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The M^X/M/1 Queue with Multiple Working Vacation

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ABSTRACT

We study a batch arrival M^X/M/1 queue with multiple working vacation. The server serves customers at a lower rate rather than completely stopping service during the service period. Using a quasi upper triangular transition probability matrix of two-dimensional Markov chain and matrix analytic method, the probability generating function (PGF) of the stationary system length distribution is obtained, from which we obtain the stochastic decomposition structure of system length which indicates the relationship with that of the M^X/M/1 queue without vacation. Some performance indices are derived by using the PGF of the stationary system length distribution. It is important that we obtain the Laplace Stieltjes transform (LST) of the stationary waiting time distribution. Further, we obtain the mean system length and the mean waiting time. Finally, numerical results for some special cases are presented to show the effects of system parameters.

Keywords: M^X/M/1 Queue; Multiple Working Vacation; Probability Generating Function; Waiting Time Distribution; Stochastic Decomposition

1. Introduction

Vacation queues have been investigated for over two decades as a very useful tool for modeling and analyzing computer systems, communication networks, manufacturing and production systems and many others. The details can be seen in the monographs of Takagi [1] and Tian and Zhang [2], the survey of Doshi [3]. However, in these models, the server stops the original work in the vacation period and can not come back to the regular busy period until the vacation period ends.

Recently, Servi and Finn [4] introduced the working vacation policy, in which the server works at a different rate rather than completely stopping service during the vacation. They studied an M/M/1 queue with working vacations, and obtained the transform formulae for the distribution of the number of customers in the system and sojourn time in steady state, and applied these results to performance analysis of gateway router in fiber communication networks. During the working vacation models, the server can not come back to the regular busy period until the vacation period ends. Subsequently, Wu and Takagi [5] generalized the model in [4] to an M/G/1 queue with general working vacations. Baba [6] studied a GI/M/1 queue with working vacations by using the matrix analytic method. Banik *et al*. [7] analyzed the GI/M/1/N queue with working vacations. Liu *et al*. [8] established a stochastic decomposition result in the M/M/1 queue with working vacations. Li *et al*. [9] established the conditional stochastic decomposition result in the M/G/1 queue with exponentially working vacations using matrix analytic approach.

For the batch arrival queues, Xu *et al*. [10] studied a batch arrival M^X/M/1 queue with single working vacation. With the matrix analytic method, they derived the PGF of the stationary system length distribution, from which they got the stochastic decomposition result for the PGF of the stationary system length which indicates the evident relationship with that of the classical M^X/M/1 queue without vacation. Furthermore, they found the upper bound and lower bound of the stationary waiting time in the Laplace transform order, from which they got the upper bound and lower bound of the waiting time.

In this paper, we study a batch arrival M^X/M/1 queue with *multiple* working vacation. We obtain the PGF of the stationary system length distribution and the stochastic decomposition structure of system length which indicates the relationship with that of M^X/M/1 queue without vacation. Although only the upper bound and lower bound of the stationary waiting time in the Laplace transform order are obtained in [10], we can obtain the exact LST of the stationary waiting time distribution.

The rest of this paper is organized as follows. In Section 2, the model of the M^X/M/1 queue with multiple working vacation is described. In Section 3, we obtain the PGF of the stationary system length and its decomposition result which indicates the evident relationship with that of the classical $M^X/M/1$ queue without vacation. Furthermore, we obtain the mean system length and several characteristic quantities. In Section 4, we obtain the LST of the stationary waiting time distribution. Numerical results for some special cases are presented in Section 5.

2. Model Description

Consider a batch arrival $M^X/M/1$ queue with multiple working vacation. Customers arrive in batches according to a Poisson process with rate λ. The batch size X is a random variable and

$$P(X=k)=g_k, \quad k=1,2,\cdots$$

The PGF, the expectation and the second order moment of X are

$$G(z)=\sum_{k=1}^{\infty}g_k z^k, |z|\le 1, g=E(X), g^{(2)}=E(X^2)$$

respectively.

Service times in the regular busy period follows an exponential distribution with parameter μ. Upon the completion of a service, if there is no customer in the system, the server begins a vacation and the vacation duration follows an exponential distribution with parameter θ. During a working vacation period, arriving customers are served at a rate of ν. When a vacation ends, if there are no customers in the queue, another vacation is taken. Otherwise, the server switches service rate from ν to μ, and a regular busy period starts. The LST's of the service time distribution in a regular busy period and the service time in a working vacation time are

$$B^*(s)=\frac{\mu}{s+\mu}, \quad C^*(s)=\frac{\nu}{s+\nu},$$

respectively.

We assume that inter-arrival times, the service times and the working vacation times are mutually independent. Furthermore, the service discipline is first come first served (FCFS).

