

On Fuzzy Random-Valued Optimization

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Abstract

In this paper, we propose a novel approach for Fuzzy random-valued Optimization. The main idea behind our approach consists of taking advantage of interplays between fuzzy random variables and random sets in a way to get an equivalent stochastic program. This helps avoiding pitfalls due to severe oversimplification of the reality. We consider a numerical example that shows the efficiency of the proposed method.

Keywords: Fuzzy Random Variables, Random Sets, Fuzzy Stochastic Optimization

1. Introduction

1.1. Background

Fuzzy Stochastic Optimization (FSO) is a worthwhile topic. It provides a glimpse into jousting with the complex and yet useful issue of handling situations where fuzziness and randomness are under one roof in a optimization setting. Here are, without any claim for exhaustivity, some examples of concrete problems necessitating consideration of both fuzziness and randomness: Linear regression problem in the presence of both random and fuzzy variables [1]; Renewal processes where inter-arrival times are only known as subjective categories of the form; almost 3 hours, around 2 hours, less than 4 hours..., the occurrence of which cannot be predicted with precision [2]. An interested reader is referred to [3] where the terrain covered by FSO is surveyed. The reader may also consult [4-9] for more insights in this emerging subfield of mathematical programming under uncertainty. The presence of both possibilistic and probabilistic information within a mathematical programming framework is a harbinger of computational nightmares if one were to approach the problem without any simplifications. Nevertheless, pitfalls due to severe oversimplification of the reality may lead to a bad caricature of the problem under consideration. In this paper the focus is on an Optimization model involving fuzzy random coefficients. This model comes up in several applications including optimal portfolio selection [10], inventory model [11], water resource management [12]. The commonly used approach for solving this model is to craft a deterministic surrogate of the fuzzy stochastic optimiza-

tion at hand, by exploiting the structure available while sticking as well as possible to uncertainty principles. This approximation paradigm is central to the literature [13-15], although some researchers have questioned both its robustness and its general validity [16]. Without a serious output analysis, it is hard to ascertain both the quality of the approximation and the viability of the obtained solutions.

1.2. Contribution

In this paper, we establish a mathematical connection between fuzzy random variables and random sets. This connection is then used to get an equivalent counterpart to the original problem. The challenging task of singling out a solution of the resulting stochastic program with infinitely many objective functions is also addressed. The paper contains a systematically solved example showing the efficiency of the proposed method.

1.3. Notation

Throughout the paper $F_{cc}(\mathbb{R})$ will denote the set of fuzzy numbers with compact supports. If $\tilde{a} \in F_{cc}(\mathbb{R})$ then \tilde{a}_α is its α -level set that is a closed interval. With $\tilde{C}[0,1]$ we will indicate the set of real-valued bounded functions f on $[0,1]$ such that:

1) f is left continuous for any $t \in [0,1]$

2) f has a right limit for any $t \in [0,1]$

Conventionally, for $\tilde{a} \in F_{cc}(\mathbb{R})$, $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ and $\mathfrak{F}_{\tilde{C}[0,1]}$ stands for the collections of families

$Z = (Z(\alpha))_{\alpha \in [0,1]}$ of closed intervals endpoints of which

are members of $\tilde{C}[0,1]$. We'll use the following notation for $Z(\alpha)$;

$$Z(\alpha) = [Z(\alpha)^L, Z(\alpha)^U].$$

For $Z^1, Z^2 \in \mathfrak{F}_{\tilde{C}[0,1]}$ the distance between Z^1 and Z^2 is given by:

$$d_{\mathfrak{F}_{\tilde{C}[0,1]}}(Z^1, Z^2) = \int_0^1 d_H(Z^1(\alpha), Z^2(\alpha)) d\alpha$$

where d_H denotes the Hausdorff metric. Moreover I and 1_A denote the interval $[0, 1]$ and the indicator function of A respectively.

1.4. Structure of the Paper

The remainder of the paper is organized as follows. In the following section, we introduce the notions of random closed set and fuzzy random variable and we briefly discuss some of their properties. In Section 3, we prove that the set of fuzzy random variables can be embedded into the set of random closed sets isomorphically and isometrically. This embedding result is then exploited in Section 4 to describe an approach for solving mathematical programs with fuzzy random coefficients. Section 5 is devoted to a numerical example for the sake of illustration. We end up in Section 6 with some concluding remarks along with lines for further developments in this field.

2. Random Set and Fuzzy Random Variable

2.1. Random Set

Consider a probability space $(\Omega, \mathfrak{B}, P)$ and let F be a set of collections of subsets of E . A random set in E is a map: $\Omega \rightarrow F$ that satisfies some measurability conditions [17]. For our purposes, $E = \mathbb{R}$ and we consider random sets of the form:

$$X : \Omega \rightarrow \mathfrak{F}_{\tilde{C}[0,1]} \\ \omega \rightsquigarrow (X_\omega(\alpha))_{\alpha \in I}$$

The class of the above random sets is denoted by $R(\Omega)$.

