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The Theory of Membership Degree of Γ-Conclusion in Several *n*-Valued Logic Systems*

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ABSTRACT

Based on the analysis of the properties of Γ-conclusion by means of deduction theorems, completeness theorems and the theory of truth degree of formulas, the present papers introduces the concept of the membership degree of formulas *A* is a consequence of Γ (or Γ-conclusion) in Łukasiewicz *n*-valued propositional logic systems, *Gödel* *n*-valued propositional logic system and the R_0 *n*-valued propositional logic systems. The condition and related calculations of formulas *A* being Γ-conclusion were discussed by extent method. At the same time, some properties of membership degree of formulas *A* is a Γ-conclusion were given. We provide its algorithm of the membership degree of formulas *A* is a Γ-conclusion by the constructions of theory root.

Keywords: *N*-Valued Propositional Logic; Γ-Conclusion; Theory; Root; Membership Degree

1. Introduction

Fuzzy logic is the theoretical foundation of fuzzy control. Spurred by the success in its applications, especially in fuzzy control, fuzzy logic has aroused the interest of many famous scholars, a series of important results have been created in documents [1-5]. For the sake of reasoning, we have to choose a subset Γ of well-formed formulas, which can reflect come essential properties, as the axioms of the logical system and we then deduce the so-called Γ-conclusion through some reasonable inference rules [6-9]. So, a natural question then arises: how to judge whether or not a general formula A is a conclusion of a given theory Γ, or to what extend the formula A is a conclusion of Γ? It is basic problem to judge one thing belong to one kind in artificial intelligence. As is well known, human reasoning is approximate rather than precise in nature. we basic starting point is to establish graded version of basic logical notions. In order to establish a solid foundation for fuzzy reasoning, professor G. J. Wang proposed the concept of root of theory [3], J. C. Zhang proposed the concept of generalized root of theory [10,11], in propositional logic systems. The graded description and properties of formulas A being Γ-conclusion were discussed. And provide its algorithm of membership degree of formulas A is a Γ-conclusion, by the constructions of theory root in the above-mentioned logic systems.

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2. Preliminaries

It is well known that different implication operators and valuation lattices L (*i.e.*, the set of truth degrees for logic) determine different logic systems (see [12]). Here valuation lattices is $L_n=\left\{0,\frac{1}{n-1},\cdots,\frac{n-2}{n-1},1\right\}$ and three popularly used implication operators and the correspond ing t-norms defined as follows:

$$R_{L_n}(x,y)=\begin{cases}1, & x\le y,\\ (1-x+y), & x>y,\end{cases}$$

$$x\otimes_{L_n} y=\max(0,x+y-1), x,y\in L_n;$$

$$R_{G_n}(x,y)=\begin{cases}1, & x\le y,\\ y, & x>y,\end{cases}$$

$$x\otimes_{G_n} y=\min(x,y), x,y\in L_n;$$

$$R_0(x,y)=\begin{cases}1, & x\le y,\\ (1-x)\vee y, & x>y,\end{cases}$$

$$x\otimes_n y=\begin{cases}x\wedge y, & x+y>1,\\ 0, & x+y\le 1,\end{cases} x,y\in L_n.$$

These three implication operators R_{L_n}, R_{G_n}, and R_0, are called Łukasiewicz implication operator $R_{Ł_n}$, Gödel implication operator R_{G_n}, and the R_0-implication operator R_0, respectively. The t-norm, which corresponds to R_0-implication operator R_0, is called also Nilpotent

Minimumtnorm [6]. If we fix a t-norm $\otimes$ above we then fix a propositional calculus (whose set of truth values is L_n): $\otimes$ is taken for the truth function of the strong conjunction &, the residuum R of $\otimes$ becomes the truth function of the implication operator and R(.,0) is the truth function of the negation. In more details, we have the following definitions.

Definition 1 [7,8]. The propositional calculus $PC(\otimes)$ given by a t-norm $\otimes$ has the set S of propositional variables $p_1, p_2, \cdots$ and connectives $\neg, \&, \to$. The set $F(S)$ of well-formed formulas in $PC(\otimes)$ is defined inductively as follows: each propositional variable is a formula; if A, B are formulas, then $\neg A$, $A \& B$ and $A \to B$ are all formulas.

Definition 2 [8,9,13]. The formal deductive systems of $PC(\otimes)$ given by $\otimes$ corresponding to R_{L_n}, R_{G_n}, and R_0, are called Łukasiewicz *n*-valued logic systems $Ł_n$, *Gödel* *n*-valued logic systems G_n, and the R_0 *n*-valued logic systems L_n^*, respectively.

