

Optimal Policy and Simple Algorithm for a Deteriorated Multi-Item EOQ Problem

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Received March 29, 2011; revised April 19, 2011; accepted May 12, 2011

Abstract

This paper considers a deteriorated multi-item economic order quantity (EOQ) problem, which has been studied in literature, but the algorithms used in the literature are limited. In this paper, we explore the optimal policy of this inventory problem by analyzing the structural properties of the model, and introduce a simple algorithm for solving the optimal solution to this problem. Numerical results are reported to show the efficiency of the proposed method.

Keywords: Inventory, EOQ, Deterioration, Multi-Item

1. Introduction

Multi-item inventory problem with resource constraints is an important topic of inventory management [1]. These constrained inventory models are still hot topics in academic and practice fields, for example, see [2-6]. The resource constraints typically arise from shipment capacity, warehouse capacity or budgetary limitation. Since some items such as fruits, vegetables, food items, drugs and fashion goods will deteriorate in the shipment or storage process, many works have been done for investigating inventory problems for deterioration items [7-11]. Since items' deterioration often takes place during the storage period, some researchers have considered economic order quantity (EOQ) models for deteriorating items, for examples see [12,13].

Recently, Mandal *et al.* [14] present a constrained multi-item EOQ model with deteriorated items. In [14], the model is firstly formulated as the transcendental form and the polynomial form, *i.e.*, without and with truncation on the deterioration terms. These two versions of the model are both solved by applying non-linear programming (NLP) method (Lagrangian multiplier method). As [14] points out, the polynomial form is an approximation of the transcendental form. Secondly, the transcendental form is converted to the minimization of a signomial expression with a posynomial constraint, which is solved by applying a modified geometric programming (MGP) method. However, we argue that the studied problem can be solved using a simple algorithm without any model

approximation or conversion.

In this paper, we prove that the deteriorated multi-item EOQ model is a special convex separable nonlinear knapsack problem studied in [15], which is characterized by positive marginal cost (PMC) and increasing marginal loss-cost ratio (IMLCR). PMC requires positive marginal cost of decision variable, and IMLCR means that the marginal loss-cost ratio is increasing in decision variable. Following [15], we explore the optimal policy for the problem, and develop a simple algorithm for solving it. The main purpose of this paper is twofold: 1) to explore the optimal policy of this inventory problem by analyzing the structural properties of the model; 2) to introduce a simple algorithm for solving the optimal solution to this problem.

The remainder of this paper is organized as follows. We formulate the problem in the next section. In Section 3, we explore the structural properties of the problem, and provide the optimal policy and algorithm. Numerical results are reported in Section 4, and Section 5 briefly concludes this paper.

2. Problem Formulation

Consider a multi-item EOQ problem with a storage space constraint, in which all items ($i = 1, \dots, n$) deteriorate after certain periods.

Before presenting the model, we list all notation as follows. Notice that the same notation used in [14] is presented.

- n = Total number of items
- Q_i = Order quantity
- c_{0i} = Purchasing cost
- c_{1i} = Holding cost per unit quantity per unit time
- c_{3i} = Set-up cost
- θ_i = Constant rate of deterioration ($0 < \theta_i < 1$)
- w_i = Required storage quantity per unit time
- D_i = Demand rate per unit time
- T_i = Time period of each cycle
- $TC(T_1, \dots, T_n)$ = Total average cost function
- W = Available storage space

Following [14], we set $a_i = c_{0i}D_i/\theta_i + c_{1i}D_i/\theta_i^2 - c_{3i}$, $b_i = c_{0i}D_i/\theta_i + c_{1i}D_i/\theta_i^2 > 0$, and $c_i = c_{1i}D_i/\theta_i$. Then the deteriorated multi-item EOQ model can be expressed as follows (denoted as problem P). We refer the reader to [14] for the details of this model.

