

Minimum Diameter Spanning Tree

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Abstract

In this paper, we discuss the simple connected graphs which have a minimum diameter spanning tree such that both have same domination number.

Keywords

Diameter, Spanning Tree, Essential Dominating Vertex, Pan Graph, Ladar Graph

1. Preliminaries

E.J. Cockayne and S.T. Hedetniemi [1] introduced the concept dominating set. Frank Harary, Robert Z. Norman and Dorwin Cartwright [2] explained an interesting application in voting situations using the concept of domination. C.L. Liu [3] also discussed the application of dominance to communication network, where a dominating set represents a set of cities which acting as transmitting stations, can transmit messages to every city in the network.

A subset S of vertices from V is called a dominating set for G if every vertex of G is either a member of S or adjacent to a member of S . A dominating set of G is called a minimum dominating set if G has no dominating set of smaller cardinality. The cardinality of minimum dominating set of G is called the dominating number for G and it is denoted by $\gamma(G)$ [4].

The eccentricity of a graph vertex v in a connected graph G is the maximum graph distance between v and any other vertex u of G . The eccentricity $e(v)$ of v is defined by $e(v) = \max\{d(u, v) : u \in V(G)\}$. The radius of a graph $r(G)$ is defined by $r(G) = \min\{e(v) : v \in V(G)\}$. The diameter of a graph $d(G)$ is defined by $d(G) = \max\{e(v) : v \in V(G)\}$. v is called a centre point if $e(v) = r(G)$ and the set of all central point is called the centre of G . A spanning tree is a subset of a graph G , which has all the vertices covered with minimum possible number of edges, hence a spanning tree doesn't have a cycle and it can't be disconnected. A spanning tree of a graph $G(V, E)$ with minimum diameter

is called the minimum diameter spanning tree.

M. Yamuna and K. Karthika [5] provided a constructive procedure to generate a spanning tree for any graph from its dominating set, γ -set.

In this paper, we discuss few simple connected graphs for which the domination numbers of the graph and that of its minimum diameter spanning tree are the same.

2. Some Definitions

In this section, we present few definitions and examples which are required for this article.

Definition 1. The spanning tree T of the simple connected graph G is said to be a minimum diameter spanning tree if there is no other spanning tree T' of G such that $d(T') < d(T)$.

Definition 2. A vertex u of a simple connected graph is an essential dominating vertex, if every minimum dominating set contains the vertex u .

Example 1. In the graph given in **Figure 1**, the vertex u is an essential dominating vertex.

Definition 3. [6] The pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge and we denote it by \mathcal{P}_n .

Example 2. The graphs given in **Figure 2** are pan graphs.

Definition 4. [6] The ladder graph L_n is a planar undirected graph with $2n$ vertices and $n + 2(n - 1)$ edges.

Example 3. The graph given in **Figure 3** is a ladder graph L_7 .

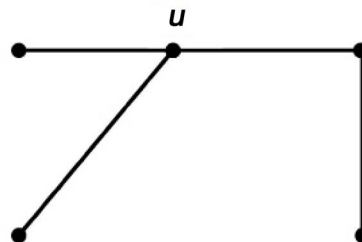


Figure 1. A Simple connected graph with essential dominating vertex u .

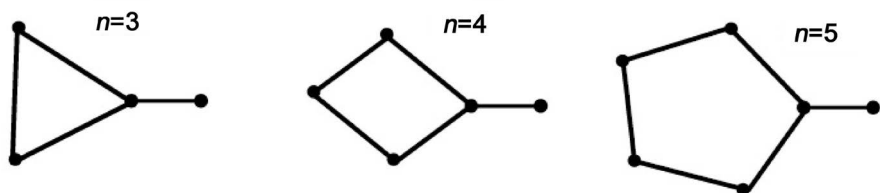


Figure 2. Pan Graphs.

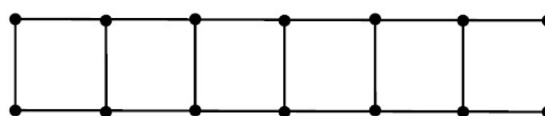


Figure 3. Ladar Graph L_7 .

3. Some Results

In this section we discuss the graphs which have minimum diameter spanning trees such that both have same domination number. It is clear to observe that cycle, regular graph, complete bipartite graph have minimum diameter spanning tree, possessing domination number same that of the graph.