Define in the above-mentioned logic systems

$$A^m := \underbrace{A \& A \& \cdots \& A}_{m}, \quad (1)$$

and in the corresponding algebras L_n

$$a^{(m)} := \underbrace{a \otimes a \otimes \cdots \otimes a}_{m}, \quad (2)$$

where $\otimes$ is the t-norm defined on L_n.

Remark 1. It is easy to verify that the following assertions are true:

(1) in G_n, $a^{(m)} = a$ for every $m \in N$; .

(2) in L_n^*, $a^{(m)} = a^{(2)}$ for every $m \in N$, and $m \geq 2$.

(3) in $Ł_n$, $a^{(n)} = (na - (n-1)) \vee 0$, for every $n \in N$.

Definition 3 [7,8]. (1) A homomorphism $v: F(S) \to L_n$ of type $(\neg, \&, \to)$ from $F(S)$ into the valuation lattice $Ł_n$, *i.e.* $v(\neg A) = \neg v(A)$, $v(A \& B) = v(A) \otimes v(B)$, $v(A \to B) = v(A) \to v(B)$, is called an R-valuation of $F(S)$. The set of all R-valuations will be denoted by Ω_R.

(2) A formula $A \in F(S)$ is called a tautology w.r.t. R if $\forall v \in \Omega_R, v(A) = 1$ holds.

Remark 2 [8,13]. It is not difficult to verify in the above-mentioned three logic systems that
$\upsilon(A \vee B) = \max\{\upsilon(A), \upsilon(B)\}$, and
$\upsilon(A \wedge B) = \max\{\upsilon(A), \upsilon(B)\}$ for every valuation
$\upsilon \in \Omega_R$. Moreover, one can check in $Ł_n$ and L_n^* that $A \& B$ and $\neg(A \to \neg B)$ are logically equivalent.

Definition 4 [8]. Assume that $A = A(p_1, p_2, \cdots, p_m)$ is a formula generated by propositional variables $p_1, p_2, \cdots, p_m$ through connectives $\neg, \&,$ and $\to$. Substitute x_i for p_i in $A(i = 1, 2, \cdots, m)$ and keep the logic connectives in A unchanged but explain them as the corresponding operators defined on the valuation lattice L_n. The we get a function $\overline{A}: L_n^m \to L_n$ and call $\overline{A}(x_1, x_2, \cdots, x_m)$ the truth degree function of A.

Definition 5 [7,8]. (1) A subset of $F(S)$ is called a theory.

(2) Let Γ be a theory, $A \in F(S)$. A deduction of A from Γ, in symbols, $\Gamma \vdash A$, is a finite sequence of formulas $A_1, \cdots, A_m = A$ such that for each $1 \leq i \leq m$, A_i is an axiom of L, or $A_i \in \Gamma$, or there are $j, k \in \{1, \cdots, i-1\}$ such that A_i follows from A_j and A_k by MP. Equivalently, we say that A is a conclusion of Γ (or Γ-conclusion). The set of all conclusions of Γ is denoted by $D(\Gamma)$. By a proof of A we shall henceforth mean a deduction of A from the empty set. We shall also write $\vdash A$ in place of $\varnothing \vdash A$ and call A a theorem.

It is easy for the reader to check the following Proposition 1.

Proposition 1. Let Γ be a theory and $A \in F(S)$ If $\Gamma \vdash A$ then there exist a finite subset of Γ say, $\{A_1, A_2, \cdots, A_m\}$ such that $\{A_1, A_2, \cdots, A_m\} \vdash A$.

Theorem 1 (Generalized deduction theorems) [7, 8,12]. Suppose that Γ is a theory, $A, B \in F(S)$, then

(1) in $Ł_n$,

$$\Gamma \cup \{A\} \vdash B \text{ iff } \exists s \in N \text{ s. t. } \Gamma \vdash A^s \to B.$$

(2) in G_n,

$$\Gamma \cup \{A\} \vdash B \text{ iff } \Gamma \vdash A \to B.$$

(3) in L_n^*,

$$\Gamma \cup \{A\} \vdash B \text{ iff } \Gamma \vdash A^2 \to B.$$

Definition 6 [8,13]. Suppose that $A = A(p_1, p_2, \cdots, p_m)$ is a formula of $F(S)$ containing m atomic formulas $p_1, p_2, \cdots, p_m$, and $\overline{A}(x_1, x_2, \cdots, x_m)$ be the truth degree function of A. Then

$$\tau(A) = \sum_{i=0}^{n-1} \frac{i}{n-1} \frac{\left|\overline{A}^{-1}\left(\frac{i}{n-1}\right)\right|}{n^m}$$

is called the truth degree of A, where $|B|$ is the cardinal of set B.

