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$H(\cdot,\cdot)$-ϕ-η-Accretive Operators and Generalized Variational-Like Inclusions*

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Abstract

In this paper, we generalize $H(\cdot,\cdot)$-accretive operator introduced by Zou and Huang [1] and we call it $H(\cdot,\cdot)$-ϕ-η-accretive operator. We define the resolvent operator associated with $H(\cdot,\cdot)$-ϕ-η-accretive operator and prove its Lipschitz continuity. By using these concepts an iterative algorithm is suggested to solve a generalized variational-like inclusion problem. Some examples are given to justify the definition of $H(\cdot,\cdot)$-ϕ-η-accretive operator.

Keywords: $H(\cdot,\cdot)$-ϕ-η-Accretive Operator, Variational-Like Inclusion, Resolvent Operator, Algorithm, Convergence

1. Introduction

Variational inclusion problems have emerged as a powerful tool for solving a wide class of unrelated problems occuring in various branches of physical, engineering, pure and applied sciences in a unified and general frame work.

In 2001, Huang and Fang [2] firstly introduced the generalized m-accretive mappings and gave the definition of resolvent operator for the generalized m-accretive mappings in Banach spaces. Also, they have shown some properties of their resolvent operator. Since then, Fang and Huang, Lan, Cho and Verma and others introduced and studied several generalized operators such as *H*-accretive, $(H\text{-}\eta)$-accretive and (A,η)-accretive mappings. For example, see [3-16] and references therein.

In 2008, Zou and Huang [1] introduced $H(\cdot,\cdot)$-accretive operator, its resolvent operator and applied them to solve a variational inclusion problem in Banach spaces. In this paper, we generalized $H(\cdot,\cdot)$-accretive operator to $H(\cdot,\cdot)$-ϕ-η-accretive operator and define its resolvent operator. Further, we prove the Lipschitz continuity of resolvent operator and apply these new concepts to solve a variational-like inclusion problem. Some example are constructed.

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2. Preliminaries

let *X* be a real Banach spaces with its dual X^*, $\langle\cdot,\cdot\rangle$ be the duality pairing between *X* and X^* and 2^X (respectively $CB(X)$) denote the family of non-empty subsets (respectively, closed and bounded subsets) of *X*. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, $J_q(x) = \|x\|^{q-1} J_2(x)$ for $x \neq 0$ and J_q is single-valued if X^* is strictly convex. If *X* is a real Hilbert space, then J_2 becomes the identity mapping on *X*.

The modulus of smoothness of *X* is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| \le 1, \|y\| \le t \right\}.$$

A Banach space *X* is called uniformly smooth, if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth, if there exists a constant $C > 0$ such that

$$\rho_X(t) \le Ct^q, \quad q > 1.$$

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Note that J_q is single-valued if X is uniformly smooth. The following inequality in q-uniformly smooth Banach spaces has been proved by Xu [17].

Lemma 2.1. Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that for all $x, y \in X$,

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q .$$

Definition 2.1. Let $A, B : X \to X$ and $\eta, H : X \times X \to X$ be the single-valued mappings.

i) A is said to be η-accretive, if

$$\langle Ax - Ay, J_q(\eta(x,y))\rangle \ge 0, \quad \forall x, y \in X ;$$

ii) A is said to be strictly η-accretive, if A is η-accretive and equality holds if and only if $x = y$;

iii) $H(A,\cdot)$ is said to be α-strongly η-accretive with respect to A, if there exists a constant $\alpha > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), J_q(\eta(x,y))\rangle \ge \alpha\|x-y\|^q, \quad \forall x, y, u \in X;$$

iv) $H(\cdot,B)$ is said to be β-relaxed η-accretive with respect to B, if there exists a constant $\beta > 0$ such that

$$\langle H(u,Bx) - H(u,By), J_q(\eta(x,y))\rangle \ge (-\beta)\|x-y\|^q, \quad \forall x, y, u \in X;$$

v) $H(\cdot,\cdot)$ is said to r_1-Lipschitz continuous with respect to A, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \le r_1\|x-y\|, \quad \forall x, y, u \in X .$$

In a similar way, we can define the Lipschitz continuity of the mapping $H(\cdot,\cdot)$ with respect to B.

vi) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \le \tau\|x-y\|, \quad \forall x, y \in X .$$

Definition 2.2. Let $N, \eta : X \times X \to X$ be the single-valued mappings. Let $M : X \times X \to 2^X$ be multi-valued mapping.