Let $L(t)$ be the number of customers in the system at time t and

$$J(t)=\begin{cases}0, & \text{the system is in a working vacation period at time } t,\\ 1, & \text{the system is in a regular busy period at time } t\end{cases}$$

Then the process $\{L(t),J(t)\}$ is a two-dimensional Markov chain with the state space

$$\Omega=\{(0,0)\}\cup\{(k,j)|k\ge 1,\ j=0,1\}.$$

Using the lexicographical order for the states, the infinitesimal generator of the process $\{L(t),J(t)\}$ can be written as the Block-Jacobi matrix

$$\mathbf{Q}=\begin{bmatrix}\mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots\\ \mathbf{C}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots\\ & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots\\ & & \mathbf{A}_0 & \mathbf{A}_1 & \cdots\\ & & \vdots & \vdots & \ddots\end{bmatrix}$$

where

$$\mathbf{B}_0=-\lambda, \quad \mathbf{B}_i=(\lambda g_i,0), \quad i\ge 1,$$

$$\mathbf{C}_0=(\nu,\mu)^T, \quad \mathbf{A}_0=\begin{bmatrix}\nu & 0\\ 0 & \mu\end{bmatrix},$$

$$\mathbf{A}_1=\begin{bmatrix}-(\lambda+\nu+\theta) & \theta\\ 0 & -(\lambda+\mu)\end{bmatrix},$$

$$\mathbf{A}_i=\begin{bmatrix}\lambda g_{i-1} & 0\\ 0 & \lambda g_{i-1}\end{bmatrix}, i\ge 2.$$

Since $\mathbf{A}=\sum_{i=0}^{\infty}\mathbf{A}_i=\begin{bmatrix}-\theta & \theta\\ 0 & 0\end{bmatrix}$ is reducible, it is found from Section 3.5 of Neuts [11] that the Markov chain is positive recurrent if and only if $\mu>\lambda\sum_{i=1}^{\infty}ig_i$, that is, $\rho=\lambda g/\mu<1$.

3. Stationary System Length Distribution

In this section, we derive the PGF of stationary distribution for $(L(t),J(t))$. Let (L,J) be the stationary limit of the process $\{L(t),J(t)\}$. Assume that

$$\boldsymbol{\pi}_k=(\boldsymbol{\pi}_{k0},\boldsymbol{\pi}_{k1}), \quad k\ge 1,$$

$$\pi_{kj}=P(L=k,J=j)=\lim_{t\to\infty}P\{L(t)=k,\ J(t)=j\}, \quad (k,j)\in\Omega.$$

Based on the stationary equations, we have

$$-\lambda\pi_{00}+\nu\pi_{10}+\mu\pi_{11}=0 \tag{1}$$

$$\lambda g_1\pi_{00}-(\lambda+\nu+\theta)\pi_{10}+\nu\pi_{20}=0 \tag{2}$$

$$\lambda g_n\pi_{00}+\lambda\sum_{k=1}^{n-1}g_k\pi_{n-k,0}-(\lambda+\nu+\theta)\pi_{n0}+\nu\pi_{n+1,0}=0, \quad n\ge 2 \tag{3}$$

$$\theta\pi_{10}-(\lambda+\mu)\pi_{11}+\mu\pi_{21}=0 \tag{4}$$

$$\lambda\sum_{k=1}^{n-1}g_k\pi_{n-k,1}+\theta\pi_{n0}-(\lambda+\mu)\pi_{n1}+\mu\pi_{n+1,1}=0, \quad n\ge 2 \tag{5}$$

Define the probability generating functions (PGF's)

$$Q_0(z)=\sum_{k=1}^{\infty}\pi_{k0}z^k, \quad |z|\le 1,$$

$$Q_1(z)=\sum_{k=1}^{\infty}\pi_{k1}z^k, \quad |z|\le 1.$$

Then the PGF of stationary system length L can be written as

$$L(z) = \pi_{00} + Q_0(z) + Q_1(z), \quad |z| \le 1.$$

In order to prove Theorem 1 which states the stochastic decomposition structure of system length, the following lemma is necessary.

Lemma 1. *The equation*

$$(\lambda + \nu + \theta)z - \nu - \lambda z G(z) = 0$$

has the unique root $z = \alpha$ in the interval $(0,1)$.

Proof: We consider the function

$$f(z) = (\lambda + \nu + \theta)z - \nu - \lambda z G(z), \quad 0 < z < 1.$$

For any $0 < z < 1$, we have

$$f''(z) = -2\lambda G'(z) - \lambda z G''(z) < 0 \tag{6}$$

Therefore $f(z)$ is a concave function in the interval $(0,1)$. Further,

$$f(0) = -\nu < 0, \quad f(1) = \theta > 0. \tag{7}$$

(6) and (7) indicate that the equation $f(z) = 0$ has the unique root $z = \alpha$ in the interval $(0,1)$.

Remark 1. Lemma 1 in this paper is the same as Lemma 1 in Xu *et al*. [10]. But the proof in this paper is simpler than the proof in [10].

We have the following Theorem which proves the stochastic decomposition structure of system length.