Theorem 2. Suppose that $A = A(p_{i_1}, p_{i_2}, \cdots, p_{i_t}) \in F(S)$ and $n \geq 2$, then in $Ł_n$ and G_n,

$T(A) = 1$ iff A is a tautology *i.e.*, $\vdash A$.

Proof. Assume that $T(A) = 1$. Since

$$T(A) \leq \frac{n-2}{n-1} \sum_{i=0}^{n-1} \frac{\left|\overline{A}^{-1}\left(\frac{i}{n-1}\right)\right|}{n^t}) + \frac{1}{n-1} \frac{\left|\overline{A}^{-1}(1)\right|}{n^t},$$

$$= \frac{n-2}{n-1} + \frac{1}{n-1} \frac{\left|\overline{A}^{-1}(1)\right|}{n^t}$$

then $\left|\overline{A}^{-1}(1)\right| \geq n^t$. By definite, $\left|\overline{A}^{-1}(1)\right| \leq n^t$, thus $\left|\overline{A}^{-1}(1)\right| = n^t$ *i.e.*, $\forall \upsilon \in \Omega_R$, $\upsilon(A) = 1$, then A is a tautology. Conversely, assume that A is a tautology *i.e.*, $\left|\overline{A}^{-1}(1)\right| = n^t$, then $\left|\overline{A}^{-1}\left(\frac{i}{n-1}\right)\right| = 0 \quad i = 0,1,\cdots,n-2$, so $T(A)=1$. This completes the proof.

Theorem 3 [8]. Suppose that $A \in F(S)$, then in $Ł_n^*$, $\tau(A)=1$ iff A is a tautology, *i.e.*, $\vDash A$.

Theorem 4. Suppose that $A, B \in F(S)$. If for every $\upsilon \in \Omega_R, \upsilon(A) \ge \upsilon(B)$, then $\tau(A) \ge \tau(B)$.

Proof. Suppose that $A = A(p_1, p_2, \cdots, p_m)$ and $B = B(p_1, p_2, \cdots, p_m)$ are all a formulas of $F(S)$ containing m atomic formulas $p_1, p_2, \cdots, p_m$, it follows from $\upsilon(A) \ge \upsilon(B)$ that

$$\begin{aligned}&\overline{A}(x_1, x_2, \cdots, x_m) = \upsilon(A)\\&= \overline{A}(\upsilon(p_1), \upsilon(p_2), \cdots, \upsilon(p_m))\\&\ge \upsilon(B) = \overline{B}(\upsilon(p_1), \upsilon(p_2), \cdots, \upsilon(p_m))\\&= \overline{B}(x_1, x_2, \cdots, x_m)\end{aligned}$$

and

$$\begin{aligned}&\overline{A}^{-1}(1) \cup \overline{A}^{-1}\left(\frac{n-2}{n-1}\right) \cup \cdots \cup \overline{A}^{-1}\left(\frac{i}{n-1}\right)\\&\supseteq \overline{B}^{-1}(1) \cup \overline{B}^{-1}\left(\frac{n-2}{n-1}\right) \cup \cdots \cup \overline{B}^{-1}\left(\frac{i}{n-1}\right),\\&(i = n-1, n-2, \cdots, 1)\end{aligned}$$

hence

$$\begin{aligned}&\left|\overline{A}^{-1}(1)\right| + \left|\overline{A}^{-1}\left(\frac{n-2}{n-1}\right)\right| + \left|\overline{A}^{-1}\left(\frac{n-3}{n-1}\right)\right| + \cdots + \left|\overline{A}^{-1}\left(\frac{i}{n-1}\right)\right|\\&\ge \left|\overline{B}^{-1}(1)\right| + \left|\overline{B}^{-1}\left(\frac{n-2}{n-1}\right)\right| + \left|\overline{B}^{-1}\left(\frac{n-3}{n-1}\right)\right|\\&\quad + \cdots + \left|\overline{B}^{-1}\left(\frac{i}{n-1}\right)\right|\\&(i = n-1, n-2, \cdots, 1, 0)\end{aligned}.$$