$$\begin{aligned} \min \quad & TC(T_1, \dots, T_n) \\ = \quad & \sum_{i=1}^n f_i(T_i) = \sum_{i=1}^n \left[-\frac{a_i}{T_i} + b_i \frac{e^{\theta_i T_i}}{T_i} - c_i \right], \end{aligned} \tag{1}$$

subject to

$$\sum_{i=1}^n g_i(T_i) = \sum_{i=1}^n (w_i \frac{D_i}{\theta_i} (e^{\theta_i T_i} - 1)) \leq W, \tag{2}$$

$$T_i > 0, \quad i = 1, \dots, n. \tag{3}$$

The order quantity is given by $Q_i = \frac{D_i}{\theta_i} (e^{\theta_i T_i} - 1)$,

$i = 1, \dots, n$. The first and second order derivatives of $f_i(T_i)$, $i = 1, \dots, n$, and the first order derivative of $g_i(T_i)$, $i = 1, \dots, n$, are calculated as follows:

$$\frac{df_i(T_i)}{dT_i} = \frac{a_i + b_i e^{\theta_i T_i} (-1 + \theta_i T_i)}{T_i^2}, \tag{4}$$

$$\frac{d^2 f_i(T_i)}{dT_i^2} = \frac{-2a_i + b_i e^{\theta_i T_i} (2 - 2\theta_i T_i + \theta_i^2 T_i^2)}{T_i^3}, \tag{5}$$

$$\frac{dg_i(T_i)}{dT_i} = D_i w_i e^{\theta_i T_i}. \tag{6}$$

3. Structural Properties and Algorithm

In this section, we first establish structural properties of problem P , and then we present an efficient procedure for solving the optimal solution to problem P .

3.1. Structural Properties

Before presenting the structural properties of problem P , we give two basic equations, which will be used in our proofs. Since Taylor expansion of exponential function is

$$e^{\theta_i T_i} = 1 + \sum_{k=1}^{+\infty} \frac{\theta_i^k T_i^k}{k!}, \text{ then we have}$$

$$1 + \theta_i T_i < 1 + \theta_i T_i + \theta_i^2 T_i^2 / 2 < e^{\theta_i T_i}, \tag{7}$$

for $T_i > 0, \quad i = 1, \dots, n$.

By comparing the definitions of a_i and b_i , we have

$$a_i < b_i, \quad i = 1, \dots, n. \tag{8}$$

Considering the objective function of problem P , we have the following proposition.

Proposition 1. *The cost $TC(T_1, \dots, T_n)$ is strictly convex.*

Proof. Since the function $TC(T_1, \dots, T_n)$ is separable, we only need to prove that $f_i(T_i)$ is strictly convex in $T_i, \quad i = 1, \dots, n$. According to Equations (7) and (8), we have.

$$\begin{aligned} & -2a_i + b_i e^{\theta_i T_i} (2 - 2\theta_i T_i + \theta_i^2 T_i^2) \\ & > -2b_i + 2b_i \left(1 + \theta_i T_i + \frac{\theta_i^2 T_i^2}{2} \right) \left(1 - \theta_i T_i + \frac{\theta_i^2 T_i^2}{2} \right) = \frac{b_i \theta_i^4 T_i^4}{2}. \end{aligned}$$

Substituting the above equation into Equation (5), we have.

$$\frac{d^2 f_i(T_i)}{dT_i^2} > \frac{b_i \theta_i^4 T_i^4}{2T_i^3} = \frac{b_i \theta_i^4 T_i}{2} > 0 \text{ for } T_i > 0.$$

Thus, $f_i(T_i)$ and $TC(T_1, \dots, T_n)$ are strictly convex.

QED.

The strictly convexity of $TC(T_1, \dots, T_n)$ ensures that the optimal solution to problem P is unique. This property has also been indicated in [14] by using a more complicated proof procedure.

Denote $\text{ProductLog}(z)$ as the principal solution for x in $z = xe^x$, which has the same function name in Mathematica to stand for the Lambert W function. Let $\hat{T}_i, \quad i = 1, \dots, n$, be a value such that $df_i(T_i)/dT_i = 0$. Denote problem CP as problem P without the constraint in Equation (2), then the following proposition characterizes the optimal solution to problem CP.