Theorem 1. *If* $\rho = \lambda g/\mu < 1$ *and* $\nu < \mu$, *the stationary system length* L *can be decomposed into the sum of two independent random variables*,

$$L = L_0 + L_d$$

where L_0 *is the stationary system length in corresponding classical* M^X/M/1 *queue without vacation and has the PGF*

$$L_0(z) = \frac{\mu(1-\rho)(z-1)}{(\lambda+\mu)z - \mu - \lambda z G(z)} \tag{8}$$

L_d *is the additional system length and has the PGF*

$$L_d(z) = \frac{\sigma(z)}{\delta\left\{(\lambda+\nu+\theta)z - \nu - \lambda z G(z)\right\}} \tag{9}$$

where

$$\sigma(z) = \mu\left\{(\lambda+\nu+\theta)z - \nu - \lambda z G(z)\right\} + \lambda z\left\{G(z) - G(\alpha)\right\}(\mu - \nu)$$

and

$$\delta = \mu + \frac{\lambda}{\theta}\left\{1 - G(\alpha)\right\}(\mu - \nu)$$

Proof: Multiplying the Equation (2) by z and each equation of (3) by z^n and summing up these equations, we have

$$\lambda \sum_{n=1}^{\infty} g_n z^n \pi_{00} + \lambda \sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} g_k \pi_{n-k,0} - (\lambda+\nu+\theta)\sum_{n=1}^{\infty}\pi_{n0} z^n + \nu \sum_{n=1}^{\infty} \pi_{n+1,0} z^n = 0.$$

After calculations, we have

$$\lambda G(z)\pi_{00} + \lambda G(z) Q_0(z) - (\lambda+\nu+\theta) Q_0(z) + \nu\left\{Q_0(z) - \pi_{10} z\right\}/z = 0.$$

Therefore, we obtain

$$Q_0(z) = \frac{\left\{\lambda G(z)\pi_{00} - \nu \pi_{10}\right\} z}{(\lambda+\nu+\theta)z - \nu - \lambda z G(z)} \tag{10}$$

Since $Q_0(z)$ is an analytic function in $(0,1)$, wherever the right-side of (10) has zeros in $(0,1)$, so must the numerator. From Lemma 1, the denominator of $Q_0(z)$ is equal to 0 if $z = \alpha$, so does the numerator. Substituting $z = \alpha$ into the numerator of the right-hand side of (10), we have

$$\lambda G(\alpha)\pi_{00} - \nu \pi_{10} = 0 \tag{11}$$

Substituting (11) into (10), we have

$$Q_0(z) = \frac{\lambda\left\{G(z) - G(\alpha)\right\}\pi_{00} z}{(\lambda+\nu+\theta)z - \nu - \lambda z G(z)} \tag{12}$$

Similarly, multiplying the Equation (4) by z and each equation of (5) by z^n and summing up these equations, we have

$$\lambda \sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} g_k \pi_{n-k,1} + \theta \sum_{n=1}^{\infty} \pi_{n0} z^n - (\lambda+\mu)\sum_{n=1}^{\infty}\pi_{n1} z^n + \mu \sum_{n=1}^{\infty}\pi_{n+1,1} z^n = 0$$

After calculations, we have

$$\lambda G(z) Q_1(z) + \theta Q_0(z) - (\lambda+\mu) Q_1(z) + \mu\left\{Q_1(z) - \pi_{11} z\right\}/z = 0$$

Therefore, we obtain

$$Q_1(z) = \frac{\left\{\theta Q_0(z) - \mu \pi_{11}\right\} z}{(\lambda+\mu)z - \mu - \lambda z G(z)} \tag{13}$$

Since $\lim_{z \to 1-0} Q_1(z)$ is finite and $\lim_{z\to 1-0}\left\{(\lambda+\mu)z - \mu - \lambda z G(z)\right\} = 0$, we have

$$\theta Q_0(1) - \mu \pi_{11} = 0. \tag{14}$$

Substituting $z = 1$ into (12), we have

$$Q_0(1) = \frac{\lambda\left\{1 - G(\alpha)\right\}\pi_{00}}{\theta} \tag{15}$$

Using (14) and (15), we have

$$Q_1(z)=\frac{\left[\theta Q_0(z)-\lambda\{1-G(\alpha)\}\pi_{00}\right]z}{(\lambda+\mu)z-\mu-\lambda zG(z)} \tag{16}$$

Using (10) and (16), we finally obtain

$$\begin{aligned}L(z)&=\pi_{00}+Q_0(z)+Q_1(z)\\&=\frac{(z-1)\sigma(z)\pi_{00}}{\{(\lambda+\mu)z-\mu-\lambda zG(z)\}\{(\lambda+\nu+\theta)z-\nu-\lambda zG(z)\}}\end{aligned} \tag{17}$$

where

$$\begin{aligned}\sigma(z)=\mu&\{(\lambda+\nu+\theta)z-\nu-\lambda zG(z)\}\\&+\lambda z\{G(z)-G(\alpha)\}(\mu-\nu)\end{aligned}$$

Using the condition that $L(1)=1$ and L'Hospital's rule, we have

$$L(1)=\lim_{z\to 1-0}L(z)=\frac{\left[\mu\theta+\lambda\{1-G(\alpha)\}(\mu-\nu)\right]\pi_{00}}{\theta(\mu-\lambda g)}=1$$

Obviously, the numerator and the denominator of the above expression are both positive since $\nu<\mu$ and $\rho=\lambda g/\mu<1$. Furthermore, we have

$$\pi_{00}=\frac{\mu(1-\rho)}{\delta} \tag{18}$$

where

$$\delta=\mu+\frac{\lambda}{\theta}\{1-G(\alpha)\}(\mu-\nu)$$

Substituting the expressions of π_{00} into (17), we finally obtain

$$\begin{aligned}L(z)&=\frac{\mu(1-\rho)(z-1)}{(\lambda+\mu)z-\mu-\lambda zG(z)}\\&\cdot\frac{\sigma(z)}{\delta(z-1)\{(\lambda+\nu+\theta)z-\nu-\lambda zG(z)\}}\\&=L_0(z)L_d(z)\end{aligned} \tag{19}$$

Since $L_d(1)=1$, $L_d(z)$ is a PGF. Therefore, we completes the proof of Theorem 1.

