

# Multi-Item Fuzzy Inventory Model Involving Three Constraints: A Karush-Kuhn-Tucker Conditions Approach

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## Abstract

In this paper, a multi-item inventory model with storage space, number of orders and production cost as constraints are developed in both crisp and fuzzy environment. In most of the real world situations the cost parameters, the objective functions and constraints of the decision makers are imprecise in nature. This model is solved with shortages and the unit cost dependent demand is assumed. Hence the cost parameters are imposed here in fuzzy environment. This model has been solved by Kuhn-Tucker conditions method. The results for the model without shortages are obtained as a particular case. The model is illustrated with numerical example.

**Keywords:** Multi-Item Inventory Model, Membership Function, Karush-Kuhn-Tucker Condition

## 1. Introduction

The literal meaning of inventory is the stock of goods for future use (production/sales). The control of inventories of physical goods is a problem common to all enterprises in any sector of an economy. The basic objective of inventory control is to reduce investment in inventories and ensuring that production process does not suffer at the same time.

In general the classical inventory problems are designed by considering that the demand rate of an item is constant and deterministic and that the unit price of an item is considered to be constant and independent in nature. But in practical situation, unit price and demand rate of an item may be related to each other. When the demand of an item is high, an item is produced in large numbers and fixed costs of production are spread over a large number of items. Hence the unit cost of the item decreases. i.e., the unit price of an item inversely relates to the demand of that item. So demand rate of an item may be considered as a decision variable.

Zadeh [1] first gave the concept of fuzzy set theory. Later on, Bellman and Zadeh [2] used the fuzzy set theory to the decision-making problem. Zimmerman [3] gave the concept to solve multi objective linear programming problem. Fuzzy set theory has made an entry into the inventory control systems. Sommer [4] applied the fuzzy concept to an inventory and production sched-

uling problem. Park [5] examined the EOQ formula in the fuzzy set theoretic perspective associating the fuzziness with the cost data. Hence we may impose warehouse space, cost parameters, number of orders, production cost etc, in fuzzy environment.

The Kuhn-Tucker conditions [6] are necessary conditions for identifying stationary points of a non linear constrained problem subject to inequality constraints. The development of this method is based on the Lagrangean method.

These conditions are also sufficient if the objective function and the solution space satisfy the conditions in the following **Table 1**.

The conditions for establishing the sufficiency of the Kuhn-Tucker conditions [7] are summarized in the following **Table 2**.

Kuhn-Tucker conditions, also known as Karush-Kuhn-Tucker (KKT) conditions was first developed by W. Karush in 1939 as part of his M.S. thesis at the University of Chicago. The same conditions were developed independently in 1951 by W. Kuhn and A. Tucker.

In this paper, a multi-item, multi-objective inventory problem with shortages along with three constraints such as limited storage space, number of orders and production cost has been formulated. The unit cost is considered here in fuzzy environment. The problem has been solved by KKT conditions method. This model is illustrated by numerical example [8-11].

**Table 1. The objective function and the solution space.**

Sense of optimization	Required conditions	
	Objective function	Solution space
Maximization	Concave	Convex Set
Minimization	Convex	Convex Set

**Table 2. The sufficiency of the Kuhn-Tucker conditions.**

Problem	Kuhn-Tucker conditions
1) Max $z = f(X)$ subject to $h^i(X) \leq 0$ $X \geq 0, i = 1, 2, \dots, m$	$\frac{\partial}{\partial x_j} f(X) - \sum_{i=1}^m \lambda_i \frac{\partial}{\partial x_j} h^i(X) = 0$ $\lambda_i h^i(X) = 0, h^i(X) \leq 0, i = 1, 2, \dots, m$ $\lambda_i \geq 0, i = 1, 2, \dots, m$
2) Min $z = f(X)$ subject to $h^i(X) \geq 0$ $X \geq 0, i = 1, 2, \dots, m$	$\frac{\partial}{\partial x_j} f(X) - \sum_{i=1}^m \lambda_i \frac{\partial}{\partial x_j} h^i(X) = 0$ $\lambda_i h^i(X) = 0, h^i(X) \geq 0, i = 1, 2, \dots, m$ $\lambda_i \geq 0, i = 1, 2, \dots, m$

**2. Assumptions and Notations**

A multi-item, multi-objective inventory model is developed under the following notations and assumptions.