Proposition 2. *The optimal solution to problem CP is*

$$\hat{T}_i = \frac{1 + \text{ProductLog}\left(-\frac{a_i}{b_i e_i}\right)}{\theta_i}, \quad i = 1, \dots, n. \tag{9}$$

Proof. Since $df_i(T_i)/dT_i = 0$ at the point \hat{T}_i , we have $a_i + b_i e^{\theta_i \hat{T}_i} (-1 + \theta_i \hat{T}_i) = 0$. This equation can be rewritten as $-\frac{a_i}{b_i e} = e^{\theta_i \hat{T}_i - 1} (\theta_i \hat{T}_i - 1)$, and hence we have

$$\hat{T}_i = \frac{1 + \text{ProductLog}\left(-\frac{a_i}{b_i e_i}\right)}{\theta_i}.$$

From Equation (8), we know $-\frac{1}{e_i} < -\frac{a_i}{b_i e_i}$. According

to [16], we know that $\text{ProductLog}(z)$ is an increasing function for $z \geq -\frac{1}{e}$, then we have $\text{ProductLog}(-\frac{a_i}{b_i e_i}) > \text{ProductLog}(-\frac{1}{e_i}) = -1$. Substituting this equation into \hat{T}_i , we have hence $\hat{T}_i > 0$, which satisfies the positive constraint in Equation (3). Thus, $\hat{T}_i, i = 1, \dots, n$, is an optimal solution to problem CP. Since the optimal solution is unique, we know that the optimal solution to problem CP is $\hat{T}_i, i = 1, \dots, n$.

QED.

Following [15], we define the marginal loss-cost ratio of item $i (i = 1, \dots, n)$ as.

$$r_i(T_i) = \frac{\frac{df_i(T_i)}{dT_i}}{\frac{dg_i(T_i)}{dT_i}} = \frac{a_i e^{-\theta_i T_i} + b_i \theta_i T_i - b_i}{D_i w_i T_i^2} \quad (10)$$

Then we have the following proposition.

Proposition 3. $r_i(T_i)$ is strictly increasing in $T_i \in (0, \hat{T}_i], i = 1, \dots, n$.

Proof. From Equation (7), we know $1 + \theta_i T_i < e^{\theta_i T_i}$ for $T_i > 0$. The convexity of $f_i(T_i)$ guarantees $a_i + b_i e^{\theta_i T_i} (-1 + \theta_i T_i) \leq 0$ for $T_i \in (0, \hat{T}_i]$. Using $a_i < b_i$ in Equation (8) and the above two equations, we have

$$\begin{aligned} & b_i e^{\theta_i T_i} (-2 + \theta_i T_i) + a_i (2 + \theta_i T_i) \\ &= [a_i + b_i e^{\theta_i T_i} (-1 + \theta_i T_i)] + [a_i (1 + \theta_i T_i) - b_i e^{\theta_i T_i}] \quad , \\ &< [a_i + b_i e^{\theta_i T_i} (-1 + \theta_i T_i)] + b_i [(1 + \theta_i T_i) - e^{\theta_i T_i}] < 0 \end{aligned}$$

for $T_i \in (0, \hat{T}_i]$. Hence we have

$$\frac{dr_i(T_i)}{dT_i} = \frac{-e^{-\theta_i T_i} [b_i e^{\theta_i T_i} (-2 + \theta_i T_i) + a_i (2 + \theta_i T_i)]}{DT_i^3 w} > 0.$$

Thus, $r_i(T_i)$ is strictly increasing in $T_i \in (0, \hat{T}_i], i = 1, \dots, n$.