It is easy to verify that

$$\begin{aligned}&\left|\overline{A}^{-1}(1)\right| + \frac{n-2}{n-1}\left|\overline{A}^{-1}\left(\frac{n-2}{n-1}\right)\right| + \frac{n-3}{n-1}\left|\overline{A}^{-1}\left(\frac{n-3}{n-1}\right)\right|\\&+\cdots+\frac{1}{n-1}\left|\overline{A}^{-1}\left(\frac{1}{n-1}\right)\right|\\&\ge \left|\overline{B}^{-1}(1)\right| + \frac{n-2}{n-1}\left|\overline{B}^{-1}\left(\frac{n-2}{n-1}\right)\right|\\&+\frac{n-3}{n-1}\left|\overline{B}^{-1}\left(\frac{n-3}{n-1}\right)\right| + \cdots + \frac{1}{n-1}\left|\overline{B}^{-1}\left(\frac{1}{n-1}\right)\right|\end{aligned}$$

then $\tau(A) \ge \tau(B)$.

3. Properties of the Roots of Theories

Definition 7 [3]. Suppose that Γ is a theory, $A \in D(\Gamma)$. If for every $B \in D(\Gamma)$ we have $\vdash A \to B$, then A is called the root of Γ.

Theorem 5. Suppose that Γ is a finite theory, say $\Gamma = \{A_1, A_2, \cdots, A_m\}$, then

(1) in $Ł_n$,

$$A_1^{n-1} \,\&\, A_2^{n-1} \,\&\cdots\&\, A_m^{n-1} \text{ is a root of } \Gamma;$$

(2) in L_n^*,

$$A_1^2 \,\&\, A_2^2 \,\&\cdots\&\, A_m^2 \text{ is a root of } \Gamma;$$

(3) in G_n, $A_1 \wedge A_2 \wedge \cdots A_m$ is a root of Γ.

Proof. (1) It following form references [4] that $A_1^{n-1} \,\&\, A_2^{n-1} \,\&\cdots\&\, A_m^{n-1} \in D(\Gamma)$, for every $B \in D(\Gamma)$, there exist $n_1, n_2, \cdots, n_m \in N$ such that $\vdash A_1^{n_1} \,\&\, A_2^{n_2} \,\&\cdots\&\, A_m^{n_m} \to B$ by Theorem 1. It is easy to check that $A_i^{n_i} \sim A_i^{n-1}, n_i \ge n-1$ by Remark 1, it following from $\vdash A \,\&\, B \to B$ that $\vdash A_1^{l_1} \,\&\, A_2^{l_2} \,\&\cdots\&\, A_m^{l_m} \to A_1^{n_1} \,\&\, A_2^{n_2} \,\&\cdots\&\, A_m^{n_m}$ where $l_1 \ge n_1, l_2 \ge n_2, \cdots, l_m \ge n_m$, thus $\vdash A_1^{n-1} \,\&\, A_2^{n-1} \,\&\cdots\&\, A_m^{n-1} \to B$ by Hypothetical, this shows that $A_1^{n-1} \,\&\, A_2^{n-1} \,\&\cdots\&\, A_m^{n-1}$ is a root of Γ.

(2) It following form references [4] that $A_1^2 \,\&\, A_2^2 \,\&\cdots\&\, A_m^2 \in D(\Gamma)$, for every $B \in D(\Gamma)$, it following from Theorem 1 that $\vdash A_1^2 \to (A_2^2 \to (\cdots(A_m^2 \to B)))$, since $A_1^2 \,\&\, A_2^2 \,\&\cdots\&\, A_m^2 \to B$ and $A_1^2 \to (A_2^2 \to (\cdots(A_m^2 \to B)))$ are provably equivalent, and so is $\vdash A_1^2 \,\&\, A_2^2 \,\&\cdots\&\, A_m^2 \to B$. This shows $A_1^2 \,\&\, A_2^2 \,\&\cdots\&\, A_m^2$ is a root of Γ.

(3) It following from references [4] that $A_1 \wedge A_2 \wedge \cdots A_m \in D(\Gamma)$, for every $B \in D(\Gamma)$, we get $\vdash A_m \to (A_{m-1} \to (\cdots(A_1 \to B)))$ by Theorem 1, it is easy to verify that $A \to (B \to C)$ and $A \wedge B \to C$ are provably equivalent, hence $A_m \wedge A_{m-1} \wedge \cdots \wedge A_1 \to B$ and $A_m \to (A_{m-1} \to (\cdots(A_1 \to B)))$ are provably equivalent, and so is $\vdash A_m \wedge A_{m-1} \wedge \cdots \wedge A_1 \to B$. This shows that $A_m \wedge A_{m-1} \wedge \cdots \wedge A_1$ is a root of Γ.