i) M is said to be η-accretive, if $\langle u - v, J_q(\eta(x,y))\rangle \ge 0$, $\forall\, x, y \in X$, $u \in M(x,z)$, $v \in M(y,z)$, for each fixed $z \in X$;

ii) M is said to be strictly η-accretive, if M is η-accretive and equality holds if and only if $x = y$;

iii) N is said to t-relaxed η-accretive in the first argument, if there exists a constant $t > 0$ such that

$$\langle N(x,u) - N(y,u), J_q(\eta(x,y))\rangle \ge -t\|x-y\|^q, \quad \forall\, x, y, u \in X;$$

iv) N is said to be ξ-Lipschitz continuous in the first argument, if there exists a constant $\xi > 0$ such that

$$\|N(x,u) - N(y,u)\| \le \xi\|x-y\|, \quad \forall x, y, u \in X .$$

Similarly, we can define the Lipschitz continuity of N in the second argument.

3. $H(\cdot,\cdot)$-ϕ-η-Accretive Operator

In this section, we generalize $H(\cdot,\cdot)$-accretive operator [1] and call it $H(\cdot,\cdot)$-ϕ-η-accretive operator and discuss some of its properties.

Definition 3.1. Let $\phi, A, B : X \to X$, $H, \eta : X \times X \to X$ be the single-valued mappings. Let $M : X \times X \to 2^X$ be a multi-valued mapping. M is said to be $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B, if for each fixed $z \in X$, $\phi \circ M(\cdot,z)$ is η-accretive in the first argument and $(H(A,B) + \phi \circ M(\cdot,z))(X) = X$.

Remark 3.1. If $\phi(x) = \lambda x$, $\forall x \in X$ and $\forall \lambda > 0$, $M(\cdot,\cdot) = M(\cdot)$ and $\eta(x,y) = x - y$ then $H(\cdot,\cdot)$-ϕ-η-accretive operator reduces to $H(\cdot,\cdot)$-accretive operator, which was introduced and studied by Zou and Huang [1].

Example 3.1. Let $X = \mathbb{R}$. Let $Ax = 0$, $Bx = \sin x$, $H(Ax, By) = Ax + By$ and $M(x,z) = x^2 + z^2$, $\forall x \in X$ and for each fixed $z \in X$. Let

$$\phi \circ M(x,z) = \frac{\partial}{\partial x}[M(x,z)] = 2x \quad \text{and} \quad \eta(x,y) = \frac{x-y}{2} .$$

Then

$$\langle \phi \circ M(x,z) - \phi \circ M(y,z), \eta(x,y)\rangle = \left[2x - 2y, \frac{x-y}{2}\right] = (x-y)^2 \ge 0,$$

which means that $\phi \circ M(\cdot,z)$ is η-accretive in the first argument. Also, for any $x \in X$, it follows from above that

$$(H(A,B) + \phi \circ M(\cdot,z))(x) = H(Ax,Bx) + \phi \circ M(x,z) = 0 + \sin x + 2x = 2x + \sin x,$$

which means that $(H(A,B) + \phi \circ M(\cdot,z))$ is surjective. Thus M is $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B.

Example 3.2. Let X, A, B, H, η and M are same as in Example 3.1. Let $\phi \circ M(x,z) = e^{x^2+z^2}$. Then

$$(H(A,B) + \phi \circ M(\cdot,z))(x) = H(Ax,Bx) + \phi \circ M(x,z) = \sin x + e^{x^2+z^2},$$

which shows that $0 \notin (H(A,B) + \phi \circ M(\cdot,z))(X)$, that is $(H(A,B) + \phi \circ M(\cdot,z))$ is not surjective, hence M is not $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings *A* and *B*.

Theorem 3.1. Let $H(A,B)$ be α-strongly η-accretive with respect to *A*, β-relaxed η-accretive with respect to *B*, $\alpha > \beta$. Let *M* be an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings *A* and *B*. Then the operator $\left(H(A,B)+\phi\circ M(\cdot,z)\right)^{-1}$ is single-valued for each fixed $z\in X$.

