American Journal of Computational Mathematics, 2018, 8, 108-119
http://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

Weak Galerkin Finite Element Method for the Unsteady Stokes Equation

Chen Ning, Haiming Gu

College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China
Email: ghm@qust.edu.cn

How to cite this paper: Ning, C. and Gu, H.M. (2018) Weak Galerkin Finite Element Method for the Unsteady Navier-Stokes Equation. *American Journal of Computational Mathematics*, **8**, 108-119.
https://doi.org/10.4236/ajcm.2018.81009

Received: February 25, 2018
Accepted: March 25, 2018
Published: March 28, 2018

Abstract

The Weak Galerkin (WG) finite element method for the unsteady Stokes equations in the primary velocity-pressure formulation is introduced in this paper. Optimal-order error estimates are established for the corresponding numerical approximation in an H^1 norm for the velocity, and L^2 norm for both the velocity and the pressure by use of the Stokes projection.

Keywords

Weak Galerkin Finite Element Methods, Unsteady Stokes Equations, Stokes Projection

1. Introduction

The finite element method for the unsteady Stokes equations developed over the last several decades is based on the weak formulation by constructing a pair of finite element spaces satisfying the inf-sup condition of Babuska [1] and Brezzi [2]. Readers are referred to [3] [4] [5] [6] [7] for specific examples and details in the different finite element methods for the Stokes equations. The idea of weak Galerkin method was first introduced by the Professor Junping Wang in June 2011. Weak Galerkin refers to a general finite element technique for partial differential equations in which differential operators are approximated by weak forms as distributions for generalized functions. Thus, two of the key features in weak Galerkin methods are 1) the approximating functions are discontinuous, and 2) the usual derivatives are taken as distributions or approximations of distributions. The method was successfully applied to the second order elliptic equations [8] [9], the Stokes equations [10], Parabolic equations [11], and Maxwell equations [12]. A posteriori error is effectively estimated, and proved the convergence of the WG finite element method in this paper.

2. Preliminaries

In this paper, we study the initial-boundary value problems of the Stokes.

$$\begin{cases} u_t - \Delta u + \nabla p = f(x,t), & t \in (0,T) \\ \nabla \cdot u = 0, & (x,t) \in \Omega \times (0,T) \\ u = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = \psi(x) & x \in \Omega, t = 0 \end{cases} \tag{1}$$

where $u = (u_1, u_2)^{\mathrm{T}}$ is fluid velocity, p is pressure, $f = (f_1, f_2)^{\mathrm{T}}$ is volumetric power density.

The solution of the Stokes equations forms an important aspect of both theoretical and computational fluid dynamics. A limited number of solutions of these non-linear partial differential equations mostly involving spatially one-dimensional problems are given in the literature. Solutions of practical interest have been obtained for cases where, with suitable approximations, the equations are reduced to linear partial differential equations.

Let Ω be a bounded domain in R^2. We introduce function spaces $X = \left[H_0^1(\Omega)\right]^2$, $V = \{u \in X, div\, u = 0\}$, $M = \left\{q \in L_2(\Omega); \int_\Omega q \mathrm{d}x\mathrm{d}y = 0\right\}$, then the unsteady Stokes problem would take the following form: seek $(u, p) \in X \times M$ satisfying

$$\begin{cases} (u_t, v) + (\nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v), & \forall v \in X \\ (q, \nabla \cdot u) = 0, & \forall q \in W \\ u(x,0) = \psi(x) \end{cases} \tag{2}$$

We use $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ to be denote the norm and Semi-norm in the Sobolev space $H^s(D)$ for any $s \geq 0$, respectively. The inner product in $H^s(D)$ is denoted by $(\cdot,\cdot)_{s,D}$. For example, for each $s \geq 0$, the Semi-norm $|\cdot|_{s,D}$ is given by

$$|\varphi|_{s,D} = \left(\sum_{|\gamma|=s} \int_D \left|\partial^\gamma \varphi\right|^2 \mathrm{d}D\right)^{\frac{1}{2}}$$

and $\|\cdot\|$ is said to be the norm of L_2.

