

An Exact Penalty Approach for Mixed Integer Nonlinear Programming Problems

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Received July 31, 2011; revised August 19, 2011; accepted September 15, 2011

Abstract

We propose an exact penalty approach for solving mixed integer nonlinear programming (MINLP) problems by converting a general MINLP problem to a finite sequence of nonlinear programming (NLP) problems with only continuous variables. We express conditions of exactness for MINLP problems and show how the exact penalty approach can be extended to constrained problems.

Keywords: Mixed Integer Nonlinear Programming, Continuous Programming, Exact Penalty Method, Exact Penalty Functions

1. Introduction

One way for relaxing the integer constraints on the variables of a problem is adding an appropriate penalty term to the objective function to create a new problem with only continuous variables. This approach was first introduced by Ragavachari [1] to solve 0-1 linear programming problems and was used by several researchers for solving real nonlinear discrete programming problems [2-5]. Recently, Murray and Ng [6] have extended this approach for large scale 0-1 nonlinear programming problems with linear constraints.

In [7], the exact penalty approach was extended to nonlinear integer programming problems. In [3,8], several penalty functions were presented and the exactness of some of them were proved in [9]. Here, using ideas of Lucidi [9] we introduce conditions for exactness of a penalty function for mixed integer nonlinear programming (MINLP) problems. Then, we extend the exact penalty approach to constrained mixed integer nonlinear programming problems.

Notation 1. Let $v(\cdot)$ denote the optimal value of problem (\cdot) .

2. Penalty Method for Unconstrained MINLP Problems

An unconstrained mixed integer nonlinear programming

problem is expressed as:

$$(UMINLP) \quad \min \quad f(x, y) \\ \text{s.t.} \quad x \in X, y \in Y,$$

where, f is a real-valued continuous function on \mathbb{R}^{n+m} , X is a finite subset of \mathbb{Z}^n in the form $\{0,1\}^n$, and Y is a compact subset in \mathbb{R}^m .

The continuous relaxation of (UMINLP) can be expressed as:

$$(R) \quad \min \quad f(x, y) \\ \text{s.t.} \quad x \in [0,1]^n, y \in Y.$$

We construct the following problem by adding some constraints to the relaxed problem (R):

$$(UNLP) \quad \min \quad f(x, y) \\ \text{s.t.} \quad q(x) = \sum_{i=1}^n q_i(x_i) = 0 \\ x \in [0,1]^n, y \in Y,$$

where, the $q_i(x_i)$ are nonnegative continuous functions as follows:

$$q_i(x_i) = \begin{cases} 0, & x_i \in \{0,1\}, \\ > 0, & x_i \in (0,1), \end{cases} \quad i = 1, \dots, n. \quad (1)$$

It is easy to see that (UMINLP) and (UNLP) are equivalent, because $q(x) = \sum_{i=1}^n q_i(x_i)$ is zero on points

in $\{0,1\}^n$ and is positive on points in $(0,1)^n$.

Some appropriate definitions for the q_i are:

$$(q1) \quad q_i(x_i) = x_i(1-x_i), i = 1, \dots, n,$$

$$(q2) \quad q_i(x_i) = 1 - \cos 2\pi x_i, i = 1, \dots, n.$$

Now, for every $r > 0$, let

$$H_r(x, y) = f(x, y) + rq(x),$$

and consider the following penalty problem for the (UNLP):

$$(UPEN_r) \quad \min H_r(x, y) = f(x, y) + rq(x) \\ \text{s.t.} \quad x \in [0, 1]^n, y \in Y.$$

Note that the problem $(UPEN_r)$ is a continuous version of the problem $(UMINLP)$.

Under certain assumptions, we show that for some finite value of penalty parameter r , problem $(UPEN_r)$ is equivalent to $(UMINLP)$.

