \varepsilon_2 \geq 0 \\ &x_1 + 6x_2 - v_1 = 9 \\ &2x_1 + 7x_2 + v_2 = 12 \\ &2x_1 - 2x_2 + \mu_1 - 2\mu_2 + y_1 = 2 \\ &-2x_1 + 4x_2 + 6\mu_1 - 7\mu_2 + y_2 = 2 \\ &\text{Where } \varepsilon_1, \varepsilon_2, v_1, v_2, \mu_1, \mu_2, y_1, y_2 \geq 0 \end{aligned}\right\} \quad (28)$$

x_1^0, x_2^0 are the free variables.

The slackness conditions are

$$x_1 y_1 = x_2 y_2 = \mu_1 v_1 = \mu_2 v_2 = 0 \quad (29)$$

Solving (28) using the simplex-method we have

$$x_1 = 1.8 \text{ and } x_2 = 1.2 \quad (30)$$

This solution satisfies the complementary slackness conditions and is optimal

In terms of the original problem the company must invest 1.8 million dollars in stock 1 and 1.2 million dollars in stock 2.

7. Conclusion

The strength of the approach presented in this paper lies within the simplicity with which linear models can be solved when there are no complementary conditions. The simplex-method performs efficiently if there is no restricted basis entry. The proposed approach significantly reduces the number of restricted basis iterations to the extent of even avoiding them completely in some problems. This is a big advantage as complementary pivot algorithm will take roughly eight times as longer time to solve a quadratic program than the full speed simplex-method solving a linear problem of the same size.

REFERENCES

[1] S. Burer and D. Vandenbussche, "A Finite Branch-and-Bound Algorithm for Nonconvex Quadratic Programs with Semidefinite Relaxations," *Mathematical Programming*, Series A, Vol. 113, No. 2, 2008, pp. 259-282. doi:10.1007/s10107-006-0080-6

[2] S. Burer and D. Vandenbussche, "Globally Solving Box-Constrained Nonconvex Quadratic Programs with Semidefinite-Based Finite Branch-and-Bound," *Computational Optimization and Applications*, Vol. 43, No. 2, 2009, pp. 181-195.

[3] R. M. Freund, "Solution Methods for Quadratic Optimization: Lecture notes," Massachussets Institute of Technology, 2002.

[4] S. T. Liu and R.-T. Wang, "A Numerical Solution Methods to Interval Quadratic Programming," *Applied Mathematics and Computation*, Vol. 189, No. 2, 2007, pp. 1274-1281. doi:10.1016/j.amc.2006.12.007

[5] B. A. McCarl, H. Moskowitz and H. Furtan, "Quadratic Programming Applications," *Omega*, Vol. 5, No. 1, 1977, pp. 43-55. doi:10.1016/0305-0483(77)90020-2

[6] J. J. Moré and G. Toraldo, "Algorithms for Bound Constrained Quadratic Programming Problems," *Numerische Mathematik*, Vol. 55, No. 4, 1989, pp. 377-400. doi:10.1007/BF01396045

[7] G. B. Dantzig, "Linear Programming and Extensions," Princeton University Press, Princeton, 1998.

[8] X.-W. Liu and Y.-X. Yuan, "A Null-Space Primal-Dual Interior-Point Algorithm for Nonlinear Optimization with Nice Convergence Properties," *Mathematical Programming*, Vol. 125, No. 1, 2010, pp. 163-193. doi:10.1007/s10107-009-0272-y

[9] S. Carvalho and A. Oliveira, "Interior-Point Methods Applied to the Predispatch Problem of a Hydroelectric System with Scheduled Line Manipulations," *American Journal of Operations Research*, Vol. 2, No. 2, 2012, pp. 266-271. doi:10.4236/ajor.2012.22032

[10] W.-Y. Sun and Y.-X. Yuan, "Optimization Theory and Methods: Nonlinear Programming," Springer, New York, 2006.

[11] O. K. Gupta, "Applications of Quadratic Programming," *Journal of Information and Optimization Sciences*, Vol. 16, No. 1, 1995, pp. 177-194.

[12] R. Horst, P. M. Pardalos and N. V. Thoai, "Introduction to Global Optimization: Nonconvex Optimization and its Applications," 2nd Edition, Kluwer Academic Publishers, Dordrecht, 2000.

[13] H. A. Taha, "Operations Research: An Introduction," 9th Edition, Pearson, 2011.

[14] W. L. Winston, "Operations Research: Applications and Algorithms," 4th Edition, Thomson Brooks/Cole, Belmont, 2004.