Proof. For any given *u* and $z\in X$, let $x,y\in\left(H(A,B)+\phi\circ M(\cdot,z)\right)^{-1}(u)$. Then

$$-H(Ax,Bx)+u\in\phi\circ M(x,z),$$
$$-H(Ay,By)+u\in\phi\circ M(y,z).$$

Since $\phi\circ M(\cdot,z)$ is η-accretive in the first argument, we have

$$\begin{aligned}0&\le\left\langle -H(Ax,Bx)+u-\left(-H(Ay,By)+u\right),J_q\left(\eta(x,y)\right)\right\rangle\\&=-\left\langle H(Ax,Bx)-H(Ay,By),J_q\left(\eta(x,y)\right)\right\rangle\\&=-\left\langle H(Ax,Bx)-H(Ay,Bx)+H(Ay,Bx)\right.\\&\quad\left.-H(Ay,By),J_q\left(\eta(x,y)\right)\right\rangle\\&=-\left\langle H(Ax,Bx)-H(Ay,Bx),J_q\left(\eta(x,y)\right)\right\rangle\\&\quad-\left\langle H(Ay,Bx)-H(Ay,By),J_q\left(\eta(x,y)\right)\right\rangle\\&\le-\alpha\|x-y\|^q+\beta\|x-y\|^q=-(\alpha-\beta)\|x-y\|^q\le 0.\end{aligned}$$

Since $\alpha>\beta$, we have $x=y$ and so $\left(H(A,B)+\phi\circ M(\cdot,z)\right)^{-1}$ is single-valued. This completes the proof.

Definition 3.2. Let $H(A,B)$ be α-strongly η-accretive with respect to *A* and β-relaxed η-accretive with respect to *B* and $\alpha>\beta$. Let *M* be an $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings *A* and *B*. Then for each fixed $z\in X$, the resolvent operator $R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}:X\to X$ is defined by

$$R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)=\left(H(A,B)+\phi\circ M(\cdot,z)\right)^{-1}(u),\quad\forall u\in X.$$

Theorem 3.2. Let $H(A,B)$ be α-strongly η-accretive with respect to *A*, β-relaxed η-accretive with respect to *B*, $\alpha>\beta$ and η is τ-Lipschitz continuous. Let $M:X\times X\to 2^X$ is a $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings *A* and *B*. Then the resolvent operator $R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}:X\to X$ is $\frac{\tau^{q-1}}{\alpha-\beta}$-Lipschitz continuous *i.e.*,

$$\left\|R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)-RR_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)\right\|\le\frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|,$$
$$\forall u,v\in X\ \text{and each fixed}\ z\in X.$$

Proof. Let $u,v\in X$, then by definition of resolvent operator, it follows that

$$R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)=\left(H(A,B)+\phi\circ M(\cdot,z)\right)^{-1}(u),$$

and

$$R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)=\left(H(A,B)+\phi\circ M(\cdot,z)\right)^{-1}(v).$$

Then

$$\begin{aligned}&u-H\left(A\left(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)\right),B\left(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)\right)\right)\\&\in\phi\circ M\left(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u),z\right),\end{aligned}$$

and

$$\begin{aligned}&v-H\left(A\left(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)\right),B\left(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)\right)\right)\\&\in\phi\circ M\left(R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v),z\right).\end{aligned}$$

Let $Pu=R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)$, $Pv=R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)$

Since $\phi\circ M(\cdot,z)$ is η-accretive in the first argument, we have

$$\begin{aligned}&\left\langle u-H\left(A(Pu),B(Pu)\right)-\left(v-H\left(A(Pv),B(Pv)\right)\right),\right.\\&\left.J_q\left(\eta(Pu,Pv)\right)\right\rangle\ge 0\\&\left\langle u-v,J_q\left(\eta(Pu,Pv)\right)\right\rangle\ge\left\langle H\left(A(Pu),B(Pu)\right)\right.\\&\qquad\left.-H\left(A(Pv),B(Pv)\right),J_q\left(\eta(Pu,Pv)\right)\right\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|u-v\|\|\eta(Pu,Pv)\|^{q-1}&\ge\left\langle u-v,J_q\left(\eta(Pu,Pv)\right)\right\rangle\\&\ge\left\langle H\left(A(Pu),B(Pu)\right)-H\left(A(Pv),B(Pv)\right),\right.\\&\quad\left.J_q\left(\eta(Pu,Pv)\right)\right\rangle\\&=\left\langle H\left(A(Pu),B(Pu)\right)-H\left(A(Pv),B(Pu)\right),\right.\\&\quad\left.J_q\left(\eta(Pu,Pv)\right)\right\rangle\\&\quad+\left\langle H\left(A(Pv),B(Pu)\right)-H\left(A(Pv),B(Pv)\right),\right.\\&\quad\left.J_q\left(\eta(Pu,Pv)\right)\right\rangle\\&\ge\alpha\|Pu-Pv\|^q-\beta\|Pu-Pv\|^q\ge(\alpha-\beta)\|Pu-Pv\|^q\end{aligned}$$

$$\|u-v\|\tau^{q-1}\|Pu-Pv\|^{q-1}\ge(\alpha-\beta)\|Pu-Pv\|^q$$

$$\|Pu-Pv\|\le\frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|$$

i.e. $\left\|R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(u)-R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}(v)\right\|\le\frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|.$

This completes the proof.