$R(\Omega)$ can be endowed with the following metric based on the Hausdorff metric d_H . For

$$X, Y \in R(\Omega),$$

$$d_{R(\Omega)}(X, Y) = \int \int_{\Omega \times I} d_H(X_\omega(\alpha), Y_\omega(\alpha)) dP(\omega) d\alpha$$

Given $X, Y \in R(\Omega)$ we say that X is less or equal than Y , in symbol:

$$X \leq_{R(\Omega)} Y$$

if $\forall \omega \in \Omega, \forall \alpha \in I, \sup X_\omega(\alpha) \leq \inf Y_\omega(\alpha)$.

For details on random sets, we refer the reader to [18, 19].

2.2. Fuzzy Random Variable

Consider again a probability space $(\Omega, \mathfrak{B}, P)$. A map

$$X : \Omega \rightarrow F_{cc}(\mathbb{R}) \\ \omega \rightsquigarrow X_\omega$$

is a fuzzy random variable if for every $\alpha \in (0,1]$ and for every Borel set B of \mathbb{R} , $X_\alpha^{-1}(B) \in \mathfrak{B}$ where $X_\alpha : \Omega \rightarrow 2^{\mathbb{R}}$ is defined as follows.

$$X_\alpha(\omega) = \{x \in \mathbb{R} \mid X_\omega(x) \geq \alpha\}.$$

In the sequel, the set of fuzzy random variables in the above sense is denoted by $F(\Omega)$. A remarkable property of a fuzzy random variable (frv) is that Zadeh's decomposition principle for fuzzy quantities extends naturally to frvs, that is for

$$Y \in F(\Omega), Y = \bigcup_{\alpha \in (0,1]} \alpha Y_\alpha.$$

Another fundamental key fact about fuzzy random variables, which is of interest on its own right and which has a huge impact on applications is that an α -level set of a fuzzy random variable is a random interval [17]. Arithmetic operations on $F(\Omega)$ are defined as follows.

Given $Y, Z \in F(\Omega)$ and $\lambda \in \mathbb{R}$ we have:

$$(Y \boxplus Z)(\omega) = Y_\omega \oplus Z_\omega; \omega \in \Omega$$

$$(\lambda \boxtimes Y)(\omega) = \lambda \odot Y_\omega; \omega \in \Omega$$

$$Y \leq Z \text{ if and only if } Y_\omega \leq Z_\omega, \forall \omega \in \Omega$$

where \square and \circ indicate that operations are on $F(\Omega)$ and on $F_{cc}(\mathbb{R})$ respectively. It is worth mentioning that \oplus and \odot are based on Zadeh's extension principle [20]. Moreover, $Y_\omega \leq Z_\omega$ is tantamount to $Y_{\omega\alpha} \leq_l Z_{\omega\alpha}$ for all $\alpha \in (0,1]$ where \leq_l stands for inequality between intervals.

We equip $F(\Omega)$ with a distance defined as follows.

Given $Y, Z \in F(\Omega)$

$$d_{F(\Omega)}(Y, Z) = \int \int_{\Omega \times I} d_H(Y_\omega(\alpha), Z_\omega(\alpha)) dP_\omega d\alpha$$

As in the case of random variables, it is efficient to describe the distribution of a frv by means of certain measures summarizing some of its most relevant characteristics. In this way, the first two moments of a fuzzy random variable Y are defined as follows. The expectation of a frv Y , in symbol EY is the fuzzy quantity whose α -level sets are given by:

$$(EY)_\alpha = \left\{ \int_{\Omega} f dP \mid f \text{ is a selection of } Y_\alpha \right\}.$$

The variance of Y , in symbol VY , is given by the following relation:

$$VY = E(d_{F(\Omega)}(Y, EY))$$

Limit theorems [22] have been obtained for frvs based on the above notions of expectation and variance.

Moreover, fuzzy random variables enjoy the Random-Nikodým property [21]. That is, if $F : \Omega \rightarrow F_{cc}(\mathbb{R})$ is a P-continuous fuzzy measure of bounded variation, then there is $Y \in F(\Omega)$ such that:

$$F(B) = \int_B Y dP, \quad \forall B \in \mathfrak{B}$$

3. Embedding Theorem for Fuzzy Random Variables

3.1. Auxiliary Mappings

The following three maps will play a starring role in the statement and the proof of an Embedding Theorem for fuzzy random variables.