**2.1. Notations**

- $n$  = number of items
- $t$  = number of orders
- $W$  = Floor (or) shelf-space available
- $B$  = Total investment cost for replenishment
- For  $i^{\text{th}}$  item: ( $i = 1, 2, \dots, n$ )
- $D_i = D_i(p_i)$  demand rate [function of unit cost price]
- $Q_i$  = lot size (decision variable)
- $M_i$  = Shortage level (decision variable)
- $S_i$  = Set-up cost per cycle
- $H_i$  = Inventory holding cost per unit item
- $m_i$  = Shortage cost per unit item
- $p_i$  = price per unit item (decision variable)
- $w_i$  = storage space per item
- $TC(p, Q, M)$  = expected annual total cost

**2.2. Assumptions**

- 1) replenishment is instantaneous
- 2) lead time is zero
- 3) demand is related to the unit price as

$$D_i = \frac{A_i}{p_i^{\beta_i}} = A_i p_i^{-\beta_i}$$

where  $A_i (>0)$  and  $\beta_i (0 < \beta_i < 1)$  are constants and real

numbers selected to provide the best fit of the estimated price function.  $A_i > 0$  is an obvious condition since both  $D_i$  and  $p_i$  must be non-negative.

**3. Formulation of Inventory Model with Shortages**

Let the amount of stock for the  $i^{\text{th}}$  item ( $i = 1, 2, \dots, n$ ) be  $R_i$  at time  $t = 0$ . In the interval  $(0, T_i (= t_{1i} + t_{2i}))$ , the inventory level gradually decreases to meet demands. By this process the inventory level reaches zero level at time  $t_{1i}$  and then shortages are allowed to occur in the interval  $(t_{1i}, T_i)$ . The cycle then repeats itself (**Figure 1**).

The differential equation for the instantaneous inventory  $q_i(t)$  at time  $t$  in  $(0, T_i)$  is given by

$$\begin{aligned} \frac{dq_i(t)}{dt} &= -D_i \text{ for } 0 \leq t \leq t_{1i} \\ &= -D_i \text{ for } t_{1i} \leq t \leq T_i \end{aligned} \tag{1}$$

with the initial conditions  $q_i(0) = R_i (= Q_i - M_i)$ ,  $q_i(T_i) = -M_i$ ,  $q_i(t_{1i}) = 0$ .

For each period a fixed amount of shortage is allowed and there is a penalty cost  $m_i$  per items of unsatisfied demand per unit time.

From (1)

$$\begin{aligned} q_i(t) &= R_i - D_i t \text{ for } 0 \leq t \leq t_{1i} \\ &= D_i (t_{1i} - t) \text{ for } t_{1i} \leq t \leq T_i \end{aligned}$$

So  $D_i t_{1i} = R_i$ ,  $M_i = D_i t_{2i}$ ,  $Q_i = D_i T_i$

$$\text{Holding cost} = H_i \int_0^{t_{1i}} q_i(t) dt = \frac{H_i (Q_i - M_i)^2}{2Q_i} T_i$$

$$\text{Shortage cost} = -m_i \int_{t_{1i}}^{T_i} q_i(t) dt = \frac{m_i M_i^2}{2Q_i} T_i$$

Production cost =  $p_i Q_i$

The total cost = Production cost + Set up cost

+ Holding cost + Shortage cost

$$= p_i Q_i + S_i + H_i \frac{(Q_i - M_i)^2}{2Q_i} T_i + \frac{m_i M_i^2}{2Q_i} T_i$$

The total average cost of the  $i^{\text{th}}$  item is

$$TC(p_i, Q_i, M_i) = p_i D_i + \frac{S_i D_i}{Q_i} + \frac{H_i (Q_i - M_i)^2}{2Q_i} + \frac{m_i M_i^2}{2Q_i}$$

$$\begin{aligned} TC_i(p_i, Q_i, M_i) &= A_i p_i^{1-\beta_i} + \frac{A_i S_i}{Q_i} p_i^{-\beta_i} \\ &+ \frac{H_i (Q_i - M_i)^2}{2Q_i} + \frac{m_i M_i^2}{2Q_i} \end{aligned} \tag{2}$$

for  $i = 1, 2, 3, \dots, n$ .