4. An Application for Solving Generalized Variational-Like Inclusions

In this section, we apply $H(\cdot,\cdot)$-ϕ-η-accretive operator for solving generalized variational-like inclusions.

Let $S,T,G: X \to CB(X)$ be the multi-valued mappings. $A,B,\phi: X \to X$, $H,N,\eta: X \times X \to X$ be single-valued mappings. Suppose $M: X \times X \to 2^X$ be a multi-valued mapping such that M is $H(\cdot,\cdot)$-ϕ-η-accretive operator.

We consider the following problem of finding $x \in X$, $u \in S(x)$, $v \in T(x)$ and $z \in G(x)$ and

$$0 \in N(u,v) + M(x,z). \tag{4.1}$$

Problem (4.1) is called generalized variational-like inclusion problem.

Below are some special cases of our problem:

i) If X is real Hilbert space and $M(\cdot,z)$ is maximal monotone operator then a problem similar to (4.1) was introduced and studied by Huang [18].

ii) If $T \equiv G \equiv 0$, S is single-valued and identity mapping and $N(\cdot,\cdot) = N(\cdot)$ and $M(\cdot,\cdot) = M(\cdot)$ then our problem reduces to the problem considered by Bi e*t al*. [19], that is find $u \in X$ such that

$$0 \in N(u) + M(u).$$

It is clear that for suitable choices of operators involved in the formulation of problem (4.1), one can obtain many variational-like inclusions studied in recent past.

Lemma 4.1. Let X be a q-uniformly smooth Banach space. $G,S,T: X \to CB(X)$ be multi-valued mappings, $A,B: X \to X$ be single-valued mappings and $\phi: X \to X$ be a mapping satisfying $\phi(x+y) = \phi(x) + \phi(y)$ and $\ker(\phi) = \{0\}$, where $\ker(\phi) = \{x \in X : \phi(x) = 0\}$. Let $H,N,\eta: X \times X \to X$ be the single-valued mappings. Let $M: X \times X \to 2^X$ be a multi-valued mapping such that M is $H(\cdot,\cdot)$-ϕ-η-accretive operator. Then (x,u,v,z) where $x \in X$, $u \in S(x)$, $v \in T(x)$ and $z \in G(x)$ is a solution of problem (4.1) if and only if (x,u,v,z) satisfies

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax,Bx) - \phi \circ N(u,v)\right]. \tag{4.2}$$

Proof. Let (x,u,v,z) where $x \in X$, $u \in S(x)$, $v \in T(x)$ and $z \in G(x)$ satisfies the Equation (4.2), *i.e.*,

$$x = R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax,Bx) - \phi \circ N(u,v)\right].$$

Using the definition of resolvent operator, we have

$$x = \left(H(A,B) + \phi \circ M(\cdot,z)\right)^{-1}\left[H(Ax,Bx) - \phi \circ N(u,v)\right]$$
$$\Leftrightarrow H(Ax,Bx) - \phi \circ N(u,v) \in H(Ax,Bx) + \phi \circ M(x,z)$$
$$\Leftrightarrow 0 \in \phi \circ N(u,v) + \phi \circ M(x,z)$$
$$\Leftrightarrow 0 \in \phi\left(N(u,v) + M(x,z)\right)$$
$$\Leftrightarrow \phi^{-1}(0) \in N(u,v) + M(u,z) \Leftrightarrow 0 \in N(u,v) + M(x,z).$$

This completes the proof.

Based on Lemma 4.1, we define the following algorithm.

Algorithm 4.1. Let G, S, T, A, B, H, N, ϕ, η and M all are same as in Lemma 4.1. For any given $x_0 \in X$, $u_0 \in S(x_0)$, $v_0 \in T(x_0)$ and $z_0 \in G(x_0)$ and $0 < \varepsilon < 1$, compute the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ by the following iterative scheme:

$$x_{n+1} = R_{M(\cdot,z_n)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_n,Bx_n) - \phi \circ N(u_n,v_n)\right]; \tag{4.3}$$

$$u_n \in S(x_n),\quad \|u_n - u_{n+1}\| \le \mathcal{D}\left(S(x_n), S(x_{n+1})\right) + \epsilon^{n+1}\|x_n - x_{n+1}\|; \tag{4.4}$$

$$v_n \in T(x_n),\quad \|v_n - v_{n+1}\| \le \mathcal{D}\left(T(x_n), T(x_{n+1})\right) + \epsilon^{n+1}\|x_n - x_{n+1}\|; \tag{4.5}$$

$$z_n \in G(x_n),\quad \|z_n - z_{n+1}\| \le \mathcal{D}\left(G(x_n), G(x_{n+1})\right) + \epsilon^{n+1}\|x_n - x_{n+1}\|; \tag{4.6}$$

$n = 0,1,2,\cdots$, where $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$.