From Theorem 1, we can obtain two important characteristic quantities in the following two Corollaries.

Corollary 1. *The mean of stationary system length* L *is given by*

$$\begin{aligned}E(L)&=\frac{\lambda\left(g^{(2)}+g\right)}{2\mu(1-\rho)}\\&+\frac{\lambda(\mu-\nu)\left[g\theta+(\lambda g-\nu)\{1-G(\alpha)\}\right]}{\delta\theta^2}\end{aligned}$$

Proof: From (8), we have

$$E(L_0)=\left.\frac{\mathrm{d}L_0(z)}{\mathrm{d}z}\right|_{z=1}=\frac{\lambda\left(g^{(2)}+g\right)}{2\mu(1-\rho)}$$

From (9), we have

$$\begin{aligned}E(L_d)&=\left.\frac{\mathrm{d}L_d(z)}{\mathrm{d}z}\right|_{z=1}\\&=\frac{\lambda(\mu-\nu)\left[g\theta+(\lambda g-\nu)\{1-G(\alpha)\}\right]}{\delta\theta^2}\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&E(L)\\&=E(L_0)+E(L_d)\\&=\frac{\lambda\left(g^{(2)}+g\right)}{2\mu(1-\rho)}+\frac{\lambda(\mu-\nu)\left[g\theta+(\lambda g-\nu)\{1-G(\alpha)\}\right]}{\delta\theta^2}\end{aligned}$$

Corollary 2. *The probability* $P(J=0)$ *that the system is in a working vacation period and the probability* $P(J=1)$ *that the system is in a regular busy period are given by*

$$P(J=0)=\frac{\mu(1-\rho)\left[\theta+\lambda\{1-G(\alpha)\}\right]}{\theta\mu+\lambda\{1-G(\alpha)\}(\mu-\nu)}$$

and

$$P(J=1)=\frac{\lambda\left[g\theta+(\lambda g-\nu)\{1-G(\alpha)\}\right]}{\theta\mu+\lambda\{1-G(\alpha)\}(\mu-\nu)}$$

respectively.

Proof: Using (12) and (18), we have

$$P(J=0)=\pi_{00}+Q_0(1)=\frac{\mu(1-\rho)\left[\theta+\lambda\{1-G(\alpha)\}\right]}{\theta\mu+\lambda\{1-G(\alpha)\}(\mu-\nu)}$$

then we have

$$P(J=1)=1-P(J=0)=\frac{\lambda\left[g\theta+(\lambda g-\nu)\{1-G(\alpha)\}\right]}{\theta\mu+\lambda\{1-G(\alpha)\}(\mu-\nu)}$$

4. Stationary Waiting Time

In this Section, we can obtain the LST of the stationary waiting time of an arbitrary customer. Let W and $W^*(s)$ denote the stationary waiting time of an arbitrary customer and its LST, respectively.

Theorem 2. *If* $\rho=\lambda g/\mu<1$ *and* $\nu<\mu$, $W^*(s)$ *is given by*

$$\begin{aligned}&W^*(s)\\&=\left[\frac{\{\pi_{00}+Q_0(B^*(s))\}\mu\theta}{(\mu-\nu)s+\mu\theta}+Q_1(B^*(s))\right]\frac{\{1-G(B^*(s))\}}{g\{1-B^*(s)\}}\\&+\frac{(\mu-\nu)s\{\pi_{00}+Q_0(C^*(s+\theta))\}\{1-G(C^*(s+\theta))\}}{g\{(\mu-\nu)s+\mu\theta\}\{1-C^*(s+\theta)\}}\end{aligned} \tag{20}$$

Proof: To compute $W^*(s)$, we consider three possible cases as follows.

Case 1: A batch of customers including the tagged customer arrive in the state $(k,1), k\ge 1$. There are k

customers in front of this batch of customers in the system. In this case, a tagged customer's waiting time in this batch is the sum of the service times of k customers outside of his batch and a period of waiting time inside of his batch. Let $r_j\ (j=1,2,\cdots)$ be the probability of the tagged customer being in the jth position of arriving batch. Using the result in renewal theory (Burke [12]), we have

$$r_j = \frac{1}{g}\sum_{n=j}^{\infty} g_n$$

Since all the customers are served at the normal service level, the tagged customer's waiting time conditioned that a batch of customers arrive in the state $(k,1)$, denoted by W_{k1}, has the LST