4. Membership Degree of Formulas *A* Is Γ-Conclusion

In following, let us first take an analysis on the conditions of formulas A is a Γ-conclusion in $Ł_n$. Suppose that Γ is a theory and A is a Γ-conclusion, it follows from Proposition 1 and Theorem 1 that there exit a finite string of formulas $A_1, A_2 \cdots, A_l \in \Gamma$ and $n_1, n_2, \cdots, n_l \in N$ such that $\vdash A_1^{n_1} \,\&\, A_2^{n_2} \,\&\cdots\&\, A_l^{n_l} \to A$ holds, *i.e.*, the formula $A_1^{n_1} \,\&\, A_2^{n_2} \,\&\cdots\&\, A_l^{n_l} \to A$ is a theorem of $Ł_n$, let $B = A_1^{n_1} \,\&\, A_2^{n_2} \,\&\cdots\&\, A_l^{n_l}$, hence $B \to A$ is a tautology, it follows from Theorem 2 that $\tau(B \to A) = 1$. Conversely, if there exist a Γ-conclusion B such that $\tau(B \to A) = 1$, then following from

Theorem 2 that $B \to A$ is a tautology, thus $B \to A$ is a theorem of $Ł_n$, *i.e.*, $\vdash B \to A$ holds and $\Gamma \vdash B$, we have that $\Gamma \vdash A$ by MP, *i.e.*, A is a Γ-conclusion. Moreover, the larger the membership degree of such formulas are, the more closer A is to be Γ-conclusion. Hence it is natural and reasonable for us using the supremum of truth degree of all formulas with the form $B \to A$ where $B \in D(\Gamma)$ to measure A is a Γ-conclusion.

Definition 8. Suppose that Γ is a theory, $A \in F(S)$. Define

$$T(A|\Gamma) = sup\{\tau(B \to A) | B \in D(\Gamma)\},$$

then $T(A|\Gamma)$ is called the membership degree of formulas A is a Γ-conclusion.

It is easy to verify that $0 \le T(A|\Gamma) \le 1$ and following Proposition 2 by Definition 8.

Proposition 2. In $Ł_n$, G_n, and L_n^*,

If A is a Γ-conclusion, then $T(A|\Gamma) = 1$.

Theorem 6. In $Ł_n$, G_n, and L_n^*, if Γ is a finite theory, say $\Gamma = \{A_1, A_2, \cdots, A_m\}$, then A is a Γ-conclusion iff $T(A|\Gamma) = 1$.

Proof. The necessity part by proposition 2, it is only necessary to prove the sufficiency. Let $T(A|\Gamma) = 1$. For every number $\varepsilon > 0$, there exist a formulas $B \in D(\Gamma)$ such that $\tau(B \to A) > 1 - \varepsilon$ by Definition 8.

(1) In L_n^*, it follows from Theorem 5 that $A_1^2 \& A_2^2 \& \cdots \& A_m^2$ is a root of Γ and $\vdash A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to B$ hold. Hence for every $\upsilon \in \Omega_R$, we have $\upsilon(A_1^2 \& A_2^2 \& \cdots \& A_m^2) \le \upsilon(B)$, it follows from properties of implication operators that $\upsilon(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) \ge \upsilon(B \to A) > 1 - \varepsilon$, since ε is arbitrary, we have $\upsilon(A_1^2 \& A_2^2 \& \cdots \& A_s^2 \to A) = 1$, thus $A_1^2 \& A_2^2 \& \cdots \& A_s^2 \to A$ is a tautology, and $A_1^2 \& A_2^2 \& \cdots \& A_s^2 \to A$ is a theorem , together with the result $\Gamma \vdash A_1^2 \& A_2^2 \& \cdots \& A_s^2$, then $\Gamma \vdash A$ by MP, *i.e.*, $A \in D(\Gamma)$.

(2) In $Ł_n$, notice that $A_1^{n-1} \& A_2^{n-1} \& \cdots \& A_s^{n-1}$ is a root of Γ by Theorem 5, hence the proof of (2) is similar to that the proof of (1) and so is omitted.