Theorem 4.1. Let X be q-uniformly smooth Banach space and $A,B,\phi: X \to X$ be the single-valued mappings. Let $H,N,\eta: X \times X \to X$ be the single-valued mappings and $G,S,T: X \to CB(X)$ be multi-valued mappings. Suppose $M: X \times X \to 2^X$ be a multi-valued mapping such that M is $H(\cdot,\cdot)$-ϕ-η-accretive operator with respect to mappings A and B. Assume that

i) $\phi(x+y) = \phi(x) + \phi(y)$ and $\ker(\phi) = \{0\}$;

ii) $H(A,B)$ is α-strongly η-accretive with respect to A and β-relaxed η-accretive with respect to B;

iii) $H(\cdot,\cdot)$ is r_1-Lipschitz continuous in the first argument and r_2-Lipschitz continuous in the second argument;

iv) $\phi \circ N(\cdot,\cdot)$ is ξ_1-Lipschitz continuous in the first argument and ξ_2-Lipschitz continuous in the second argument;

v) $\phi \circ N(\cdot,\cdot)$ is t-relaxed η-accretive in the first argument;

vi) η is τ-Lipschitz continuous;

vii) S, T and G are $\mathcal{D}$-Lipschitz continuous with constant λ_S, λ_T and λ_G respectively;

viii) $\left\|R_{M(\cdot,z_n)}^{H(\cdot,\cdot)-\varphi-\eta}(x) - R_{M(\cdot,z_{n-1})}^{H(\cdot,\cdot)-\varphi-\eta}(x)\right\| \le \lambda\|z_n - z_{n-1}\|$, $\lambda > 0$, $\forall z_n, z_{n-1} \in X$;

ix) $\sqrt[q]{(r_1+r_2)^q + qt\lambda_S^q + q\xi_1\lambda_S\left[(r_1+r_2)^{q-1}+\tau^{q-1}\lambda_S^{q-1}\right]+C_q\xi_1^q\lambda_S^q} < \frac{\alpha-\beta(1-\lambda\lambda_G)}{\tau^{q-1}} - \xi_2\lambda_T$.

Then (x,u,v,z) where $x\in X$, $u\in S(x)$, $v\in T(x)$ and $z\in G(x)$ is a solution of problem (4.1), and the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ defined in Algorithm 3.1 converge strongly to x, u, v and z, respectively in X.

Proof. Using Algorithm 4.1, Lipschitz continuity of resolvent operator and condition (viii), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \Big\|R_{M(\cdot,z_n)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_n,Bx_n)-\phi\circ N(u_n,v_n)\right] \\ &\quad - R_{M(\cdot,z_{n-1})}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_{n-1},Bx_{n-1})-\phi\circ N(u_{n-1},v_{n-1})\right]\Big\| \\ &= \Big\|R_{M(\cdot,z_n)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_n,Bx_n)-\phi\circ N(u_n,v_n)\right] \\ &\quad - R_{M(\cdot,z_n)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_{n-1},Bx_{n-1})-\phi\circ N(u_{n-1},v_{n-1})\right]\Big\| \\ &\quad + \Big\|R_{M(\cdot,z_n)}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_{n-1},Bx_{n-1})-\phi\circ N(u_{n-1},v_{n-1})\right] \\ &\quad - R_{M(\cdot,z_{n-1})}^{H(\cdot,\cdot)-\phi-\eta}\left[H(Ax_{n-1},Bx_{n-1})-\phi\circ N(u_{n-1},v_{n-1})\right]\Big\| \\ &\le \frac{\tau^{q-1}}{\alpha-\beta}\Big\|H(Ax_n,Bx_n)-\phi\circ N(u_n,v_n) \\ &\quad -\left(H(Ax_{n-1},Bx_{n-1})-\phi\circ N(u_{n-1},v_{n-1})\right)\Big\| + \lambda\|z_n-z_{n-1}\| \\ &= \frac{\tau^{q-1}}{\alpha-\beta}\Big\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1}) \\ &\quad -\left(\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n)\right)\Big\| \\ &\quad + \frac{\tau^{q-1}}{\alpha-\beta}\left\|\phi\circ N(u_{n-1},v_n)-\phi\circ N(u_{n-1},v_{n-1})\right\| \\ &\quad + \lambda\|z_n-z_{n-1}\|. \end{aligned} \tag{4.7}$$