3.1.1. Mapping Π

Π maps $F_{cc}(\mathbb{R})$ into $\mathfrak{F}_{\tilde{C}[0,1]}$ as follows.

$$\begin{aligned} \Pi : F_{cc}(\mathbb{R}) &\rightarrow \mathfrak{F}_{\tilde{C}[0,1]} \\ \tilde{a} &\rightsquigarrow [\tilde{a}^L(\alpha), \tilde{a}^U(\alpha)]_{\alpha \in I} \end{aligned}$$

where $\tilde{a}^L(\alpha) = \tilde{a}_\alpha^L$ and $\tilde{a}^U(\alpha) = \tilde{a}_\alpha^U$.

It is well known (see e.g [23,24]) that Π thus defined is injective, isometric and satisfies the following relation:

For $\tilde{a}, \tilde{b} \in F_{cc}(\mathbb{R})$ and $s, t \in \mathbb{R}, s \geq 0, t \geq 0$

$$\Pi \left[(1_{\{s\}} \odot \tilde{a}) \oplus (1_{\{t\}} \odot \tilde{b}) \right] = s \Pi(\tilde{a}) + t \Pi(\tilde{b}).$$

3.1.2. Mappings f_ω and σ_ω

The two other auxiliary maps are given below.

$$\begin{aligned} f_\omega : F(\Omega) &\rightarrow F_{cc}(\mathbb{R}) \\ X &\rightsquigarrow X_\omega \end{aligned}$$

$$\begin{aligned} \sigma_\omega : F(\Omega) &\rightarrow \mathfrak{F}_{\tilde{C}[0,1]} \\ X &\rightsquigarrow \Pi(X_\omega) \end{aligned}$$

3.1.3. Remark

Relationships between auxiliary mappings are shown in **Figure 1**. The three auxiliary mappings make the diagram given in **Figure 1** commutative.

As a matter of fact, $\prod f_\omega(X) = \prod(X_\omega) = \sigma_\omega(X)$

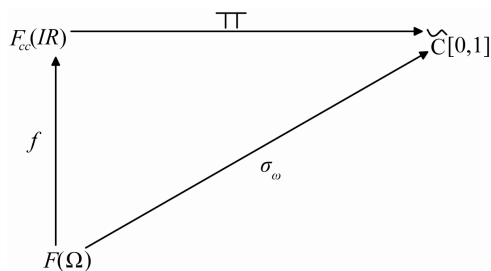


Figure 1. Diagram involving auxiliary functions.

Therefore $\prod f_\omega = \sigma_\omega$.

3.2. Main mapping σ

The mapping σ that is used in our Embedding Theorem for fuzzy random variables is defined as follows.

$$\begin{aligned} \sigma : F(\Omega) &\rightarrow R(\Omega) \\ X &\rightsquigarrow \sigma X \end{aligned}$$

where

$$\begin{aligned} \sigma X : \Omega &\rightarrow \mathfrak{F}_{\tilde{C}[0,1]} \\ \omega &\rightsquigarrow [X_\omega^L(\alpha), X_\omega^U(\alpha)] \end{aligned}$$

and

$$[X_\omega^L(\alpha), X_\omega^U(\alpha)] = [X_{\omega\alpha}^L, X_{\omega\alpha}^U]_{\alpha \in I}$$

Relationships between the main mapping and auxiliary mappings is given in **Figure 2**. Mappings involved in **Figure 2** make the diagram given in **Figure 2** commutative.

As a matter of fact,

For $\omega \in \Omega$

$$\prod X(\omega) = \prod(X_\omega) = \sigma X(\omega)$$

Hence $\prod \circ X = \sigma X$.

For the sake of convenience, we identify a real number a with the following degenerate fuzzy number.

$$\tilde{a} : \mathbb{R} \rightarrow [0,1]$$

with

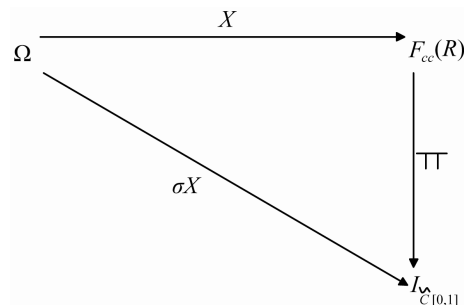


Figure 2. Diagram involving the main mapping.

$$\tilde{a}(t) = \begin{cases} 1 & \text{if } t = a \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we denote by $\tilde{I}_A (A \subseteq \mathbb{R})$, the degenerate fuzzy random variable:

$$\begin{aligned} \tilde{I}_A : \Omega &\rightarrow F_{cc}(A) \\ \omega &\rightsquigarrow 1_A \end{aligned}$$

Building on the mappings $\Pi, f_\omega, \sigma_\omega, \sigma$ and on the above terminological conventions, we are now ready to present the main result of this section.

3.3. Statement and Proof of the Embedding Theorem

3.3.1. Theorem 1

Consider $X, Y \in F(\Omega)$; $\lambda, \mu \in \mathbb{R}, \lambda \geq 0, \mu \geq 0$ and let σ be as in §3.2. Then the following statements hold true.