For ε , $0 < \varepsilon < 1/2$, define a punctured neighborhood of $\{0,1\}$ in $[0,1]$ as follows:

$$J_\varepsilon = (0, \varepsilon) \cup (1 - \varepsilon, 1). \tag{2}$$

Assumption 1. *There exist $\Delta > 0$ and $\gamma > 0$ such that i) for every $(x, y) \in J_\varepsilon^n \times Y \subset [0, 1]^n \times Y$, we have*

$$\left| \frac{\partial}{\partial x_i} f(x, y) \right| < \Delta, i = 1, \dots, n,$$

ii) each q_i is differentiable on J_ε and for each $x_i \in J_\varepsilon$, we have

$$q_i'(x_i) = \begin{cases} > \gamma, & x_i < \varepsilon, \\ < -\gamma, & x_i > 1 - \varepsilon, \end{cases} i = 1, \dots, n.$$

Note that if f has bounded derivatives, then it satisfies Assumption 1(i), and as an example, (q1) satisfies Assumption 1(ii).

The following theorem shows that we can find a solution of an unconstrained MINLP problem by solving a finite sequence of NLP problems.

Theorem 1. *Under Assumption 1, there exists a finite r_0 such that for any $r > r_0$, any solution of $(UPEN_r)$ also solves $(UMINLP)$ with $v(UPEN_r) = v(UMINLP)$.*

Proof. For any feasible point (x, y) for $(UMINLP)$, we have

$$H_r(x, y) = f(x, y) + rq(x) = f(x, y).$$

Since any feasible point for $(UMINLP)$ is also feasible for $(UPEN_r)$, the above relation implies:

$$v(UMINLP) \geq v(UPEN_r). \tag{3}$$

For any $r > 0$, let (x^r, y^r) be an optimal solution of $(UPEN_r)$. Suppose that $\{(x^r, y^r)\}$ is a convergent sub-

sequence of optimal solutions of $(UPEN_r)$ and (x^*, y^*) is its limit. Note that since $\{(x^r, y^r)\} \subset [0, 1]^n \times Y$ and $[0, 1]^n \times Y$ is compact, at least one convergent subsequence exists.

Since $x^r \rightarrow x^*$, there exists an \bar{r} such that for every $r > \bar{r}$, we have: $\|x^r - x^*\|_\infty < \varepsilon$. Therefore, (2) implies that $x_i^r \in \{0, 1\}$ or $x_i^r \in J_\varepsilon$.

Now, let $r_0 = \max(\bar{r}, \Delta/\gamma)$ and suppose that $r > r_0$. If $x_i^r \notin \{0, 1\}$, then $x_i^r \in J_\varepsilon$. Since f and q_i are differentiable on J_ε and the first-order necessary conditions for problem $(UPEN_r)$ holds in subspace J_ε , then we have

$$\frac{\partial}{\partial x_i} H_r(x^r, y^r) = 0,$$

$$\frac{\partial}{\partial x_i} f(x^r, y^r) + r \frac{\partial}{\partial x_i} q(x^r, y^r) = 0,$$

$$\frac{\partial}{\partial x_i} f(x^r, y^r) + r \frac{\partial}{\partial x_i} q_i(x_i^r) = 0.$$

Assumption 1(ii) implies:

$$\left| \frac{\partial}{\partial x_i} f(x^r, y^r) \right| = r \left| q_i'(x_i^r) \right| > \frac{\Delta}{\gamma} \times \gamma = \Delta.$$

This is clearly a contradiction to Assumption 1(i). Therefore, for $r > r_0$, we have $x^r \in \{0, 1\}^n$. Thus, (x^r, y^r) is feasible for $(UMINLP)$ and $q(x^r) = 0$. Furthermore,

$$\begin{aligned} v(UPEN_r) &= H_r(x^r, y^r) \\ &= f(x^r, y^r) + rq(x^r) \\ &= f(x^r, y^r) \\ &\geq v(UMINLP). \end{aligned} \tag{4}$$

Relations (3) and (4) imply:

$$\begin{aligned} f(x^r, y^r) &= v(UMINLP) = v(UPEN_r) \\ &= H_r(x^r, y^r). \end{aligned}$$

Therefore, for any $r > r_0$, (x^r, y^r) is an optimal solution for both problems.