Now, we estimate

$$\begin{aligned}&\Big\|H((Ax_n,Bx_n))-H(Ax_{n-1},Bx_{n-1}) \\ &\quad -\left(\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n)\right)\Big\|^q \\ &\le \left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\|^q - q\big\langle \phi\circ N(u_n,v_n) \\ &\quad -\phi\circ N(u_{n-1},v_n), J_q(\eta(u_n,u_{n-1}))\big\rangle - q\big\langle\phi\circ N(u_n,v_n) \\ &\quad -\phi\circ N(u_{n-1},v_n), J_q\left[H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right] \\ &\quad -J_q(\eta(u_n,u_{n-1}))\big\rangle + C_q\left\|\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n)\right\|^q \\ &\le \left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\|^q - q\big\langle\phi\circ N(u_n,v_n) \\ &\quad -\phi\circ N(u_{n-1},v_n), J_q(\eta(u_n,u_{n-1}))\big\rangle + q\big\|\phi\circ N(u_n,v_n) \\ &\quad -\phi\circ N(u_{n-1},v_n)\big\|\times\left[\left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\|^{q-1}\right. \\ &\quad \left.+\left\|\eta(u_n,u_{n-1})\right\|^{q-1}\right] + C_q\left\|\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n)\right\|^q. \end{aligned} \tag{4.8}$$

Since $H(A,B)$ is r_1-Lipschitz continuous in the first argument and r_2-Lipschitz continuous in the second argument, we have

$$\begin{aligned}&\left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\| \\ &= \left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_n)\right\| \\ &\quad + \left\|H(Ax_{n-1},Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\| \\ &\le r_1\|x_n-x_{n-1}\| + r_2\|x_n-x_{n-1}\| = (r_1+r)_2\|x_n-x_{n-1}\| \end{aligned}$$

and hence

$$\left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\|^q \le (r_1+r_2)^q\|x_n-x_{n-1}\|^q. \tag{4.9}$$

Since S is $\mathcal{D}$-Lipschitz continuous with constant λ_S and using (4.4), we have

$$\begin{aligned}\|u_n-u_{n-1}\| &\le \mathcal{D}(S(x_n),S(x_{n-1})) + \epsilon^n\|x_n-x_{n-1}\| \\ &\le \lambda_S\|x_n-x_{n-1}\| + \varepsilon^n\|x_n-x_{n-1}\| \le (\lambda_S+\varepsilon^n)\|x_n-x_{n-1}\|. \end{aligned} \tag{4.10}$$

Since, $\phi\circ N(\cdot,\cdot)$ is t-relaxed η-accretive in the first argument and using (4.10), we have

$$\begin{aligned}&\left\langle\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n), J_q(\eta(u_n,u_{n-1}))\right\rangle \\ &\ge -t\|u_n-u_{n-1}\|^q \ge -t(\lambda_S+\varepsilon^n)^q\|x_n-x_{n-1}\|^q. \end{aligned} \tag{4.11}$$

As, $\phi\circ N(\cdot,\cdot)$ is ξ_1-Lipschitz continuous in the first argument, $H(\cdot,\cdot)$ is r_1-Lipschitz continuous in the first argument and r_2-Lipschitz continuous in the second argument and η is τ-Lipschitz continuous, we have

$$\begin{aligned}&\left\|\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n)\right\| \\ &\times\left[\left\|H(Ax_n,Bx_n)-H(Ax_{n-1},Bx_{n-1})\right\|^{q-1}+\left\|\eta(u_n,u_{n-1})\right\|^{q-1}\right] \\ &\le \xi_1\|u_n-u_{n-1}\|\Big[(r_1+r_2)^{q-1}\|x_n-x_{n-1}\|^{q-1} \\ &\quad +\tau^{q-1}\|u_n-u_{n-1}\|^{q-1}\Big] \\ &\le \xi_1(\lambda_S+\epsilon^n)\|x_n-x_{n-1}\|\Big[(r_1+r_2)^{q-1}\|x_n-x_{n-1}\|^{q-1} \\ &\quad +\tau^{q-1}(\lambda_S+\epsilon^n)^{q-1}\left\|x_n-x_{n-1}{}^{q-1}\right\|\Big] \\ &= \xi_1(\lambda_S+\epsilon^n)\left[(r_1+r_2)^{q-1}+\tau^{q-1}(\lambda_S+\epsilon^n)^{q-1}\right]\|x_n-x_{n-1}\|^q. \end{aligned} \tag{4.12}$$