- 1) σ is injective
- 2) $\sigma[\tilde{I}_{\{\lambda\}} \boxtimes X] \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y) = \lambda\sigma X + \mu\sigma Y$
- 3) $d_{F(\Omega)}(X, Y) = d_{R(\Omega)}(\sigma X, \sigma Y)$
- 4) $X \leq Y \Leftrightarrow \sigma X \leq_{R(\Omega)} \sigma Y$

In other terms σ maps $F(\Omega)$ into $R(\Omega)$ isomorphically and isometrically.

Moreover, σ is order preserving.

3.3.2. Proof of Theorem 1

1) Let $X, Y \in F(\Omega)$ and assume $\sigma X = \sigma Y$. Then for $\omega \in \Omega$ we have that:

$$\begin{aligned} (\sigma X)(\omega) &= (\sigma Y)(\omega); \\ \sigma_\omega(X) &= \sigma_\omega(Y). \end{aligned}$$

This means that for $\omega \in \Omega$ we have:

$$(\Pi \circ f_\omega)(X) = \Pi(X_\omega) = (\Pi \circ f_\omega)(Y) = \Pi(Y_\omega)$$

As Π is injective, we conclude that $X_\omega = Y_\omega$ for all $\omega \in \Omega$.

Therefore $X = Y$ and we are done.

2)

$$\begin{aligned} &\sigma \left[(\tilde{I}_{\{\lambda\}} \boxtimes X) \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y) \right] (\omega) \\ &= \sigma_\omega \left[(\tilde{I}_{\{\lambda\}} \boxtimes X) \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y) \right] \\ &= (\Pi \circ f_\omega) \left[(\tilde{I}_{\{\lambda\}} \boxtimes X) \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y) \right] \\ &= \Pi \left[(\tilde{I}_{\{\lambda\}} \boxtimes X) \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y) \right]_\omega \\ &= \Pi \left[(\tilde{I}_{\{\lambda\}} \boxtimes X)(\omega) \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y)(\omega) \right] \end{aligned}$$

$$\begin{aligned} &= \Pi \left[(\tilde{I}_{\{\lambda\}} \odot X(\omega)) \boxplus (\tilde{I}_{\{\mu\}} \odot Y(\omega)) \right] \\ &= \lambda \Pi(X(\omega)) + \mu \Pi(Y(\omega)) = \lambda \sigma_\omega(X) + \mu \sigma_\omega(Y) \\ &= \lambda(\sigma X)(\omega) + \mu(\sigma Y)(\omega) = [\lambda\sigma X + \mu\sigma Y](\omega). \end{aligned}$$

Therefore

$$\sigma \left[(\tilde{I}_{\{\lambda\}} \boxtimes X) \boxplus (\tilde{I}_{\{\mu\}} \boxtimes Y) \right] = \lambda\sigma X + \mu\sigma Y$$

as desired.

3)

$$\begin{aligned} &d_{R(\Omega)}(\sigma X, \sigma Y) \\ &= \int \int_{\Omega \times I} d_H((\sigma X)_\omega(\alpha), (\sigma Y)_\omega(\alpha)) dP(\omega) d\alpha \\ &= \int \int_{\Omega \times I} d_H(\sigma_\omega(X)(\alpha), \sigma_\omega(Y)(\alpha)) dP(\omega) d\alpha \\ &= \int \int_{\Omega \times I} d_H(\Pi(X_\omega)(\alpha), \Pi(Y_\omega)(\alpha)) dP(\omega) d\alpha. \end{aligned}$$

As Π is isometric we have that:

$$\begin{aligned} d_{R(\Omega)}(\sigma X, \sigma Y) &= \int \int_{\Omega \times I} d_H(X_\omega(\alpha), Y_\omega(\alpha)) dP(\omega) d\alpha \\ &= d_{F(\Omega)}(X, Y). \end{aligned}$$

4) $X \leq Y$ if and only if $X_\omega \leq Y_\omega, \forall \omega \in \Omega$.

This is tantamount to say that:

$$X \leq Y \text{ if and only if } X_\omega^{\alpha U} \leq Y_\omega^{\alpha L} \quad \forall \alpha, \forall \omega \in \Omega \quad (1)$$

As $X_\omega^{\alpha L} \leq X_\omega^{\alpha U}$ and $Y_\omega^{\alpha L} \leq Y_\omega^{\alpha U}$, we have that (1) can be written:

$$\begin{aligned} &X \leq Y \text{ if and only if} \\ &[X_\omega^{\alpha L}, X_\omega^{\alpha U}] \leq [Y_\omega^{\alpha L}, Y_\omega^{\alpha U}] \quad \forall \alpha, \forall \omega \quad (2) \end{aligned}$$

(2) is equivalent to $X \leq Y$ if and only if

$$X \leq Y \text{ if and only if } (\sigma X)(\omega) \leq (\sigma Y)(\omega), \quad \forall \omega \quad (3)$$

But $\sigma X(\omega) \leq (\sigma Y)(\omega), \forall \omega$ is equivalent to

$$\sup(\sigma X)_\omega(\alpha) \leq \inf(\sigma Y)_\omega(\alpha) \quad \forall \omega, \forall \alpha$$

or

$$\sigma X \leq_{R(\Omega)} \sigma Y.$$

Therefore (3) can be written: $X \leq Y$ if and only if $\sigma X \leq_{R(\Omega)} \sigma Y$ and we are done.