$$\begin{aligned} W_{k1}^*(s) &= \sum_{j=1}^{\infty} r_j \left\{B^*(s)\right\}^{k+j-1} \\ &= \sum_{j=1}^{\infty}\frac{1}{g}\sum_{n=j}^{\infty} g_n \left\{B^*(s)\right\}^{k+j-1} \\ &= \frac{\left\{B^*(s)\right\}^{k}}{g}\sum_{n=1}^{\infty} g_n \sum_{j=1}^{n}\left\{B^*(s)\right\}^{j-1} \\ &= \frac{\left\{B^*(s)\right\}^{k}}{g}\sum_{n=1}^{\infty} g_n \cdot \frac{1-\left\{B^*(s)\right\}^{n}}{1-B^*(s)} \\ &= \frac{\left\{B^*(s)\right\}^{k}\left\{1-G\left(B^*(s)\right)\right\}}{g\left\{1-B^*(s)\right\}} \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sum_{k=1}^{\infty}\pi_{k1}W_{k1}^*(s) \\ &= \sum_{k=1}^{\infty}\pi_{k1}\frac{\left\{B^*(s)\right\}^{k}\left\{1-G(B^*(s))\right\}}{g\left\{1-B^*(s)\right\}} \\ &= \frac{Q_1\left(B^*(s)\right)\left\{1-G\left(B^*(s)\right)\right\}}{g\left\{1-B^*(s)\right\}} \end{aligned} \tag{21}$$

Case 2: A batch of customers including the tagged customer arrive in the state $(k,0), k \ge 1$. There are k customers in front of this batch of customers in the system. If the tagged customer is the jth position in his batch, the LST of waiting time of the tagged customer is given by

$$\begin{aligned} &\sum_{i=0}^{k+j-2}\frac{\theta}{\theta+\nu}\left(\frac{\nu}{\theta+\nu}\right)^{i}\left(\frac{\theta+\nu}{s+\theta+\nu}\right)^{i+1} B^*(s)^{k+j-i-1} \\ &+\left(\frac{\nu}{\theta+\nu}\right)^{k+j-1}\left(\frac{\theta+\nu}{s+\theta+\nu}\right)^{k+j-1} \\ &= \frac{\theta\mu}{(\mu-\nu)s+\mu\theta}\left(B^*(s)^{k+j-1} - C^*\left(s+\theta\right)^{k+j-1}\right) \\ &+C^*\left(s+\theta\right)^{k+j-1} \end{aligned}$$

Let W_{k0} and $W_{k0}^*(s)$ denote the tagged customer's waiting time conditioned that a batch of customers arrive in the state $(k,0)$ and its LST, respectively. Then we have

$$\begin{aligned} &\sum_{k=1}^{\infty}\pi_{k0}W_{k0}^*(s) \\ &= \sum_{k=1}^{\infty}\pi_{k0}\sum_{j=1}^{\infty} r_j\left\{\frac{\mu\theta}{(\mu-\nu)s+\mu\theta}\left(B^*(s)^{k+j-1} - C^*\left(s+\theta\right)^{k+j-1}\right) + C^*\left(s+\theta\right)^{k+j-1}\right\} \\ &= \sum_{k=1}^{\infty}\pi_{k0}\sum_{j=1}^{\infty}\frac{1}{g}\sum_{n=j}^{\infty} g_n\left\{\frac{\mu\theta}{(\mu-\nu)s+\mu\theta}\left(B^*(s)^{k+j-1} - C^*\left(s+\theta\right)^{k+j-1}\right) + C^*\left(s+\theta\right)^{k+j-1}\right\} \\ &= \frac{1}{g}\sum_{k=1}^{\infty}\pi_{k0}\sum_{n=1}^{\infty} g_n\sum_{j=1}^{n}\left\{\frac{\mu\theta}{(\mu-\nu)s+\mu\theta}(B^*(s)^{k+j-1} - C^*\left(s+\theta\right)^{k+j-1}) + C^*\left(s+\theta\right)^{k+j-1}\right\} \\ &= \frac{1}{g}\sum_{k=1}^{\infty}\pi_{k0}\sum_{n=1}^{\infty} g_n\left\{\frac{\mu\theta}{\left(\mu-\nu\right)s+\mu\theta}\left(\frac{B^*(s)^{k}-B^*(s)^{k+n}}{1-B^*(s)} - \frac{C^*(s+\theta)^{k}-C^*(s+\theta)^{k+n}}{1-C^*(s+\theta)}\right) + \frac{C^*(s+\theta)^{k}-C^*(s+\theta)^{k+n}}{1-C^*(s+\theta)}\right\} \\ &= \frac{\mu\theta}{g\left\{\left(\mu-\nu\right)s+\mu\theta\right\}}\left[\frac{Q_0\left(B^*(s)\right)\left\{1-G(B^*(s))\right\}}{1-B^*(s)} - \frac{Q_0\left(C^*\left(s+\theta\right)\right)\left\{1-G\left(C^*(s+\theta)\right)\right\}}{1-C^*(s+\theta)}\right] \\ &+\frac{Q_0(C^*\left(s+\theta\right))\left\{1-G\left(C^*(s+\theta)\right)\right\}}{g(1-C^*(s+\theta))} \end{aligned} \tag{22}$$