QED

Since this proposition illustrates that $r_i(T_i)$ is strictly increasing in $T_i \in (0, \hat{T}_i], i = 1, \dots, n$, we let $T_i(r_i), r_i \in (-\infty, 0]$, be the inverse function of $r_i(T_i)$. Although it is difficult to write $T_i(r_i)$ in a closed form, the strict monotony of $T_i(r_i)$ ensures that $T_i(r_i)$ can be easily determined by applying a bi-section search procedure over $T_i \in (0, \hat{T}_i]$, for any given $r_i \in (-\infty, 0]$.

3.2. Optimal Policy and Algorithm

We now demonstrate that the deteriorated multi-item EOQ model is a special convex separable nonlinear knapsack problem studied in [15]. Firstly, Proposition 1 illustrates that P is a convex problem; Secondly, from Equation (6), we know $\frac{dg_i(T_i)}{dT_i} > 0$, for $T_i > 0, i = 1, \dots, n$,

which means there are positive marginal costs in problem P ; Finally, Proposition 3 ensures that the marginal loss-cost ratio $r_i(T_i)$ is increasing in $T_i \in (0, \hat{T}_i], i = 1, \dots, n$. Therefore, the theoretical results and solution procedure proposed in [15] are both applicable for problem P .

Denote by $T_i^*, i = 1, \dots, n$, the optimal solution to problem P . By directly applying the theoretical results in [15], we can summarize the optimal policy for the deteriorated multi-item EOQ problem in the following proposition.

Proposition 4. The optimal policy of problem P is (a)

$T_i^* = \hat{T}_i, i = 1, \dots, n$, if $\sum_{i=1}^n g_i(\hat{T}_i) \leq W$; (b) $\sum_{i=1}^n g_i(T_i^*) = W$ and $r_i(T_i^*) = r_k(T_k^*), i, k = 1, \dots, n$, specify the optimal solution $T_i^*, i = 1, \dots, n$, if $\sum_{i=1}^n g_i(\hat{T}_i) > W$.

This proposition is obtained by directly applying the theoretical results in [15] to problem P , since problem P has PMC and IMLCR. Based on Propositions 2-4, the idea of the algorithm proposed in [15] can be used for solving problem P . The basic idea of the algorithm is as follows: If the constraint in Equation (2) is inactive, i.e., $\sum_{i=1}^n g_i(\hat{T}_i) \leq W$, then the optimal solution to problem P equals to the optimal solution to the unconstrained problem, i.e., $T_i^* = \hat{T}_i, i = 1, \dots, n$; Otherwise, Proposition 4(b) means that obtaining the exact value of $r^* = r_i(T_i^*) = r_k(T_k^*), i, k = 1, \dots, n$, is the key to solving the optimal solution to problem P . The optimal value r^* can be searched by applying a binary search method over the interval $r \in (-M, 0]$, where M is a sufficient large value such that $\sum_{i=1}^n (w_i \frac{D_i}{\theta_i} (e^{\theta_i T_i(M)} - 1)) \leq W$.

Main steps of the above solution procedure for solving the optimal solution to problem P are summarized in the following algorithm.

The Algorithm

Step 1: Solve $\hat{T}_i, i = 1, \dots, n$, from Equation (9);

Step 2: If $\sum_{i=1}^n g_i(\hat{T}_i) \leq W$, then

let $T_i^* = \hat{T}_i, i = 1, \dots, n$, go to Step 8;

Step 3: Let $r_L = -M, r_U = 0$;

Step 4: Let $r = (r_L + r_U)/2$;

Step 5: Calculate $T_i = T_i(r), i = 1, \dots, n$;

Step 6: If $\sum_{i=1}^n g_i(T_i) < W$, then let $r_L = r$;

If $\sum_{i=1}^n g_i(T_i) > W$, then let $r_U = r$;

Go to Step 4;

Step 7: Let $T_i^* = T_i, i = 1, \dots, n$;

Step 8: Calculate $TC(T_1^*, \dots, T_n^*)$, and output.

In comparison with the two methods in [14], there are two main advantages of our algorithm: 1) It is a polynomial algorithm of $O(n)$ order, which ensures that the algorithm is applicable for large-scale problems; 2) It does not need any approximation or conversion of the original model, thus it always solves the optimal solution to problem P .