4. Solving Fuzzy Random-Valued Optimization Problems

4.1. Case of Deterministic Feasible Set

Here we are interested in solving the following Optimization problem:

$$(P1) \left\{ \begin{array}{l} \min \tilde{f}(x) \\ x \in X \end{array} \right.$$

where $\tilde{f}: \mathbb{R}^n \rightarrow F(\Omega)$ and X is a convex and bounded subset of \mathbb{R}^n

As σ is an isomorphism isometric and order preserving, solving (P1) is tantamount to find a solution of the mathematical program:

$$(P1)' \left\{ \begin{array}{l} \min \sigma \tilde{f}(x) \\ x \in X \end{array} \right.$$

More formally we have,

Proposition 1

x^* is an optimal solution of (P1) if and only if x^* is an optimal solution of (P1)' .

Proof

Assume x^* is an optimal solution for (P1). Then $x^* \in X$ and $\tilde{f}(x^*) \leq \tilde{f}(x) \quad \forall x \in X$.

Then by Theorem 1 (d) we have that:

$$\sigma \tilde{f}(x^*) \leq \sigma \tilde{f}(x) \quad \forall x \in X$$

This means x^* is optimal for (P1)' .

Assume now that x^* is optimal for (P1)' . Then $x^* \in X$ and $\sigma \tilde{f}(x^*) \leq \sigma \tilde{f}(x) \quad \forall x \in X$. By Theorem 1(d) again, we have that:

$$\tilde{f}(x^*) \leq \tilde{f}(x) \quad \forall x \in X$$

and we are done ■

By definition of σ , (P1)' is equivalent to:

$$(P1)'' \left\{ \begin{array}{l} \min \left\{ \left[\tilde{f}_\omega^L(x)(\alpha), \tilde{f}_\omega^U(x)(\alpha) \right] \right\} \\ x \in X \\ \alpha \in [0,1]; \omega \in \Omega. \end{array} \right.$$

Worthy to note here is the fact that (P1)'' is a stochastic multiobjective mathematical program with infinitely many objective functions.

To the best of our knowledge, there is no available solution technique for it. This is the price to pay for considering an equivalent approach to treat fuzziness instead of an approximate one. To be able to carry out a fairly discussion of (P1)'' we find it convenient to assume that:

1) the expectation model is acceptable for tackling randomness;

2) minimizing an interval can be well handled by minimizing its midpoint.

It might be pointed out in passing, that assumption 1) is often used in the literature for derandomization purposes [25,26]. Moreover, assumption 2) grants us a way for transforming intervals into real numbers. This

transformation generalizes quite canonically the real case. As a matter of fact the midpoint of $[a, a]$ is a .

Bearing in mind 1), 2) and considering the fact that multiplying an objective function by a constant does not alter the localisation of an optimum, (P1)'' may be written as follows.

$$(P2) \left\{ \begin{array}{l} \min \left[E(\tilde{f}_\omega^L(x))(\alpha) + E(\tilde{f}_\omega^U(x))(\alpha) \right] \\ x \in X \\ \alpha \in [0,1]; \omega \in \Omega. \end{array} \right.$$

from now on, $(f(x))(\alpha)$ stands for

$$E(\tilde{f}_\omega^L(x))(\alpha) + E(\tilde{f}_\omega^U(x))(\alpha)$$

Therefore (P2) reads merely:

$$(P2)' \left\{ \begin{array}{l} \min \left[(f(x))(\alpha) \right] \\ x \in X \\ \alpha \in I \end{array} \right.$$

Let's now select a finite subset of $I, S = \{\alpha_1, \dots, \alpha_m\}$ and let $\theta_1, \dots, \theta_n$ be real-valued functions such that:

1) $\theta_j(\alpha) \geq 0; \alpha \in I; j = 1, \dots, m$

2) $\theta_j(\alpha_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Define a positive interpolating operator K with nodes $\alpha_1, \dots, \alpha_m$ as follows:

$$(Kh)(\alpha) = \sum_{j=1}^m \theta_j(\alpha) h(\alpha_j)$$

Consider now the following mathematical programs.

$$(P3) \left\{ \begin{array}{l} \min \left[(Kf(x))(\alpha) \right] \\ x \in X \\ \alpha \in I. \end{array} \right.$$

$$(P4) \left\{ \begin{array}{l} \min \left[f(x)(\alpha_j) \right] \\ x \in X \\ j \in \{1, \dots, m\} \end{array} \right.$$

The following result bridge the gap between (P3) and (P4).