(3) In G_n, notice that $A_1 \wedge A_2 \wedge \cdots \wedge A_s$ is a root of Γ by Theorem 5, hence the proof of (2) is similar to that the proof of (1). In fact $A_1 \wedge A_2 \wedge \cdots \wedge A_s \to B$ is a theorem by Definition 7, hence $\forall \upsilon \in \Omega_R$ we have $\upsilon(A_1 \wedge A_2 \wedge \cdots \wedge A_s) \le \upsilon(B)$ and $\upsilon(A_1 \wedge A_2 \wedge \cdots \wedge A_s \to A) \ge \upsilon(B \to A) > 1 - \varepsilon$, thus $\forall \upsilon \in \Omega_R$, $\upsilon(A_1 \wedge A_2 \wedge \cdots \wedge A_s \to A) = 1$ holds, then $A_2 \wedge \cdots \wedge A_s \to A$ is a theorem , together with the result $\Gamma \vdash A_1 \wedge \cdots \wedge A_s$, we have $\Gamma \vdash A$ by MP. The proof is completed.

Theorem 7. Suppose that $\Gamma = \{A_1, A_2, \cdots, A_m\}$, then

(1) in L_n^*,

$$T(A|\Gamma) = \tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A);$$

(2) in $Ł_n$,

$$T(A|\Gamma) = \tau(A_1^{n-1} \& A_2^{n-1} \& \cdots \& A_m^{n-1} \to A);$$

(3) in G_3,

$$T(A|\Gamma) = \tau(A_1 \wedge A_2 \wedge \cdots \wedge A_m \to A).$$

Proof. (1) Since $A_1^2 \& A_2^2 \& \cdots \& A_m^2$ is a root of Γ by Theorem 5, hence for every $B \in D(\Gamma)$, we have $\vdash A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to B$. Thus for every $\upsilon \in \Omega_R$, $\upsilon(A_1^2 \& A_2^2 \& \cdots \& A_m^2) \le \upsilon(B)$, and $\upsilon(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) \ge \upsilon(B \to A)$ holds, then $\tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) \ge \tau(B \to A)$ by Theorem 4. It follows from $A_1^2 \& A_2^2 \& \cdots \& A_m^2 \in D(\Gamma)$ that $\tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) = sup\{\tau(B \to A) | B \in D(\Gamma)\}$, *i.e.*, $T(A|\Gamma) = \tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A)$.

(2) Notice that in $Ł_n$, $A_1^{n-1} \& A_2^{n-1} \& \cdots \& A_m^{n-1}$ is a root of Γ by Theorem 5, the proof of (2) is similar to that the proof of (1) and so is omitted.

(3) Notice that in G_3, $A_1 \wedge A_2 \wedge \cdots \wedge A_m$ is a root of Γ by Theorem 5, the proof of (2) is similar to that the proof of (1) and so is omitted.

Theorem 8. Suppose that Γ is a infinite theory. Then

(1) in L_n^*,

$$T(A|\Gamma) = sup\{\tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) \,|\, \{A_1, A_2, \cdots A_m\} \subset \Gamma, m \in N\};$$

(2) $Ł_n$,

$$T(A|\Gamma) = sup\{\tau(A_1^{n-1} \& A_2^{n-1} \& \cdots \& A_m^{n-1} \to A) \,|\, \{A_1, A_2, \cdots A_m\} \subset \Gamma, m \in N\};$$

(3) in G_n,

$$T(A|\Gamma) = sup\{\tau(A_1 \wedge A_2 \wedge \cdots \wedge A_m \to A) \,|\, \{A_1, A_2, \cdots A_m\} \subset \Gamma, m \in N\}.$$

Proof. (1) For every $B \in D(\Gamma)$, it following from Proposition 1 that there exist a finite string of formulas $A_1, A_2, \cdots A_m \in \Gamma$ such that $\{A_1, A_2, \cdots A_m\} \vdash B$. It follows from Theorem 1 that $A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to B$ is a theorem, hence $A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to B$ is a tautology by completeness theorem, and for every $\upsilon \in \Omega_R$, $\upsilon(A_1^2 \& A_2^2 \& \cdots \& A_m^2) \le \upsilon(B)$, we have $\tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) \ge \tau(B \to A)$ by Theorem 4.

It following form references [14] that $A_1^2 \& A_2^2 \& \cdots \& A_m^2 \in D(\Gamma)$, then

$$T(A|\Gamma) = sup\{\tau(A_1^2 \& A_2^2 \& \cdots \& A_m^2 \to A) \,|\, \{A_1, A_2, \cdots A_m\} \subset \Gamma, m \in N\}.$$

(2) Notice that in $Ł_n$, $A_i^{n_i} \sim A_i^{n-1}, n_i \ge n-1$ by Remark 1, the Proof of (2) is similar to that the Proof of (1)

and so is omitted.

(3) Notice that in G_n, $A_1 \wedge A_2 \to B$ and $A_1 \to (A_2 \to B)$ is Provably equivalent, the Proof of (3) is similar to that the Proof of (1) and so is omitted.