Case 3: A batch of customers including the tagged customer arrive in the state $(0,0)$. The tagged customer's waiting time is equal to the waiting time inside of his batch. Therefore, the tagged customer's waiting time conditioned that a batch of customers arrive in the state $(0,0)$, denoted by W_0, has the LST

$$\begin{aligned}
W_0^*(s) &= \sum_{j=1}^{\infty} r_j \left\{ \sum_{i=0}^{j-2} \frac{\theta}{\theta+\nu}\left(\frac{\nu}{\theta+\nu}\right)^i \left(\frac{\theta+\nu}{s+\theta+\nu}\right)^{i+1} B^*(s)^{j-i-1} \right. \\
&\quad \left. + \left(\frac{\nu}{\theta+\nu}\right)^{j-1}\left(\frac{\theta+\nu}{s+\theta+\nu}\right)^{j-1} \right\} \\
&= \sum_{j=1}^{\infty} \frac{1}{g} \sum_{n=j}^{\infty} g_n \left\{ \frac{\mu\theta}{(\mu-\nu)s+\mu\theta}\left(B^*(s)^{j-1} - C^*(s+\theta)^{j-1}\right) \right. \\
&\quad \left. + C^*(s+\theta)^{j-1} \right\} \\
&= \frac{1}{g}\sum_{n=1}^{\infty} g_n \sum_{j=1}^{n} \left\{ \frac{\mu\theta}{(\mu-\nu)s+\mu\theta}\left(B^*(s)^{j-1} - C^*(s+\theta)^{j-1}\right) \right. \\
&\quad \left. + C^*(s+\theta)^{j-1} \right\} \\
&= \frac{1}{g}\sum_{n=1}^{\infty} g_n \left\{ \frac{\mu\theta}{(\mu-\nu)s+\mu\theta}\left(\frac{1-B^*(s)^n}{1-B^*(s)} - \frac{1-C^*(s+\theta)^n}{1-C^*(s+\theta)}\right) \right. \\
&\quad \left. + \frac{1-C^*(s+\theta)^n}{1-C^*(s+\theta)} \right\} \\
&= \frac{\mu\theta}{g\{(\mu-\nu)s+\mu\theta\}}\left\{ \frac{1-G(B^*(s))}{1-B^*(s)} - \frac{1-G(C^*(s+\theta))}{1-C^*(s+\theta)} \right\} \\
&\quad + \frac{1-G(C^*(s+\theta))}{g(1-C^*(s+\theta))}. \qquad (23)
\end{aligned}$$

From (21), (22) and (23), we finally obtain

$$\begin{aligned}
W^*(s) &= \pi_{00} W_0^*(s) + \sum_{k=1}^{\infty} \pi_{k0} W_{k0}^*(s) + \sum_{k=1}^{\infty} \pi_{k1} W_{k1}^*(s) \\
&= \frac{\pi_{00}\mu\theta}{g\{(\mu-\nu)s+\mu\theta\}}\left\{ \frac{1-G(B^*(s))}{1-B^*(s)} - \frac{1-G(C^*(s+\theta))}{1-C^*(s+\theta)} \right\} \\
&\quad + \frac{\pi_{00}\{1-G(C^*(s+\theta))\}}{g\{1-C^*(s+\theta)\}} \\
&\quad + \frac{\mu\theta}{g\{(\mu-\nu)s+\mu\theta\}}\left[\frac{Q_0(B^*(s))\{1-G(B^*(s))\}}{1-B^*(s)} \right. \\
&\quad \left. - \frac{Q_0(C^*(s+\theta))\{1-G(C^*(s+\theta))\}}{1-C^*(s+\theta)} \right] \\
&\quad + \frac{Q_0\left(C^*(s+\theta)\right)\{1-G(C^*(s+\theta))\}}{g\{1-C^*(s+\theta))\}} \\
&\quad + \frac{Q_1(B^*(s))\{1-G(B^*(s))\}}{g\{1-B^*(s))\}} \\
&= \left[\frac{\{\pi_{00}+Q_0(B^*(s))\}\mu\theta}{(\mu-\nu)s+\mu\theta} + Q_1(B^*(s)) \right] \frac{\{1-G(B^*(s))\}}{g\{1-B^*(s)\}} \\
&\quad + \frac{(\mu-\nu)s\{\pi_{00}+Q_0(C^*(s+\theta))\}\{1-G(C^*(s+\theta))\}}{g\{(\mu-\nu)s+\mu\theta\}\{1-C^*(s+\theta)\}}
\end{aligned}$$

Remark 2. We can obtain the mean waiting time of an arbitrary customer, $E(W) = -\left.\frac{\mathrm{d}W^*(s)}{\mathrm{d}s}\right|_{s=0}$, by differentiating (20) and substituting $s=0$. On the other hand, the mean waiting time of an arbitrary customer, $E(W)$, can also be obtained by Little's formula, that is,

$$E(L) - (1-\pi_{00}) = \lambda g E(W)$$

However, in order to obtain the higher moments of the waiting time of an arbitrary customer, $E(W^n)$ $(n \geq 2)$, we must differentiate (20) n times and substitute $s=0$.

Remark 3. If $\nu = 0$, our model reduces to a batch arrival M^X/M/1 queue with multiple vacation. If the batch size of arrivals is always equal to 1, that is, $g_1 = P(X=1) = 1$, our model reduces to an M/M/1 queue with multiple working vacation studied in Servi and Finn [4] and Liu *et al.* [8]. These results correspond to the known results in existing literature.