Also

$$\begin{aligned}&\left\|\phi\circ N(u_n,v_n)-\phi\circ N(u_{n-1},v_n)\right\| \\ &\le \xi_1\|u_n-u_{n-1}\| \le \xi_1(\lambda_S+\epsilon^n)\|x_n-x_{n-1}\| \end{aligned}$$

$$\left\|\phi\circ N\left(u_n,v_n\right)-\phi\circ N\left(u_{n-1},v_n\right)\right\|^q\le\xi_1^q\left(\lambda_S+\epsilon^n\right)^q\left\|x_n-x_{n-1}\right\|^q. \quad (4.13)$$

Using (4.9), (4.11), (4.12) and (4.13), (4.8) becomes

$$\begin{aligned}&\left\|H\left(Ax_n,Bx_n\right)-H\left(Ax_{n-1},Bx_{n-1}\right)-\left(\phi\circ N\left(u_n,v_n\right)-\phi\circ N\left(u_{n-1},v_n\right)\right)\right\|^q\\ &\le\left(r_1+r_2\right)^q\left\|x_n-x_{n-1}\right\|^q+qt\left(\lambda_S+\epsilon^n\right)^q\left\|x_n-x_{n-1}\right\|^q+q\xi_1\left(\lambda_S+\epsilon^n\right)\left[\left(r_1+r_2\right)^{q-1}+\tau^{q-1}\left(\lambda_S+\epsilon^n\right)^{q-1}\right]\left\|x_n-x_{n-1}\right\|^q\\ &\quad+C_q\xi_1^q\left(\lambda_S+\epsilon^n\right)^q\left\|x_n-x_{n-1}\right\|^q\\ &=\left[\left(r_1+r_2\right)^q+qt\left(\lambda_S+\epsilon^n\right)^q+q\xi_1\left(\lambda_S+\epsilon^n\right)\left[\left(r_1+r_2\right)^{q-1}+\tau^{q-1}\left(\lambda_S+\epsilon^n\right)^{q-1}\right]+C_q\xi_1^q\left(\lambda_S+\epsilon^n\right)^q\right]\left\|x_n-x_{n-1}\right\|^q\end{aligned}$$

$$\begin{aligned}&\left\|H\left(Ax_n,Bx_n\right)-H\left(Ax_{n-1},Bx_{n-1}\right)-\left(\phi\circ N\left(u_n,v_n\right)-\phi\circ N\left(u_{n-1},v_n\right)\right)\right\|\\ &\le\sqrt[q]{\left[\left(r_1+r_2\right)^q+qt\left(\lambda_S+\epsilon^n\right)^q+q\xi_1\left(\lambda_S+\epsilon^n\right)\left[(r_1+r_2)^{q-1}+\tau^{q-1}(\lambda_S+\epsilon^n)^{q-1}\right]+C_q\xi_1^q\left(\lambda_S+\epsilon^n\right)^q\right.}\left\|x_n-x_{n-1}\right\|.\end{aligned} \quad (4.14)$$

Since $\phi\circ N(\cdot,\cdot)$ is ξ_2-Lipschitz continuous in the second argument, and using the $\mathcal{D}$-Lipschitz continuity of T with constant λ_T and (4.5), we have

$$\begin{aligned}&\left\|\phi\circ N\left(u_{n-1},v_n\right)-\phi\circ N\left(u_{n-1},v_{n-1}\right)\right\|\le\xi_2\left\|v_n-v_{n-1}\right\|\\ &\le\xi_2\left(\mathcal{D}\left(T\left(x_n\right),T\left(x_{n-1}\right)\right)+\epsilon^n\left\|x_n-x_{n-1}\right\|\right)\\ &\le\xi_2\left(\lambda_T\left\|x_n-x_{n-1}\right\|+\epsilon^n\left\|x_n-x_{n-1}\right\|\right)\\ &=\xi_2\left(\lambda_T+\epsilon^n\right)\left\|x_n-x_{n-1}\right\|.\end{aligned} \quad (4.15)$$