3. Exact Penalty Functions

The following penalty functions have been suggested [3, 9] for zero-one problems ($0 \leq x_i \leq 1$):

$$(q3) \quad q_i(x_i) = 4x_i(1-x_i),$$

$$(q4) \quad q_i(x_i) = \begin{cases} 2x_i, & x_i \leq \frac{1}{2}, \\ 2(1-x_i), & x_i > \frac{1}{2}, \end{cases}$$

$$(q5) \quad q_i(x_i) = \log(x_i + \theta) + \log(1 - x_i + \theta) - \log \theta - \log(1 + \theta),$$

$$\forall x_i \in \left(0, \frac{1}{3}\right), q'_i(x_i) > q'_i(1/3) > 0,$$

$$(q6) \quad q_i(x_i) = -(x_i + \theta)^{-p} - [(1 - x_i) + \theta]^{-p} + \theta^{-p} + (1 + \theta)^{-p},$$

and

$$\forall x_i \in \left(\frac{2}{3}, 1\right), q'_i(x_i) < q'_i(2/3) < 0.$$

$$(q7) \quad q_i(x_i) = 1 - \exp(-\alpha x_i) + 1 - \exp(-\alpha(1 - x_i)) - 1 - \exp(-\alpha),$$

$$(q8) \quad q_i(x_i) = x_i^q + (1 - x_i)^q - 1,$$

$$(q9) \quad q_i(x_i) = [1 + \exp(-\alpha x_i)]^{-1} + [1 + \exp(-\alpha(1 - x_i))]^{-1} - \frac{1}{2} - 1 - \exp(-\alpha)^{-1},$$

Therefore, Assumption 1(ii) is satisfied for (q3)–(q11). Thus, the penalty problem (UPEN_r) with any one of the functions (q3)–(q11) is exact for (UMINLP).

4. Extension to Constrained Problems

A constrained mixed integer nonlinear programming problem is expressed as:

$$(MINLP) \quad \min \quad f(x, y) \\ \text{s.t.} \quad g_j(x, y) \leq 0, \quad j = 1, \dots, l, \\ x \in \{0, 1\}^n, \quad y \in Y,$$

where, $\alpha, p > 0$, $0 < \theta < 1/2$ and $0 < q < 1$. Penalty functions (q5)–(q9) were introduced in [9]. Here, to have (1) satisfied, we add a fixed number to every function (q5)–(q9).

where, Y is a compact subset in \mathbb{R}^m .

Let $S = \{(x, y) \in \{0, 1\}^n \times Y \mid g_j(x, y) \leq 0, j = 1, \dots, l\}$.

Also, two other penalty functions for zero-one problems can be defined as follows:

$$(q10) \quad q_i(x_i) = \sin(\pi x_i),$$

$$(q11) \quad q_i(x_i) = x_i(1 - x_i)(x_i + \tau)(1 - x_i + \tau),$$

A penalty function for (MINLP) is defined as follows:

$$p(x, y) = \begin{cases} 0, & (x, y) \in S \\ > 0, & (x, y) \notin S. \end{cases}$$

where, $\tau > 0$.

A typical penalty function for the constraints g_j in (MINLP) is:

Note that any bounded MINLP problem can be reformulated as a mixed zero-one programming problem by using the following representation for the integer variables (see [7]):

$$x_i = \sum_{k=0}^M 2^k y_k^{(i)}, \quad y_k^{(i)} \in \{0, 1\}, \quad i = 1, \dots, n,$$

$$p(x, y) = \sum_{j=1}^l \max(0, g_j(x, y)). \quad (5)$$

Consider the penalty problem of (MINLP) as:

$$(PEN_{\mu}) \quad \min \quad H_{\mu}(x, y) = f(x, y) + \mu p(x, y) \\ \text{s.t.} \quad x \in \{0, 1\}^n, \quad y \in Y,$$

where, M is an upper integer bound for $\log x_i$. Thus, we can use the penalty functions for all bounded integer problems.

and define the following continuous penalty problem for (MINLP):

Also, note that direct use of penalty functions for MINLP problems (not zero-one) is not suitable, because due to the structure of the q_i (see (1)), the resulting nonconvex optimization problem, in general, may have many local minimizers (see [4]).

$$(PEN_{\mu,r}) \quad \min \quad H_{\mu,r}(x, y) = f(x, y) + \mu p(x, y) + r q(x) \\ \text{s.t.} \quad x \in [0, 1]^n, \quad y \in Y.$$

Now, we show that (UPEN_r) with the penalty functions (q3)–(q11) are exact for (UMINLP). Note that exactness of (q5)–(q9) have already been proved in [9]. Here, by using Theorem 1, we prove the exactness corresponding to all of (q3)–(q11).