Theorem 9. Suppose that Γ is a theory, $T(A|\Gamma) \ge \alpha$ and $T(A \to C|\Gamma) \ge \beta$, then $T(C|\Gamma) \ge (\alpha+\beta-1) \vee 0$.

Proof. (1) If $\alpha+\beta \le 1$, we get $(\alpha+\beta-1)\vee 0 = 0$, then $T(C|\Gamma) \ge (\alpha+\beta-1)\vee 0$.

(2) If $\alpha+\beta>1$, we get $\alpha>0$ and $\beta>0$, for any given positive number ε such that $\alpha-\varepsilon>0$ and $\beta-\varepsilon>0$, there exists formulas $B_i \in D(\Gamma), i=1,2,$ such that $\tau(B_1 \to A) > \alpha-\varepsilon,$ and $\tau(B_2 \to (A \to C)) > \beta-\varepsilon$. It follows from properties of Regular implication operators that $\tau(B_1 \& B_2 \to A) \ge \tau(B_1 \to A)$ and $\tau(B_1 \& B_2 \to (A \to C)) \ge \tau(B_2 \to (A \to C))$. It is easy to verify that $B_1 \& B_2 \to (A \to C)$ and $A \to (B_1 \& B_2 \to C)$ are provably equivalent (*i.e.*, logically equivalent), hence $\tau(A \to (B_1 \& B_2 \to C)) = \tau(B_1 \& B_2 \to (A \to C))$. It follows from the theory of truth degrees of formulas and $\tau(B_1 \& B_2 \to A) > \alpha-\varepsilon,$ $\tau(B_1 \& B_2 \to (A \to C)) > \beta-\varepsilon$ that $\tau(B_1 \& B_2 \to (B_1 \& B_2 \to C)) \ge (\alpha+\beta-2\varepsilon-1)\vee 0$.

Bucas $B_1 \& B_2 \to (B_1 \& B_2 \to C)$ and $((B_1 \& B_2)\&(B_1 \& B_2)) \to C$ are provably equivalent (*i.e.*, logically equivalent), hence $\tau((B_1 \& B_2)^2 \to C) \ge (\alpha+\beta-2\varepsilon-1)\vee 0$, it is easy to verify that $(B_1 \& B_2)^2 \in D(\Gamma)$, then $T(C|\Gamma) \ge (\alpha+\beta-1)\vee 0$ by the definition of the membership degree of formulas.

Example 1. Suppose that $\Gamma = \{p_1, p_2\} \subset S, p_3 \in S$. In Ł$_3$, L_3^* and G_3, compute $T(p_3|\Gamma)$, respectively.

Solution. (1) In L_3, assume that $A = p_1 \& p_2 \to p_3$. Since $\overline{A}(x_1,x_2,x_3) = x_1 \otimes x_2 \to x_3$ and

$$x_i^2 = (2x_i - 1)\vee 0 = \begin{cases} 1, & x_i = 1, \\ 0, & \text{otherwise}; \end{cases}$$

thus

$$x_1^2 \otimes x_2^2 = \begin{cases} 1, & x_1 = x_2 = 1, \\ 0, & \text{otherwise}; \end{cases}$$

and

$$x_1^2 \otimes x_2^2 \to x_3 = \begin{cases} \frac{1}{2}, & x_1 = x_2 = 1, x_3 = \frac{1}{2}; \\ 0, & x_1 = x_2 = 1, x_3 = 0; \\ 1, & \text{otherwise}; \end{cases}$$

We have $\left|\overline{A}^{-1}\left(\frac{1}{2}\right)\right| = 1$ and $\left|\overline{A}^{-1}(1)\right| = 3^3 - 2 = 25$,

hence

$$\tau(A) = \tau(p_1^2 \& p_2^2 \to p_3) = \frac{1}{3^3}\left[\frac{1}{2}\left|\overline{A}^{-1}\left(\frac{1}{2}\right)\right| + \left|\overline{A}^{-1}(1)\right|\right] = \frac{17}{18},$$

then $T(p_3|\Gamma) = \tau(p_1^2 \& p_2^2 \to p_3) = \frac{17}{18}$.