Also, using $\mathcal{D}$-Lipschitz continuity of G with constant λ_G and (4.6), we have

$$\begin{aligned}\left\|z_n-z_{n-1}\right\|&\le\mathcal{D}\left(G\left(x_n\right),G\left(x_{n-1}\right)\right)+\epsilon^n\left\|x_n-x_{n-1}\right\|\\ &\le\lambda_G\left\|x_n-x_{n-1}\right\|+\epsilon^n\left\|x_n-x_{n-1}\right\|\le\left(\lambda_G+\epsilon^n\right)\left\|x_n-x_{n-1}\right\|.\end{aligned} \quad (4.16)$$

Using (4.14),(4.15) and (4.16), (4.7) becomes

$$\begin{aligned}\left\|x_{n+1}-x_n\right\|\le\left[\frac{\tau^{q-1}}{\alpha-\beta}\left[\sqrt[q]{\left[\left(r_1+r_2\right)^q+qt\left(\lambda_S+\epsilon^n\right)^q+q\xi_1\left(\lambda_S+\epsilon^n\right)\left[\left(r_1+r_2\right)^{q-1}+\tau^{q-1}\left(\lambda_S+\epsilon^n\right)^{q-1}\right]+C_q\xi_1^q\left(\lambda_S+\epsilon^n\right)^q\right]}\right.\right.\\ \left.\left.+\xi_2\left(\lambda_T+\epsilon^n\right)\right]+\lambda\left(\lambda_G+\epsilon^n\right)\right]\left\|x_n-x_{n-1}\right\|,\end{aligned}$$

or $\left\|x_{n+1}-x_n\right\|\le\theta\left(\varepsilon^n\right)\left\|x_n-x_{n-1}\right\|$, where

$$\begin{aligned}\theta\left(\epsilon^n\right)=\left[\frac{\tau^{q-1}}{\alpha-\beta}\left[\sqrt[q]{\left[\left(r_1+r_2\right)^q+qt\left(\lambda_S+\epsilon^n\right)^q+q\xi_1\left(\lambda_S+\epsilon^n\right)\left[\left(r_1+r_2\right)^{q-1}+\tau^{q-1}\left(\lambda_S+\epsilon^n\right)^{q-1}\right]+C_q\xi_1^q\left(\lambda_S+\epsilon^n\right)^q\right]}\right.\right.\\ \left.\left.+\xi_2\left(\lambda_T+\epsilon^n\right)\right]+\lambda\left(\lambda_G+\epsilon^n\right)\right].\end{aligned}$$

Since $0<\varepsilon<1$, it follows that $\theta\left(\varepsilon^n\right)\to\theta$, as $n\to\infty$ where

$$\theta=\left[\frac{\tau^{q-1}}{\alpha-\beta}\left[\sqrt[q]{\left[\left(r_1+r\right)_2{}^q+qt\lambda_S^q+q\xi_1\lambda_S\left[\left(r_1+r_2\right)^{q-1}+\tau^{q-1}\lambda_S^{q-1}\right]+C_q\xi_1^q\lambda_S^q\right]}+\xi_2\lambda_T\right]+\lambda\lambda_G\right].$$

From (ix), it follows that $\theta<1$, and consequently $\{x_n\}$ is a Cauchy sequence in X. Since X is a Banach space, there exists $x\in X$, such that $x_n\to x$ as $n\to\infty$.

From (4.4), (4.5) and (4.6) of Algorithm 4.1, it follows that $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ all are Cauchy sequences in X, that is there exist u,v and $z\in X$ such that $u_n\to u$, $v_n\to v$ and $z_n\to z$ as $n\to\infty$. Now, using the continuity of operators S, T, G, A, B, H, $\phi\circ N$, η and M and by Algorithm 4.1, we have

$$x=R_{M(\cdot,z)}^{H(\cdot,\cdot)-\phi-\eta}\left[H\left(Ax,Bx\right)-\phi\circ N\left(u,v\right)\right].$$

Now, we shall show that $u\in S\left(x\right)$

$$\begin{aligned}d\left(u,S\left(x\right)\right)&\le\left\|u-u_n\right\|+d\left(u,S\left(x\right)\right)\\ &\le\left\|u-u_n\right\|+\mathcal{D}\left(S\left(x_n\right),S\left(x\right)\right)\\ &\le\left\|u-u_n\right\|+\lambda_S\left\|x_n-x\right\|\to0\text{ as }n\to\infty.\end{aligned}$$

This implies that $d(u, S(x)) = 0$, since $S(x) \in CB(X)$, it follows that $u \in S(x)$. Similarly, we can prove that $v \in T(x)$, $z \in G(x)$. This completes the proof.

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