5. Numerical Results

In Section 3 and Section 4, we obtain the mean system length and the mean waiting time of an arbitrary customer. In this section, we assume that the arrival batch size X follows a geometric distribution with parameter q, that is, $P(X=k) = (1-q)^{k-1}q$ $(0<q<1; k=1,2,\cdots)$. Then it is easy to verify that

$$g = \frac{1}{q}, \quad g^{(2)} = \frac{2-q}{q^2}, \quad G(z) = \frac{qz}{1-(1-q)z} \quad (|z| \leq 1).$$

First, we consider an M^X/M/1 queue with multiple working vacation where the system parameters are $\lambda = 0.3, \mu = 2, \theta = 1$. In **Figure 1**, we demonstrate the effect of the service rate in the working vacation period ν on the mean system length $E(L)$ for different g. **Figure 1** indicates that $E(L)$ decreases as ν increases when g is equal to 2, 3 and 4, respectively. On the other hand, if ν is fixed, $E(L)$ increases as g increases, that is, $\rho = \lambda g/\mu$ increases.

Secondly, we assume that $\lambda = 0.3, g = 3, \mu = 2$. In **Figure 2**, we demonstrate the effect of ν on $E(L)$ for different vacation rate θ. **Figure 2** indicates that $E(L)$ decreases as ν increases when θ is equal to 0.5, 1.0 and 1.5, respectively. On the other hand, if ν is fixed, $E(L)$ decreases as θ increases.

Thirdly, in **Figure 3**, we present the comparison of three queueing model, that is, the M^X/M/1 queue without vacation, the M^X/M/1 queue with multiple vacation and the M^X/M/1 queue with multiple working vacation. Assume that $g = 2, \mu = 2, \nu = 1, \theta = 1$ for the M^X/M/1 queue with multiple working vacation. **Figure 3** indicates that $E(L)$ and $E(W)$ increase as ρ increases. On the other hand, $E(L)$ and $E(W)$ of M^X/M/1 queue

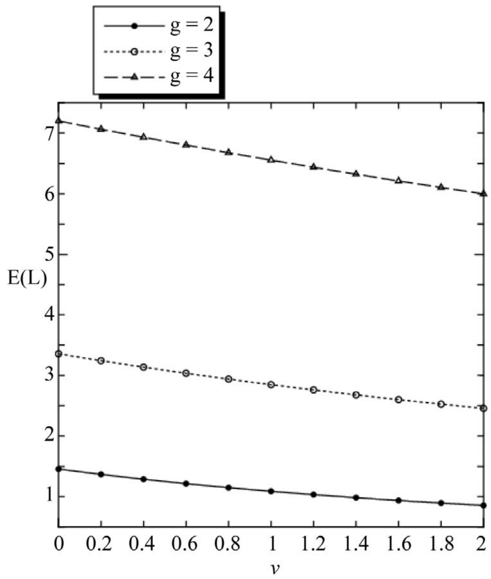

Figure 1. *E*(*L*) versus *v* for different *g*.

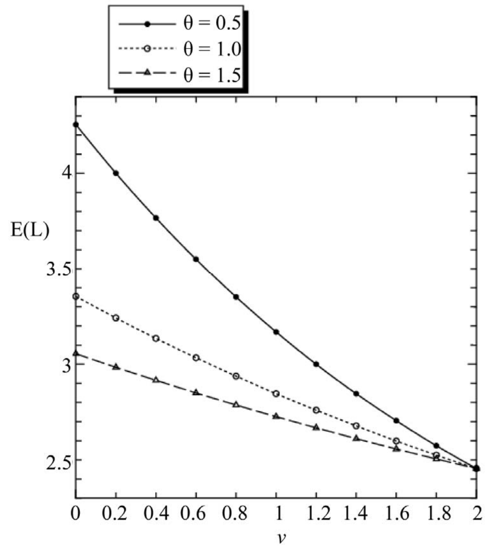

Figure 2. *E*(*L*) versus *v* for different *θ*.