(2) In G_3, assume that $A = p_1 \wedge p_2 \to p_3$. Since $\overline{A}(x_1,x_2,x_3) = x_1 \wedge x_2 \to x_3$, and

$$x_1 \wedge x_2 = \begin{cases} 1, & x_1 = x_2 = 1, \\ \frac{1}{2}, & x_1 = 1, x_2 = \frac{1}{2} \\ & \text{or } x_1 = \frac{1}{2}, x_2 = 1 \text{ or } x_1 = x_2 = \frac{1}{2} \\ 0, & \text{otherwise}; \end{cases}$$

thus

$$x_1 \wedge x_2 \to x_3 = \begin{cases} \frac{1}{2}, & x_1 = x_2 = 1, x_3 = \frac{1}{2}; \\ 0, & x_3 = 0, x_1 \ne 0, x_2 \ne 0; \\ 1, & \text{otherwise}; \end{cases}$$

then

$$\tau(A) = \tau(p_1 \wedge p_2 \to p_3) = \frac{1}{3^3}\left[\frac{1}{2}\left|\overline{A}^{-1}\left(\frac{1}{2}\right)\right| + \left|\overline{A}^{-1}(1)\right|\right] = \frac{1}{27}\left[\frac{1}{2} + 22\right] = \frac{15}{18},$$

then $T(p_3|\Gamma) = \frac{15}{18}$.

(3) In L_3^*, assume that $A = p_1^2 \& p_2^2 \to p_3$. Since $\overline{A}(x_1,x_2,x_3) = x_1^2 \otimes x_2^2 \to x_3$, and

$$x_i^2 = \begin{cases} 1, & x_i = 1, \\ 0, & x_i = 0 \text{ or } x_i = \frac{1}{2}; \end{cases}$$

$$x_1^2 \otimes x_2^2 = \begin{cases} 1, & x_1 = x_2 = 1, \\ 0, & \text{otherwise}; \end{cases}$$

$$x_1^2 \otimes x_2^2 \to x_3 = \begin{cases} 0, & x_1 = x_2 = 1, x_3 = 0, \\ \frac{1}{2}, & x_1 = x_2 = 1, x_3 = \frac{1}{2}, \\ 1, & \text{otherwise}; \end{cases}$$

thus

$$\tau(p_1^2 \& p_2^2 \to p_3) = \frac{1}{3^3}\left[\frac{1}{2}\left|\overline{A}^{-1}\left(\frac{1}{2}\right)\right| + \left|\overline{A}^{-1}(1)\right|\right]$$

$$= \frac{\frac{1}{2} + 25}{27} = \frac{17}{18},$$

then $T(p_3 \mid \Gamma) = \frac{17}{18}$.

Example 2. Suppose that $\Gamma = \{p_1, p_1 \to p_2\}$, $p_1, p_2 \in S$, in $Ł_3$, compute $T(p_2|\Gamma)$.

Solution. (1) Assume that $A = p_1^2 \& (p_1 \to p_2)^2 \to p_2$. Since $\overline{A}(x_1, x_2) = x_1^2 \otimes (x_1 \to x_2)^2 \to x_2$, and

$$x_1^2 = (2x_1 - 1) \vee 0 = \begin{cases} 1, & x_1 = 1, \\ 0, & \text{otherwise;} \end{cases}$$

$$(x_1 \to x_2) = \begin{cases} 0, & x_1 = 1, x_2 = 0, \\ \frac{1}{2}, & x_1 = 1, x_2 = \frac{1}{2} \text{ or } x_1 = \frac{1}{2}, x_2 = 0, \\ 1, & \text{otherwise;} \end{cases}$$

$$(x_1 \to x_2)^2 = \begin{cases} 0, & x_1 = 1, x_2 = 0, \\ & \text{or } x_1 = 1, x_2 = \frac{1}{2} \text{ or } x_1 = \frac{1}{2}, x_2 = 0 \\ 1, & \text{otherwise} \end{cases}$$

$$x_1^2 \otimes (x_1 \to x_2)^2 = \begin{cases} 0, & x_1 = 1, x_2 = 0, \text{or } x_1 = 1, x_2 = \frac{1}{2} \\ & \text{or } x_1 = \frac{1}{2}, x_2 = 0 \\ 1, & \text{otherwise} \end{cases}$$

$$x_1^2 \otimes (x_1 \to x_2)^2 \to x_2 = 1,$$

$$\tau(A) = \tau\left[p_1^2 \& (p_1 \to p_2)^2 \to p_2\right] = \frac{1}{3^n}\left[\frac{1}{2}\left|\overline{A}^{-1}\left(\frac{1}{2}\right)\right| + \left|\overline{A}^{-1}(1)\right|\right] = \frac{1}{9}(0+9) = 1$$

thus $T(p_2|\Gamma) = \tau\left[p_1^2 \& (p_1 \to p_2)^2 \to p_2\right] = 1$, then p_2 is a Γ-conclusion.

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