Proposition 1. *Every pan graph has at least one essential dominating vertex. In fact, the vertex, which is adjacent to the pendent vertex is an essential dominating vertex.*

Proof: By definition 3 of a pan graph, it is a cycle in which one of the nodes in the cycle is adjacent to a node which is not in the cycle. Let u be the such vertex in the cycle, which is adjacent to the pendent vertex v . Since u is in the cycle, u is also adjacent to two other vertices, say x and y , in the cycle. Clearly, the vertex u dominates the three vertices v , x and y . Suppose u is not an essential dominating set, then there must exist another vertex in the cycle, which dominates the four vertices u , v , x and y . But obviously, no such vertex exists. Hence, u is an essential dominating vertex.

Let us recall the following results $\gamma(\mathcal{P}_n) = \left\lfloor \frac{n+2}{3} \right\rfloor$ and $\gamma(P_n) = \left\lfloor \frac{n+2}{3} \right\rfloor$

Proposition 2. *A pan graph \mathcal{P}_n has a minimum diameter spanning tree T for which $\gamma(\mathcal{P}_n) = \gamma(T)$ if $n \not\equiv 0$ or $5 \pmod{6}$*

Proof: Consider \mathcal{P}_n where $n > 3$. To obtain a spanning tree of \mathcal{P}_n , we must remove an edge from the cycle. To obtain a minimum diameter spanning tree T , we must remove an edge which is far away from the pendant vertex. In other words, the path P_{n-3} , obtained after the removal of the vertex u , must be separated in to two paths of length equal or one path has length 1 more than the other. Now, let us discuss the following cases.

Case 1: $n \equiv 0 \pmod{6}$

Let $n = 6m$ for some positive integer. Then $n - 3 = 6m - 3$. Therefore the path $P_{n-3} = P_{6m-3}$ can be separated into two paths P_{3m-2} and P_{3m-1} . Since

$$\gamma(P_{3m-2}) = \left\lfloor \frac{3m}{3} \right\rfloor = m \quad \text{and} \quad \gamma(P_{3m-1}) = \left\lfloor \frac{3m+1}{3} \right\rfloor = m, \quad \text{we have } \gamma(T) = 2m + 1.$$

$$\text{But } \gamma(\mathcal{P}_n) = \left\lfloor \frac{6m+2}{3} \right\rfloor = 2m.$$

Case 2: $n \equiv 1 \pmod{6}$

Let $n = 6m + 1$ for some positive integer. Then $n - 3 = 6m - 2$. Therefore the path $P_{n-3} = P_{6m-2}$ can be separated into two equal paths P_{3m-1} . But

$$\gamma(P_{3m-1}) = \left\lfloor \frac{3m+1}{3} \right\rfloor = m \quad \text{and hence we have } \gamma(T) = 2m + 1. \quad \text{Also}$$

$$\gamma(\mathcal{P}_n) = \left\lfloor \frac{6m+3}{3} \right\rfloor = 2m + 1.$$

Case 3: $n \equiv 2 \pmod{6}$

Let $n = 6m + 2$ for some positive integer. Then $n - 3 = 6m - 1$. Therefore the path $P_{n-3} = P_{6m-1}$ can be separated into two paths P_{3m-1} and P_{3m} . But

$$\gamma(P_{3m-1}) = \left\lfloor \frac{3m+1}{3} \right\rfloor = m \quad \text{and} \quad \gamma(P_{3m}) = \left\lfloor \frac{3m+2}{3} \right\rfloor = m, \quad \text{and hence we have}$$

$$\gamma(T) = 2m + 1. \quad \text{Also} \quad \gamma(\mathcal{P}_n) = \left\lfloor \frac{6m+4}{3} \right\rfloor = 2m + 1.$$

Case 4: $n \equiv 3 \pmod{6}$

Let $n = 6m + 3$ for some positive integer. Then $n - 3 = 6m$. Therefore the path $P_{n-3} = P_{6m}$ can be separated into two equal paths P_{3m} . But

$$\gamma(P_{3m}) = \left\lfloor \frac{3m+2}{3} \right\rfloor = m, \quad \text{hence} \quad \gamma(T) = 2m + 1. \quad \text{Also} \quad \gamma(\mathcal{P}_n) = \left\lfloor \frac{6m+5}{3} \right\rfloor = 2m + 1.$$

Case 5: $n \equiv 4 \pmod{6}$

Let $n = 6m + 4$ for some positive integer. Then $n - 3 = 6m + 1$. Therefore the path $P_{n-3} = P_{6m+1}$ can be separated into two paths P_{3m} and P_{3m+1} . But

$$\gamma(P_{3m}) = \left\lfloor \frac{3m+2}{3} \right\rfloor = m \quad \text{and} \quad \gamma(P_{3m+1}) = \left\lfloor \frac{3m+3}{3} \right\rfloor = m + 1, \quad \text{hence} \quad \gamma(T) = 2m + 2.$$

$$\text{Also} \quad \gamma(\mathcal{P}_n) = \left\lfloor \frac{6m+6}{3} \right\rfloor = 2m + 2.$$