For w is $[0,T]$ to $H^s(D)$, the definition is given by

$$\|w\|_{L^q(0,T;H^s(D))} = \left(\int_0^T \|w(\cdot,t)\|_{s,D}^q \,\mathrm{d}t\right)^{\frac{1}{q}}$$

for $1 \leq q \leq \infty$, we have

$$\|w\|_{L^\infty(0,T;H^s(D))} = \sup_{0 \leq t \leq T} \|w(\cdot,t)\|_{s,D}$$

The space $H^s(D)$ and the norm defined in the $H^s(D)$ defined as

$$H(div, \Omega) = \left\{q : q \in \left[L^2(\Omega)\right], \nabla \cdot q \in L^2(\Omega)\right\}$$

$$\|q\|_{H(div,\Omega)} = \left(\|q\|^2 + \|\nabla \cdot q\|^2\right)^{\frac{1}{2}}$$

3. Weak Galerkin Finite Element Approximation Scheme

Let K be any polygonal or polyhedral domain with boundary ∂K. A weak vector-valued function on the region K refers to a vector-valued function $v=\{v_0,v_b\}$ such that $v_0\in\left[L^2(K)\right]^d$ and $v_b\in\left[H^{\frac{1}{2}}(\partial K)\right]^d$. The first component v_0 can be understood as the value of v in K, and the second component v_b represents v on the boundary of K. Note that v_b may not necessarily be related to the trace of v_0 on ∂K should a trace be well-defined. Denote by $v(K)$ the space of weak functions on K;

$$v(K)=\left\{v=\{v_0,v_b\}:v_0\in\left[L^2(K)\right]^d,v_b\in\left[H^{\frac{1}{2}}(\partial K)\right]^d\right\} \tag{3}$$

Definition 1. For any $v\in v(K)$, the weak gradient of v is defined as a linear functional $\nabla_w v$ in the dual space of $\left[H(div,K)\right]^d$, whose action on each $q\in\left[H(div,K)\right]^d$ is given by

$$(\nabla_w v,q)_K=-(v_0,\nabla\cdot q)_K+\langle v_b,q\cdot n\rangle_{\partial K} \tag{4}$$

where n is the outward normal direction to ∂K, $(v_0,\nabla\cdot q)_K=\int_K v_0(\nabla\cdot q)\mathrm{d}K$ is the action of v_0 on $\nabla\cdot q$, and $\langle v_b,q\cdot n\rangle_{\partial K}=\int_{\partial K}v_b q\cdot n\mathrm{d}s$ is the action of $q\cdot n$ on $v_b\in\left[H^{\frac{1}{2}}(\partial K)\right]$.

The Sobolev space $\left[H^1(K)\right]^d$ can be embedded into the space $v(K)$ by an inclusion map $iv:\left[H^1(K)\right]^d\to v(K)$ defined as follows

$$iv(\phi)=\left\{\phi|_K,\phi|_{\partial K}\right\},\phi\in\left[H^1(K)\right]^d$$

With the help of the inclusion map iv, the Sobolev space $\left[H^1(K)\right]^d$ can be viewed as a subspace of $v(K)$ by identifying each $\phi\in\left[H^1(K)\right]^d$ with $iv(\phi)$.

Let $P_r(K)$ be the set of polynomials on K with degree no more than r.

Definition 2. The discrete weak gradient operator, denoted by $\nabla_{w,r,K}$, is defined as the unique polynomial $(\nabla_{w,r,K}v)\in\left[P_r(K)\right]^{d\times d}$ satisfying the following equation,

$$(\nabla_{w,r,K}v,q)_K=-(v_0,\nabla\cdot q)_K+\langle v_b,q\cdot n\rangle_{\partial K}, \tag{5}$$

for all $q\in\left[P_r(K)\right]^{d\times d}$.

In what follows, we give the definition of weak divergence, first of all, we require weak function $v=\{v_0,v_b\}$ such that $v_0\in\left[L^2(K)\right]^d$ an $v_b\cdot n\in L^2(\partial K)$ Denote by $V(K)$ the space of weak vector-valued functions on K;

$$V(K)=\left\{v=\{v_0,v_b\}:v_0\in\left[L^2(K)\right]^d,v_b\cdot n\in L^2(\partial K)\right\} \tag{6}$$

Definition 3. For any $v\in V(K)$, the weak divergence of v is defined as a linear functional $\nabla_w\cdot v$ in the dual space of $H^1(K)$ whose action on each $\varphi\in H^1(K)$ is given by

$$(\nabla_w\cdot v,\varphi)_K=-(v_0,\nabla\varphi)_K+\langle v_b\cdot n,\varphi\rangle_{\partial K} \tag{7}$$

where n is the outward normal direction to ∂K, $(v_0,\nabla\varphi)_K$ is the action of v_0

on $\nabla\varphi$, and $\langle v_b\cdot n,\varphi\rangle_{\partial K}$ is the action of $v_b\cdot n$ on $\varphi\in H^{\frac{1}{2}}(\partial K)$.