Proposition 2

If x^* is efficient for (P3) then x^* is efficient for (P4).

Proof

Suppose that x^* is an efficient solution for (P3) and not efficient for (P4) Then there is no $x \in X$ such that

$$\left[Kf(x) \right](\alpha) \leq \left[Kf(x^*) \right](\alpha) \quad \forall \alpha \in I \quad (\$)$$

and

$$[Kf(x)](\bar{\alpha}) < [Kf(x^*)](\bar{\alpha}) \text{ for some } \bar{\alpha} \in I \text{ (§§)}$$

As x^* is not efficient for (P4) we have also that, there is

$$x \in X \text{ such that } f(x)(\alpha_j) \leq f(x^*)(\alpha_j) \\ \forall j \in \{1, \dots, m\}$$

and

$$f(x)(\alpha_s) < f(x^*)(\alpha_s) \text{ for some } s \in \{1, \dots, m\}.$$

Consider now $\alpha \in I$ arbitrarily chosen. As $\theta_j(\alpha) \geq 0$ and $\theta_j(\alpha_j) = 1$ we have that:

$$\theta_j(\alpha) [f(x)(\alpha_j) - f(x^*)(\alpha_j)] \leq 0 \quad \forall j \in \{1, \dots, m\}$$

and

$$\theta_s(\alpha) [f(x)(\alpha_s) - f(x^*)(\alpha_s)] < 0 \text{ for some } s \in \{1, \dots, m\}$$

Therefore we can say that there is $x \in X$ such that:

$$\sum_{j=1}^m \theta_j(\alpha) [f(x)(\alpha_j) - f(x^*)(\alpha_j)] < 0$$

This means, as α has been chosen arbitrarily, that there is $x \in X$ such that:

$$[Kf(x)](\alpha) < [Kf(x^*)](\alpha) \quad \forall \alpha \in I$$

This contradicts the fact that there is no $x \in X$ such that (§) and (§§) hold.

Therefore x^* is efficient for (P4).

It is a common place to say that (P3) is the same as (P2)' where $f(x)$ is replaced by $Kf(x)$. Unfortunately both (P2)' and (P3) are too cumbersome for mathematical tractability. For practical purposes, we'll resort to (P4) that is a discretization of (P2)

Thanks to the contrapositive of Proposition 2, we know that only an efficient solution of (P4) can be efficient for (P3). It might also be pointed out in passing that the discretization error decreases when the grid $S = \{\alpha_1, \dots, \alpha_m\}$ is refined [30].

This means we should keep the roughness of the grid, i.e.

$$h = h(\alpha_1, \dots, \alpha_m) = \max_{\alpha \in I} \min_{1 \leq i \leq m} |\alpha - \alpha_i|$$

as low as possible.

The foregoing discussion leads us to describe the following algorithm for solving (P1).

Description of the algorithm

Step 0: Fix $\varepsilon > 0$ an acceptable bound of error for h .

Step 1: Read data of (P1).

Step 2: Frame (P1) as (P2)'.

Step 3: Put $\ell = 0$.

Step 4: Take a discretization of $I, S_\ell = \{\alpha_{\ell_1}, \dots, \alpha_{\ell_{m_\ell}}\}$.

Step 5: Write (P4).

Step 6: Find an efficient solution for (P4).

Step 7: Compute $h = \max_{\alpha \in I} \min_{1 \leq i \leq m_\ell} |\alpha - \alpha_i|$ and check whether $|h| < \varepsilon$.

If this is true, go to Step 9,

Otherwise go to Step 8.

Step 8: Take a finer discretization S' , put $S = S'$ and go to Step 5.

Step 9: Print the solution obtained.

Step 10: Stop.

4.2. Case of Fuzzy Random Constraints

Here we are interested in the following optimization problem.

$$(P5) \begin{cases} \min \tilde{f}(x) \\ \tilde{g}_i(x) \leq \tilde{b}_i; i = 1, \dots, m \end{cases}$$

where $\tilde{f}, \tilde{g}_i, i = 1, \dots, m$ are fuzzy random-valued functions of \mathbb{R}^n and $\tilde{b}_i \in F(\Omega)$; $i = 1, \dots, m$.