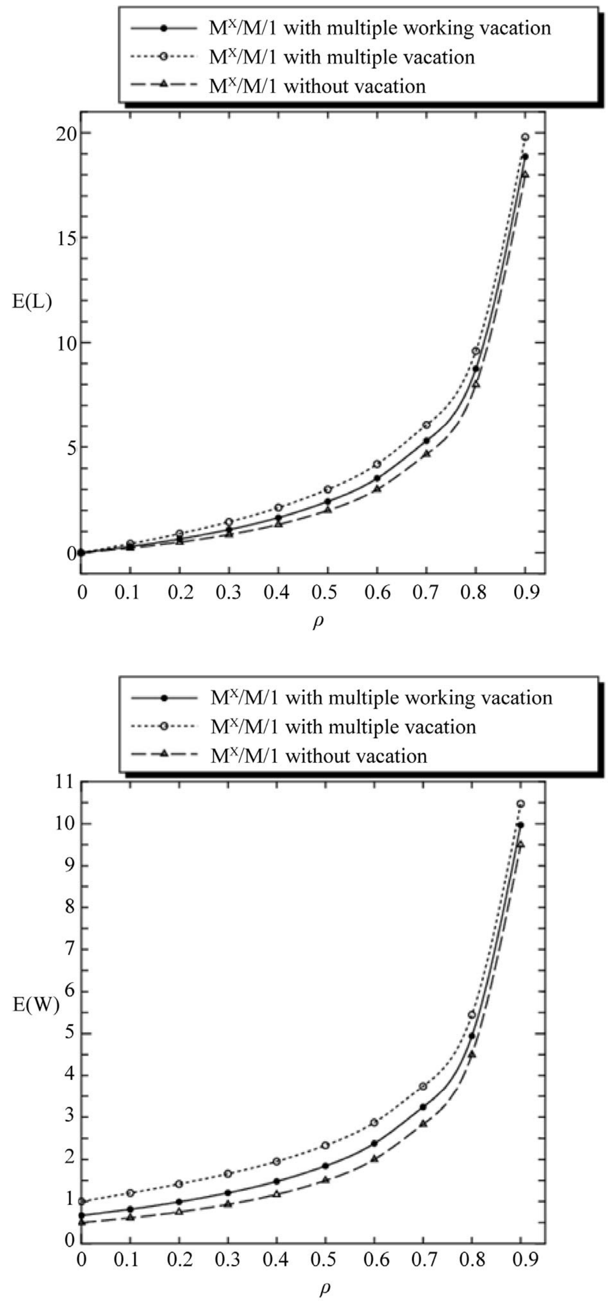

Figure 3. *E*(*L*) and *E*(*W*) versus *ρ* for different queueing model.

without vacation are shortest and those of the M^X/M/1 queue with multiple vacation are longest, which is identical with the intuition. Furthermore, **Figure 3** indicates that $\lim_{\rho\to+0}E(L)=0$ and $\lim_{\rho\to+0}E(W)>0$, which is a well known result for batch arrival queues.

6. Conclusion

This paper studied the M^X/M/1queue with multiple working vacation. We obtained the PGF of the stationary system length distribution and the stochastic decomposition structure of system length which indicates the relationship with that of the M^X/M/1 queue without vacation. Performance indices such as the mean of stationary system length, the probability that the system is in a working vacation period and the probability that the system is in a regular busy period were also presented. Further, we obtained the LST of the stationary waiting time distribution of anarbitrary customer.We obtained the mean system length and the mean waiting time. Some numerical results for special cases showed efficiency of service in this multi-purpose batch arrival model.

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REFERENCES

[1] H. Takagi, "Queueing Analysis: A Foundation of Performance Evaluation, Vol. 1: Vacation and Priority Systems, Part 1," Elsevier Science Publishers, Amsterdam, 1991.

[2] N. Tianand and G. Zhang, "Vacation Queueing Models-Theory and Applications," Springer-Verlag, New York, 2006.

[3] B. Doshi, "Queueing Systems with Vacations—A Survey," *Queueing Systems*, Vol. 1, No. 1, 1986, pp. 29-66. doi:10.1007/BF01149327

[4] L. Servi and S. Finn, "M/M/1 Queue with Working Vacations (M/M/1/WV)," *Performance Evaluation*, Vol. 50, No. 1, 2002, pp. 41-52. doi:10.1016/S0166-5316(02)00057-3

[5] D. Wuand and H. Takagi, "M/G/1 Queue with Multiple Working Vacations," *Performance Evaluation*, Vol. 64, 2006, pp. 654-681.

[6] Y. Baba, "Analysis of a GI/M/1 Queue with Multiple Working Vacations," *Operations Research Letters*, Vol. 33, No. 2, 2005, pp. 201-209. doi:10.1016/j.orl.2004.05.006

[7] A. Banik, U. Gupta and S. Pathak, "On the GI/M/1/N Queue with Multiple Working Vacations-Analytic Analysis and Computation," *Applied Mathematical Modelling*, Vol. 31, No. 9, 2007, pp. 1701-1710. doi:10.1016/j.apm.2006.05.010

[8] W. Liu, X. Xu and N. Tian, "Some Results on the M/M/1 Queue with Working Vacations," *Operations Research Letters*, Vol. 35, No. 5, 2007, pp. 595-600. doi:10.1016/j.orl.2006.12.007

[9] J. Li, N. Tian, Z. G. Zhang and H. P. Lu, "Analysis of the M/G/1 Queue with Exponential Working Vacations-A Matrix Analytic Approach," *Queueing Systems*, Vol. 61, No. 2-3, 2009, pp. 139-166. doi:10.1007/s11134-008-9103-8

[10] X. Xu, Z. Zhang and N. Tian, "Analysis for the $M^{[X]}$/M/1 Working Vacation Queue," *International Journal of Information and Management Sciences*, Vol. 20, 2009, pp. 379-394.

[11] M. F. Neuts, "Structured Stochastic Matrices of M/G/1 Type and Their Applications," MarcelDekker Inc., New York, 1989.

[12] P. J. Burke, "Delay in Single-Server Queues with Batch Arrivals," *Operations Research*, Vol. 23, No. 4, 1975, pp. 830-833. doi:10.1287/opre.23.4.830