The Sobolev space $\left[H^1(K)\right]^d$ can be embedded into the space $V(K)$ by an inclusion map $iv:\left[H^1(K)\right]^d\to V(K)$ defined as follows

$$iv(\phi)=\left\{\phi|_K,\phi|_{\partial K}\right\},\phi\in\left[H^1(K)\right]^d$$

Definition 4. A discrete weak divergence operator, denoted by $\nabla_{w,r,K}$, is defined as the unique polynomial $(\nabla_{w,r,K}\cdot v)\in P_r(K)$ that satisfies the following equation.

$$(\nabla_{w,r,K}\cdot v,\varphi)_K=-(v_0,\nabla\varphi)_K+\langle v_b\cdot n,\varphi\rangle_{\partial K}, \tag{8}$$

for all $\varphi\in P_r(K)$.

4. Weak Galerkin Finite Element Scheme

Let $\mathcal{T}_h$ be a partition of the domain Ω with mesh size *h* that consists of arbitrary polygons/polyhedra. In this paper, we assume that the partition $\mathcal{T}_h$ is WG shape regular-defined by a set of conditions as detailed in references. Denote by ε_h the set of all edges/flat faces in $\mathcal{T}_h$, and let $\varepsilon_h^0=\varepsilon_h\partial\Omega$ be the set of all interior edges/faces. For any integer $k\ge 1$, we define a weak Galerkin finite element space for the velocity variable as follows,

$$V_h=\left\{v=\{v_0,v_b\}:\{v_0,v_b\}|_T\in\left[P_k(T)\right]^d\times\left[P_{k-1}(e)\right]^d,e\subset\partial T\right\}$$

We would like to emphasize that there is only a single value v_b defined on each edge $e\in\varepsilon_h$. For the pressure variable, we have the following finite element space

$$W_h=\left\{q:q\in L_0^2(\Omega),q|_T\in P_{k-1}(T)\right\}$$

Denote by V_h^0 the subspace of V_h consisting of discrete weak functions with vanishing boundary value;

$$V_h^0=\left\{v=\{v_0,v_b\}\in V_h,v_b=0\text{ on }\partial\Omega\right\}$$

The discrete weak gradient $\nabla_{w,k-1}$ and the discrete weak divergence $(\nabla_{w,k-1})$ on the finite element space V_h can be computed by using (5) and (8) on each element *T*, respectively. More precisely, they are given by

$$(\nabla_{w,k-1}v)|_T=\nabla_{w,k-1,T}(v|_T),\ \forall v\in V_h$$

$$(\nabla_{w,k-1}\cdot v)|_T=\nabla_{w,k-1,T}\cdot(v|_T),\ \forall v\in V_h$$

For simplicity of notation, from now on we shall drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ and $(\nabla_{w,k-1})$ for the discrete weak gradient and the discrete weak divergence. The usual L^2 inner product can be written locally on each element as follows

$$(\nabla_w v,\nabla_w w)=\sum_{T\in\mathcal{T}_h}(\nabla_w v,\nabla_w w)_T$$

$$(\nabla_w \cdot v, q) = \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot v, q)_T$$

Denote by Q_0 the L^2 projection operator from $\left[L^2(T)\right]^d$ onto $\left[P_k(T)\right]^d$. For each edge/face $e \in \varepsilon_h$, denote by Q_b the L^2 projection from $\left[L^2(e)\right]^d$ onto $\left[P_{k-1}(e)\right]^d$. We shall combine Q_0 with Q_b by writing $Q_h = \{Q_0 Q_b\}$.

We are now in a position to describe a weak Galerkin finite element scheme for the Stokes Equations (1). To this end, we first introduce three bilinear forms as follows

$$s(v, w) = \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial_T}$$

$$a(v, w) = (\nabla_w v, \nabla_w w) + s(v, w)$$

$$b(v, q) = (\nabla_w \cdot v, q)$$

WG Algorithm. Seek $(u_h = \{u_0, u_b\}, p_h) \in V_h \times W_h$ satisfying

$$\begin{cases} (u_{h,t}, v) + a(u_h, v) - b(v, p_h) = (f, v), \forall v \in V_h^0 \\ b(u_h, q) = 0, \ \forall q \in W_h \\ u_h(x, 0) = \psi_0(x) \end{cases} \tag{9}$$

In the following, the proof process of Lemma 1-6 refers to reference [10] [11] [12].

Lemma 1. For any $v \in V_h^0$, the following equation hold true,

$$|||v|||^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w v\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b v_0 - v_b\|_{\partial T}^2$$

Lemma 2. For any $v, w \in V_h^0$ we have

$$|a(v, w)| \le |||v||| |||w|||$$

$$a(v, v) = |||v|||^2.$$

In addition to the projection $Q_h = \{Q_0, Q_b\}$ defined in the previous section, let Q_h and Q_h be two local L^2 projections onto $P_{k-1}(T)$ and $\left[P_{k-1}(T)\right]^{d \times d}$, respectively.