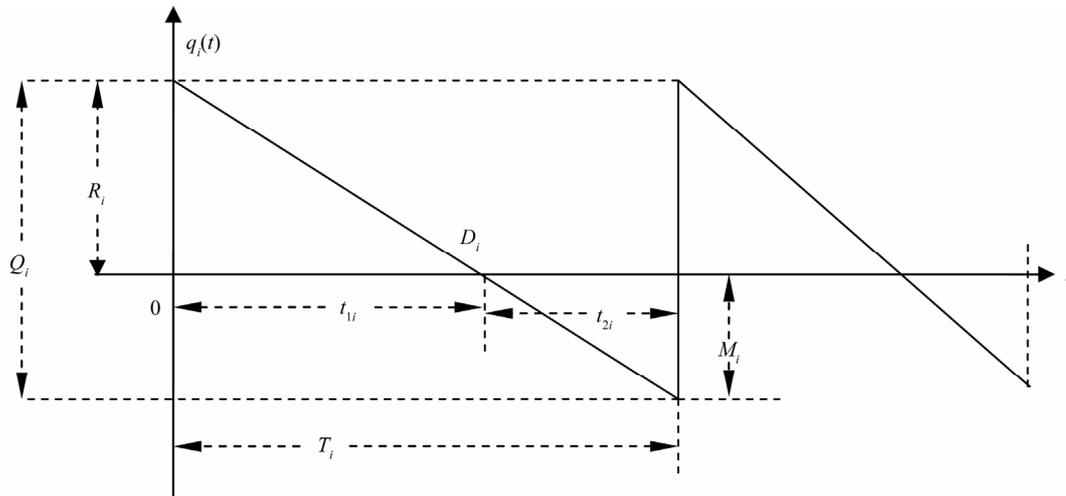


Figure 1. Inventory level of the  $i^{\text{th}}$  item.

There are some restrictions on available resources in inventory problems that cannot be ignored to derive the optimal total cost.

1) There is a limitation on the available warehouse floor space where the items are to be stored *i.e.*

$$\sum_{i=1}^n w_i Q_i \leq W ;$$

2) Investment amount on total production cost cannot be infinite, it may have an upper limit on the maximum investment. *i.e.*

$$\sum_{i=1}^n p_i Q_i \leq B ;$$

3) An upper limit on the number of orders that can be made in a time cycle on the system (*i.e.*)

$$\sum_{i=1}^n \frac{D_i}{Q_i} \leq t .$$

The problem is to find price per unit item, the lot size, the shortage amount so as to minimize the total average cost function (2) subject to the total space and total production cost restrictions.

It may be written as

$$\text{Min } TC_i(p_i, Q_i, M_i) \text{ for all } i = 1, 2, 3, \dots, n .$$

Subject to the inequality constraints

$$\sum_{i=1}^n w_i Q_i \leq W$$

$$\sum_{i=1}^n p_i Q_i \leq B$$

$$\sum_{i=1}^n \frac{D_i}{Q_i} \leq t$$

#### 4. Fuzzy Inventory Model with Shortages

When  $p_i^s$  are fuzzy decision variables, the above crisp

model under fuzzy environment reduces to

$$\text{Min } TC(\tilde{p}, Q, M) = \sum_{i=1}^n \left[ A_i \tilde{p}_i^{1-\beta_i} + \frac{A_i S_i}{Q_i} \tilde{p}_i^{-\beta_i} + \frac{H_i (Q_i - M_i)^2}{2Q_i} + \frac{m_i M_i^2}{2Q_i} \right] \leq t$$

subject to the constraints

$$\sum_{i=1}^n w_i Q_i \leq W$$

$$\sum_{i=1}^n \tilde{p}_i Q_i \leq B$$

$$\sum_{i=1}^n \tilde{p}_i^{1-\beta_i} A_i Q_i^{-1} \leq t$$

[Here cap ‘ $\sim$ ’ denotes the fuzzification of the parameters.]