To prove the exactness of (PEN_{μ,r}) for (MINLP), first we show that for some penalty function p , (PEN_μ) is exact for (MINLP). Note that exactness for some penalty functions, such as absolute-value penalty function, for the nonlinear programming (NLP) problems or nonlinear integer programming (NLIP) problems has already been proved (see [10,11]), that is, it has been shown that for a finite value of the penalty parameter, the main problem and the corresponding penalty problem are equivalent. Here, we prove exactness for the constrained

Let us suppose that f satisfies Assumption 1 1). We then need to show that Assumption 1 2) holds for every one of (q3)–(q11). For $\varepsilon = 1/3$, we have

$J_{\varepsilon} = (0, 1/3) \cup (2/3, 1)$. It is easy to show that for the functions (q3)–(q11), we have

MINLP problems.

Theorem 2. Consider (MINLP) and for every x , define the following NLP problem,

$$(NLP^x) \quad \begin{aligned} \min \quad & f(x, y) \\ \text{s.t.} \quad & g_j(x, y) \leq 0, j=1, \dots, l, \\ & y \in Y, \end{aligned}$$

and its corresponding penalty problem,

$$(PEN_\mu^x) \quad \begin{aligned} \min \quad & H_\mu(x, y) = f(x, y) + \mu p(x, y) \\ \text{s.t.} \quad & y \in Y. \end{aligned}$$

Suppose that for any x feasible to (NLP^x) , there exists a μ_x such that for every $\mu > \mu_x$, problems (NLP^x) and (PEN_μ^x) are equivalent. Then, there exists a μ_0 such that for every $\mu > \mu_0$, any solution of (PEN_μ) also solves (MINLP) and $v(PEN_\mu) = v(MINLP)$.

Proof. For any feasible point (x, y) of (MINLP), we have

$$H_\mu(x, y) = f(x, y) + \mu p(x, y) = f(x, y).$$

Since any feasible point for (MINLP) is also feasible for (PEN_μ) , it follows from the above equality that

$$v(MINLP) \geq v(PEN_\mu). \tag{6}$$

We know:

$$\begin{aligned} v(PEN_\mu) &= \min_{(x, y) \in \{0,1\}^n \times Y} H_\mu(x, y) \\ &= \min_{(x, y) \in \{0,1\}^n \times Y} f(x, y) + \mu p(x, y) \\ &= \min_{x \in \{0,1\}^n} \min_{y \in Y} f(x, y) + \mu p(x, y). \end{aligned}$$

For any fixed x , define

$$S_x = \{y \in Y \mid g_j(x, y) \leq 0, j=1, \dots, l\}.$$

Consider the following two cases.

Case 1: $S_x \neq \emptyset$. Consider the following problem:

$$\min_{y \in Y} f(x, y) + \mu p(x, y).$$

From the assumption of the theorem, there exists μ_x such that for any $\mu > \mu_x$, any solution of the above problem is also a solution of the following problem,

$$\min_{y \in S_x} f(x, y).$$

Case 2: $S_x = \emptyset$. From the definition of S_x , for any $y \in Y$, we have $p(x, y) > 0$. Since Y is compact, then

$$\min_{y \in Y} p(x, y) = \delta_x > 0.$$

Let \underline{f} be a lower bound of f on $\{0,1\}^n \times Y$, and $\mu_x = (v(MINLP) - \underline{f} + 1) / \delta_x$. For any $\mu > \mu_x$, we have

$$\begin{aligned} f(x, y) + \mu p(x, y) &> f(x, y) + \mu_x p(x, y) \\ &\geq \underline{f} + \mu_x \delta_x = v(MINLP) + 1. \end{aligned}$$