4. Numerical Results

In this section, numerical experiments are provided to show the efficiency of the proposed algorithm for solving problem P . The instances of problem P are all randomly generated. We use the notation $x \sim U(\alpha, \beta)$ to denote that x is uniformly generated over $[\alpha, \beta]$. The parameters of test instances are generated as follows: $w_i \sim U(1,10), c_{0i} \sim U(1,10), c_{1i} \sim U(0.5,1.0), c_{3i} \sim U(40,100), \theta_i \sim U(0.01,0.10), D_i \sim U(200,500), i = 1, \dots, n$, and $W = 100 \times n$.

In this numerical study, we set $n=100$ and 1000 , respectively. For each problem size n , 100 test instances are randomly generated. The statistical results on number of iterations of the binary search and computation time (in milliseconds) are reported in **Table 1**, where 95% C.I. stands for 95% confidence interval.

From **Table 1**, we can conclude that the proposed algorithm can solve large-scale deteriorated multi-item EOQ models very quickly in few iteration times. Since the ranges of parameters are large, the standard deviations of number of iterations and computation time are quite low, reflecting the fact that the algorithm is quite effective and robust.

We also use our algorithm to solve the illustrative example studied in [14], which outputs the optimal solution: $T_1^* = 0.2899, T_2^* = 0.2176, Q_1^* = 102.6397, Q_2^* = 98.6801, TC^* = 2587.1382$ with $r^* = -0.5925$. Unfortunately, this result cannot be directly compared with that solved by [14], because there is something wrong with the values of T_i^* and $Q_i^*, i = 1, 2$, shown in **Tables 2** and **3** of [14], since they violate the basic equation

Table 1. Performance of our algorithm for solving the randomly generated instances.

		Number of iterations		Computation time	
Problem size n		100	1000	100	1000
	Mean	27.9	31.8	217.0	2619.3
	Std. Dev.	1.7	1.7	8.9	38.8
95%	Lower	27.6	31.4	215.2	2611.6
C.I.	Upper	28.3	32.1	218.8	2627.0

$$Q_i = \frac{D_i}{\theta_i}(e^{\theta_i T_i} - 1), i = 1, 2.$$

For example, in the **Table 2** of [14],

when $T_1^* = 0.2414712$ and $T_2^* = 0.2419020, Q_i = \frac{D_i}{\theta_i}(e^{\theta_i T_i} - 1)$ gives $Q_1 = 85.3365, Q_2 = 109.7828$. In addition, their mistake can also be verified by our proved optimal policy $r_1^*(T_1^*) = r_2^*(T_2^*)$. For example, the values of $r_i^*(T_i^*), i = 1, 2$ for the MGP solution presented in **Table 3** of [14] are $r_1^*(T_1^*) = -0.6017$, and $r_2^*(T_2^*) = -0.5857$, which does not satisfy $r_1^*(T_1^*) = r_2^*(T_2^*)$.

From the above analysis, we illustrate that the solution provided in [14] does not satisfy the optimal policy proved in this paper, which is easily used to verify the optimality of a solution to problem P . Thus, the numerical results in [14] are incorrect. Since some comparison of NLP and MGP given by [14] were established based on the numerical results, especially the results in **Tables 2** and **3** of [14], we argue that the comparison of NLP and MCP in [14] are questionable.

5. Conclusions

In this paper, we explore the structural properties of deteriorated multi-item EOQ model and propose a simple algorithm for solving the optimal solution by proving that the studied problem is a special convex separable nonlinear knapsack problem. In addition, it is obvious that the basic idea and obtained results in this paper can be simply modified for solving the classical constrained multi-item EOQ problem.

6. Acknowledgements

The authors would like to thank the two reviewers for their insightful comments, which helped to improve the manuscript. This work is supported by national Natural Science Foundation of China (No. 70801065), the Fundamental Research Funds for the Central Universities of China and Natural Science Foundation of Guangdong Province, China (No. 10451027501005059).

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