The above fuzzy non-linear programming can be solved using Kuhn-Tucker conditions.

#### 4.1. Membership Function

The membership function for the fuzzy variable  $p_i$  is defined as follows

$$\mu_{p_i}(X) = \begin{cases} 1, & p_i \leq L_{L_i} \\ \frac{U_{L_i} - p_i}{U_{L_i} - L_{L_i}}, & L_{L_i} \leq p_i \leq U_{L_i} \\ 0, & p_i \geq U_{L_i} \end{cases}$$

Here  $U_{L_i}$  and  $L_{L_i}$  are upper limit and lower limit of  $p_i$  respectively.

**4.2. Fuzzy Inventory Model without Shortages**

When  $p_i^s$  are fuzzy decision variables, the above crisp model without shortages under fuzzy environment reduces to

$$\text{MinTC}(\tilde{p}, Q) = \sum_{i=1}^n \left[ A_i \tilde{p}_i^{1-\beta_i} + \frac{A_i S_i}{Q_i} \tilde{p}_i^{-\beta_i} + \frac{H_i Q_i}{2} \right]$$

subject to the constraints

$$\sum_{i=1}^n w_i Q_i \leq W$$

$$\sum_{i=1}^n \tilde{p}_i Q_i \leq B$$

$$\sum_{i=1}^n \tilde{p}_i^{-\beta_i} A_i Q_i^{-1} \leq t$$

**5. Numerical Example**

To solve the above non-linear programming using Kuhn-Tucker conditions, the following values are assumed.

$$\begin{aligned} n = 1, t = 3, A_1 = 100, S_1 = \$100, H_1 = \$1, \\ w_1 = 2 \text{ sq. ft.}, W = 150 \text{ sq. ft.}, B = \$1200, \\ m_1 = \$1 \text{ and } \$10 \leq p_1 \leq \$20 \end{aligned}$$

By the method of Kuhn-Tucker conditions, consider the four cases

- 1)  $\lambda_1 = 0, \lambda_2 = 0$
- 2)  $\lambda_1 \neq 0, \lambda_2 = 0$
- 3)  $\lambda_1 = 0, \lambda_2 \neq 0$
- 4)  $\lambda_1 \neq 0, \lambda_2 \neq 0$

Here Kuhn-Tucker conditions are used as trial and error method by taking different values for  $\beta_1$  until an optimum result is obtained.

Optimal solutions for the fuzzy model with shortages

$\beta_1$	$p_1$	$\mu_{p_1}$ value	$Q_1$	$D_1$	$M_1$	Expected Total cost
0.88	10.179	0.982	72.046	12.978	36.023	168.133
0.89	12.406	0.759	65.220	10.85	32.609	164.484
0.90	15.405	0.4595	58.428	8.533	29.214	160.664
0.91	19.561	0.0439	51.693	6.681	25.847	156.533

From the above table it follows that 10.179 has the maximum membership value 0.982.

Hence the required optimum solution is

$$p_1 = 10.179, Q_1 = 72.046, M_1 = 36.023$$

Minimum expected Total cost = \$168.133.

Optimal solutions for the fuzzy model without shortages

$\beta_1$	$p_1$	$\mu_{p_1}$ value	$Q_1$	$D_1$	Expected Total cost
0.89	10.788	0.921	75.000	12.042	183.460
0.85	11.178	0.882	50.693	12.850	194.327
0.92	16.00	0.400	75.000	07.802	172.733

From the above table it follows that 10.788 has the maximum membership value 0.921.

Hence the required optimum solution is

$$p_1 = 10.788, Q_1 = 75$$

Minimum expected Total cost = \$183.460.

**6. Conclusions**

In this paper we have proposed a concept of the optimal solution of the inventory problem with fuzzy cost price per unit item. Fuzzy set theoretic approach of solving an inventory control problem is realistic as there is nothing like fully rigid in the world. By solving the above fuzzy inventory model using Kuhn-Tucker condition method we have the values of imprecise variable for decision making. The above discussed model can be developed with many limitations, such as their inventory level, Warehouse space and budget limitations, etc.

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