This means that if $\mu > \mu_x$, then $f(x, y) + \mu p(x, y) > v(MINLP) + 1$. Thus, (6) implies that if $\mu > \mu_x$, then minimum does not occur in this case. Now, let $\mu_0 = \max_{x \in \{0,1\}^n} \mu_x$. For any $\mu > \mu_0$, if (x^*, y^*) is an optimal solution of (PEN_μ) , then from the previous implication we get that Case 2 does not occur. From Case 1, we have

$$\begin{aligned} f(x^*, y^*) + \mu p(x^*, y^*) &= v(PEN_\mu) \\ &= \min_{x \in \{0,1\}^n} \min_{y \in Y} f(x, y) + \mu p(x, y) \\ &= \min_{x \in \{0,1\}^n} \min_{y \in S_x} f(x, y). \end{aligned}$$

Let \bar{f} be an upper bound of f on $\{0,1\}^n \times Y$. Then

$$f(x^*, y^*) + \mu p(x^*, y^*) \leq \bar{f}.$$

Since this relation holds for any $\mu > \mu_0$, we have $p(x^*, y^*) = 0$. Therefore, (x^*, y^*) is feasible to (MINLP). Furthermore,

$$\begin{aligned} f(x^*, y^*) &= v(PEN_\mu) \\ &= \min_{x \in \{0,1\}^n} \min_{y \in S_x} f(x, y) \\ &= \min_{x \in \{0,1\}^n, y \in Y, g_j(x, y) \leq 0} f(x, y) \\ &= v(MINLP). \end{aligned}$$

Thus, (x^*, y^*) is an optimal solution for both (PEN_μ) and (MINLP). \square

Theorem 2 shows that if p is an exact penalty function for an NLP problem, then it is also exact for the MINLP problem. Thus, (5) is an exact penalty function for (MINLP).

Now, we show that $(PEN_{\mu,r})$ is exact for (MINLP), that is, for finite penalty parameter values of r and μ , $(PEN_{\mu,r})$ and (MINLP) are equivalent.

Assumption 2. Assumption 1(i) holds for each g_j , namely, for each $(x, y) \in J_\varepsilon^n \times Y$, we have

$$\left| \frac{\partial}{\partial x_i} g_j(x, y) \right| < \Delta, i=1, \dots, n, j=1, \dots, l.$$

Theorem 3. Suppose that both Assumption 1 and Assumption hold and p is an exact penalty function for (MINLP). Then, there exist r_0 and μ_0 such that for every $r > r_0$ and $\mu > \mu_0$, any solution of $(PEN_{\mu,r})$ also solves (MINLP) and $v(PEN_{\mu,r}) = v(MINLP)$.

Proof. Since p is an exact penalty function for (MINLP), there exists a μ_0 such that for each $\mu > \mu_0$, any solution for $(PEN_{\mu,r})$ also solves (MINLP) and

$$v(PEN_{\mu}) = v(MINLP).$$

Theorem 1 on $(PEN_{\mu,r})$ implies that there exists an r_0 such that for every $r > r_0$, any solution of $(PEN_{\mu,r})$ is also a solution of (PEN_{μ}) and $v(PEN_{\mu,r}) = v(PEN_{\mu})$. Therefore, for every $\mu > \mu_0$ and $r > r_0$, any solution of $(PEN_{\mu,r})$ also solves $(MINLP)$ and $v(PEN_{\mu,r}) = v(MINLP)$. \square

5. Summary

We proposed an exact penalty approach for solving mixed integer nonlinear programming (MINLP) problems and showed how to convert a MINLP problem to a finite sequence of NLP problems. We stated conditions for exactness of a penalty function for MINLP problems and showed how exact penalty functions for NLP problems could serve as exact penalty functions for MINLP problems.

6. Acknowledgements

The second author thanks the support of Research Council of Sharif University of Technology.

7. References

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