Case 6: $n \equiv 5 \pmod{6}$

Let $n = 6m + 5$ for some positive integer. Then $n - 3 = 6m + 2$. Therefore the path $P_{n-3} = P_{6m+2}$ can be separated into two equal paths P_{3m+1} . But

$$\gamma(P_{3m+1}) = \left\lfloor \frac{3m+3}{3} \right\rfloor = m + 1, \quad \text{hence we have} \quad \gamma(T) = 2m + 3. \quad \text{But}$$

$$\gamma(\mathcal{P}_n) = \left\lfloor \frac{6m+7}{3} \right\rfloor = 2m + 2.$$

Hence the proof of the proposition follows.

Proposition 3. *The ladder graphs L_n have minimum diameter spanning tree T such that $\gamma(L_n) = \gamma(T)$, only when $n \leq 3$ or n is an even integer.*

Proof: In a ladder graph L_n , the $2n$ vertices are divided into two sets of each n vertices and each vertex of one set is adjacent to exactly one vertex of the other set and vice-versa. We may call these edges as *step edges*. Further, each of the n vertices are connected by a path on $n - 1$ edges. We may note that the diameter of a minimum diameter spanning tree of L_n is n , when n is odd and $n + 1$, when n is even. In fact when n is odd, L_n has a unique minimum diameter spanning tree, which is obtained by deleting all the step edges, except the middle one. When n is even, there may be several minimum diameter spanning trees. The **Figure 4** & **Figure 5** respectively illustrate the minimum diameter spanning trees of L_5 and L_6 .

$$\text{Further} \quad \gamma(L_n) = 1 + \left\lfloor \frac{n}{2} \right\rfloor.$$

Now let us consider the two cases on n

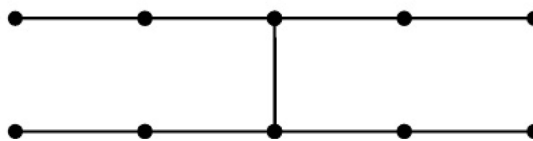


Figure 4. Unique minimum diameter spanning tree of L_5 .

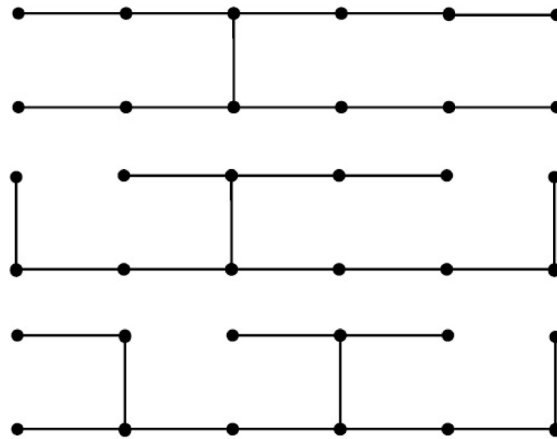


Figure 5. Minimum diameter spanning trees of L_6 .

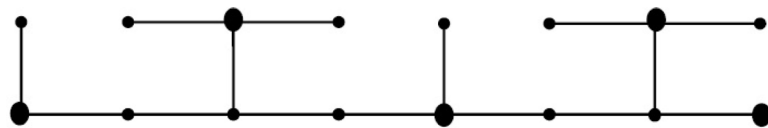


Figure 6. A Minimum diameter spanning trees of L_n with Equal dominating numbers.

Case:1 n is odd.

When n is odd, the domination number of the unique minimum diameter spanning tree of L_n is given by $2 \times \lfloor \frac{n}{2} \rfloor$. But this is equal to $1 + \lfloor \frac{n}{2} \rfloor$ only when $n = 3$.

Case:2 n is even.

When n is even, the domination number of the minimum diameter spanning tree of L_n is given by $2 + \lfloor \frac{2n-1}{4} \rfloor$, because in a ladder graph, a node can dominate maximum four nodes, including itself. Since $2n$ is divisible by 4, the dominated nodes can be grouped with 4 nodes. The final 4 nodes can be found as 1 in one end and 3 in the other end. The 1 node is dominated by itself and in the 3 nodes, one node dominates the other two. Also this is equal to $1 + \lfloor \frac{n}{2} \rfloor$.

Hence the proof follows.

An example is given in **Figure 6**.

4. Conclusion

In this article, we have discussed few graphs for which the domination number is the same that of its minimum diameter spanning tree. Further research can be done in exploring various graphs with the same property. The condition for which a graph does not possess such spanning tree may also be explored.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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