Consider the following optimization problem:

$$(P6) \begin{cases} \min \sigma \tilde{f}(x) \\ \sigma \tilde{g}_i(x) \leq_{R(\Omega)} \sigma \tilde{b}_i; i = 1, \dots, m \end{cases}$$

Before stating a result that bridges the gap between (P5) and (P6), we introduce the following respective surrogates to (P5) and (P6) respectively.

$$(P5)' \begin{cases} \min \tilde{f}(x) \\ x \in X' \end{cases}$$

$$(P6)' \begin{cases} \min \sigma \tilde{f}(x) \\ x \in X'' \end{cases}$$

where X' and X'' are deterministic counterparts of the following sets respectively:

$$\{x \in \mathbb{R}^n \mid \tilde{g}_i(x) \leq \tilde{b}_i; i = 1, \dots, m\}$$

and

$$\{x \in \mathbb{R}^n \mid \sigma \tilde{g}_i(x) \leq_{R(\Omega)} \sigma \tilde{b}_i; i = 1, \dots, m\}$$

Proposition 3

x^* is an optimal solution for (P5)' if and only if x^* is optimal for (P6)'.

Proof

By Theorem 1 (d), we have that $X' = X''$. Moreover by Proposition 1, (P5)' and (P6)' are equivalent.

In this sense, we can say that (P5) and (P6) are

equivalent. To solve (P5) we have to consider (P6)' and then apply the method described in the previous section for solving (P2)'.

5. Numerical Example

For the sake of illustration, we consider the following simple example.

$$(P_E) \begin{cases} \max(\bar{c}_1 x_1 + \bar{c}_2 x_2) \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0 \end{cases}$$

where \bar{c}_1 and \bar{c}_2 are fuzzy random variables defined on $\Omega = \{\omega_1, \omega_2\}$ with $P(\omega_1) = p_1 = \frac{2}{5}$ and

$P(\omega_2) = p_2 = \frac{3}{5}$ Details on the two frvs are given in

Table 1.

$\Delta(a, b, c)$ stands for a triangular fuzzy number with membership $\mu_{\Delta(a, b, c)}$ defined as in **Figure 3**.

According to Proposition 1, (P_E) is equivalent to the following mathematical program:

Table 1. Details on frvs \bar{c}_1 and \bar{c}_2 .

frv	fuzzy values	probabilities
\bar{c}_1	$\tilde{c}_{11} = \Delta(1, 1, 1)$	$p_{11} = \frac{2}{5}$
	$\tilde{c}_{12} = \Delta(4; 2; 0, 5)$	$p_{12} = \frac{3}{5}$
\bar{c}_2	$\tilde{c}_{21} = \Delta(2; 1, 5; 3)$	$p_{21} = \frac{2}{5}$
	$\tilde{c}_{22} = \Delta(3; 0, 5; 0, 25)$	$p_{22} = \frac{3}{5}$

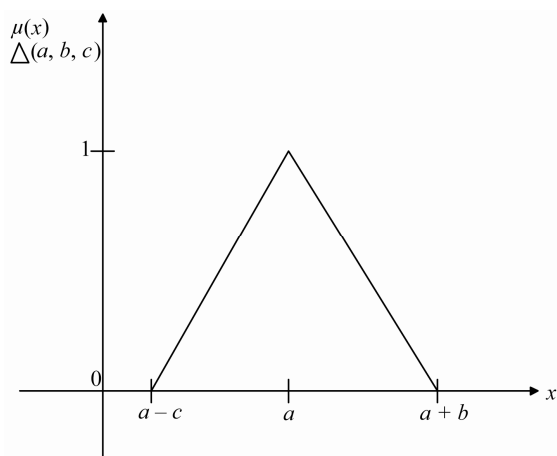


Figure 3. Triangular fuzzy number delta (a, b, c).

$$(P_E)' \begin{cases} \max(\sigma \bar{c}_1 x_1 + \sigma \bar{c}_2 x_2) \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0 \end{cases}$$

Here $f_\omega(x) = \tilde{c}_{1\omega} x_1 + \tilde{c}_{2\omega} x_2$ and (P2) reads:

$$(P_E)'' \begin{cases} \max \left[E(\tilde{c}_{1\omega}^L(\alpha) x_1 + \tilde{c}_{2\omega}^L(\alpha) x_2) \right. \\ \left. + E(\tilde{c}_{1\omega}^U(\alpha) x_1 + \tilde{c}_{2\omega}^U(\alpha) x_2) \right] \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0 \\ \alpha \in I; \omega \in \Omega \end{cases}$$

Let $\varepsilon = 0,25$ and consider the following discretization of I ;

$$S = \{0; 0,25; 0,5; 0,75; 1\}$$

with this grid, (P4) takes the form:

$$(P_E)''' \begin{cases} \max \left[p_{11} \tilde{c}_{11}^L(\alpha) x_1 + p_{12} \tilde{c}_{12}^L(\alpha) x_1 + p_{21} \tilde{c}_{21}^L(\alpha) x_2 \right. \\ \left. + p_{22} \tilde{c}_{22}^L(\alpha) x_2 + p_{11} \tilde{c}_{11}^U(\alpha) x_1 + p_{12} \tilde{c}_{12}^U(\alpha) x_1 \right. \\ \left. + p_{21} \tilde{c}_{21}^U(\alpha) x_2 + p_{22} \tilde{c}_{22}^U(\alpha) x_2 \right] \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0 \\ \alpha \in \{0; 0,25; 0,5; 0,75; 1\} \end{cases}$$

Considering data of **Tables 1-3**, (P_E)''' becomes:

$$\begin{cases} \max \{ 6,5x_1 + 4,75x_2, 6,27x_1 + 4,86x_2, 6,05x_1 + 4,97x_2, \\ 5,82x_1 + 5,08x_2, 10,4x_1 + 5,2x_2 \} \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0 \end{cases}$$

A Pareto optimal solution of this multiobjective program may be obtained by solving the following weighting program.

Table 2. Left endpoints of α -level cuts of \tilde{c}_{ij} .

$\tilde{c}_i^L(\alpha)$	0	0.25	0.5	0.75	1
$\tilde{c}_j^L(\alpha)$	0	0.25	0.5	0.75	1
$\tilde{c}_{11}^L(\alpha)$	3.5	3.625	3.75	3.875	4
$\tilde{c}_{21}^L(\alpha)$	-1	-0.25	0.5	1.25	2
$\tilde{c}_{22}^L(\alpha)$	2.75	2.812	2.875	2.937	3

Table 3. Right endpoints of α -level cuts of \tilde{c}_{ij} .

$\tilde{c}_{ij}^U(\alpha) \alpha$	0	0.25	0.5	0.75	1
$\tilde{c}_{11}^U(\alpha)$	2	1.75	1.5	1.25	1
$\tilde{c}_{12}^U(\alpha)$	6	5.5	5	4.5	4
$\tilde{c}_{21}^U(\alpha)$	3.5	3.125	2.75	2.375	2
$\tilde{c}_{22}^U(\alpha)$	3.5	3.375	3.25	3.125	3

$$\begin{cases} \max [\lambda_1 (6,5x_1 + 4,75x_2) + \lambda_2 (6,27x_1 + 4,86x_2) \\ \quad + \lambda_3 (6,05x_1 + 4,97x_2) + \lambda_4 (5,82x_1 + 5,08x_2) \\ \quad + \lambda_5 (10,4x_1 + 5,2x_2)] \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0; \lambda_s > 0; s = 1, \dots, 5 \end{cases}$$

for $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1$ we have the program:

$$\begin{cases} \max (35,04x_1 + 24,86x_2) \\ x_1 + x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 19 \\ x_1 \geq 0; x_2 \geq 0 \end{cases}$$

which yields the solution $x^* = (8,0)$ using LINGO software. As the roughness of the grid h is small, this solution is a good approximation of the solution of the original problem (P_E)

6. Concluding Remarks

Though significant progress has been made in recent years on Fuzzy Stochastic Optimization [3-7], there are from an algorithmic point of view, many challenges remaining. Developing effective and efficient techniques for handling such problems still remain an important issue. This paper has been written to address some of the above mentioned challenges. It is also filled with many references for those whose appetite has been sufficiently whetted that they are hungry for more.

It might be pointed out that a general methodology for solving Fuzzy Stochastic Optimization problems has been outlined in [3]. The quintessential of that methodology is to perform a couple of transformations (possibilistic and probabilistic) say f_1 and f_2 either sequentially or in parallel in a way to put the original problem into deterministic terms.

To be in tune with uncertainty principles [22], these transformations should be able to introduce possibilistic and probabilistic information at a more fundamental level. They should also capture the essence of involved

fuzziness and randomness. This is the reason why, in this paper, we found it convenient not to let both f_1 and f_2 be mere approximations. The possibilistic transformation is an equivalence obtained from connections between fuzzy random variables and random closed sets. Therefore our approach contrasts markedly with those where approximation of fuzzy values by real ones is followed by approximation of random variables by their moments [7]. It also differs from approaches based on fuzzy-stochastic simulation [28]. Moreover, our approach can handle both linear and non linear optimization problems. It is also less demanding in terms of information that the decision maker should provide before having his problem solved. This departs strongly with extant methods as illustrated by the following sample. In [4,5] for example, emphasis is placed on linear optimization problems. The method described in [28] is based on the assumption that involved fuzzy random variables are of the $L-R$ type. Techniques discussed in [6,29] require that the decision maker be able to set appropriate targets and suitable thresholds or to manipulate complex indexes. The price to pay for using the method described here is the computational challenge brought up by the resulting problem that is a stochastic program with infinitely many objective functions. An algorithm for solving this problem has been presented. An efficient implementation and numerical testing of the proposed algorithm is a topic of future research.

7. References

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