Lemma 3. The projection operators Q_h, R_h, and S_h satisfy the following commutative properties

$$\nabla_w (Q_h v) = R_h (\nabla v), \forall v \in \left[H^1(\Omega)\right]^d$$

$$\nabla_w \cdot (Q_h v) = S_h (\nabla \cdot v), \forall v \in H(div\,\Omega)$$

Lemma 4. There exists a positive constant β independent of h such that

$$\sup_{v \in V_h^0} \frac{b(v, \rho)}{|||v|||} \ge \beta \|\rho\|$$

for all $\rho \in W_h$.

Lemma 5. Poincare inequality of Weak gradient operator: If $v \in V_h^0$, then exists a constant c satisfying

$$\|v\|^2 \le c\|\nabla_w v\|^2 \le c\|\|v\|\|^2$$

First of all, we study the existence and uniqueness of the solution for (9). The space defined as follows

$$T_h = \{v_h \in V_h(j); (q, \nabla_w \cdot v) = 0, \forall q \in W_h\}.$$

Then we need to seek $u_h(x,t):(0,T) \to T_h$ satisfying

$$\begin{cases} (u_{h,t}, v) + a(u_h, v) = (f, v), v \in T_h \\ u_h(x,0) = \psi_0(x) \end{cases} \tag{10}$$

Let $u_h(x,t)$ be the solution of (10) and which is unique, the linear bounded functional $l = l(u_h)$ on V_h defined as follows.

$$\langle l, v \rangle = (u_{h,t}, v) + a(u_h, v) - (f, v) \tag{11}$$

Then problem (9) is equivalent to seek $P_h \in W_h$ satisfying

$$(P_h, \nabla_w \cdot v) = \langle l, v \rangle, \forall v \in V_h \tag{12}$$

Using LBB condition and Lax-Milgram Lemma, we know that the solution $P_h \in W_h$ of (12) is unique.

Combing (11) and (12), it is concluded that if initial approximation $u_h(x,0) = \psi_0(x) \in T_h$, the solution $(u_h, P_h) \in V_h \times M_h$ of (9) is unique.

In what follows, we introduce Stokes projection, which is the important approximation of projection.

Lemma 6. First of all, we introduce Stokes projection of $(u,p) \in X \times W$, which is $(Q_h^1 u, S_h^1 p) \in V_h \times W_h$ need satisfying

$$\begin{cases} a(u_h - Q_h^1 u, v) - b(v, p_h - S_h^1 p) = -\varphi_{u,p}(v), v \in V_h \\ b(u_h - Q_h^1 u, q) = 0, q \in W_h \end{cases} \tag{13}$$

If let $f^* = -\Delta u + \Delta p$, easy to know that $(Q_h^1 u, S_h^1 p) \in V_h \times W_h$ satisfying

$$\begin{cases} a(Q_h^1 u, v) - b(v, S_h^1 p) = (f^*, v_0), \ v \in V_h \\ b(Q_h^1 u, q) = 0, \ q \in W_h \end{cases} \tag{14}$$

Then $(Q_h^1 u, S_h^1 p) \in V_h \times W_h$ is the finite element approximation of $(u,p) \in X \times W$, so we have

$$\begin{cases} \|u_h - Q_h^1\| + h\|\|u_h - Q_h^1\|\| \le Ch^{k+1}(\|u\|_{k+1} + \|p\|_k) \\ \|p_h - S_h^1 p\| \le Ch^k(\|u\|_{k+1} + \|p\|_k) \\ \|u_h - Q_h^1\| + h\|\|(u_h - Q_h^1)_t\|\| \le Ch^{k+1}(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k) \end{cases} \tag{15}$$

5. Error Equations

In what follows, we list Lemma 7 to prove the error estimation of approximate solution for Semi-discrete scheme.

We know that $(u,p) \in X \times M$ and $(u_h, p_h) \in X_h \times M_h$ be solution of (1) and Galerkin finite element solution of (9), respectively. The L^2 projection of u in the finite element space V_h is given by $Q_h u = \{Q_0 u, Q_b u\}$. Similarly, the pressure p is projected into W_h as $S_h p$. Denote by e_h and ε_h the corresponding error given by

$$\begin{cases} e_h = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\} \\ \varepsilon_h = S_h p - p_h \end{cases} \tag{16}$$

Lemma 7. Let $(w, p) \in \left[H^1(\Omega)\right]^d \times L^2(\Omega)$ be sufficiently smooth and satisfy the following equation

$$w_t - \Delta w + \nabla \rho = \eta \tag{17}$$

in the domain Ω. Let $Q_h w = \{Q_0 w, Q_b w\}$ and $S_h \rho$ be the L^2 projection of (w, p) into the finite element space $V_h \times W_h$. Then, the following equation holds true

$$\begin{aligned} &(Q_h w_t, v_0) + (\nabla_w (Q_h w), \nabla_w v) - (\nabla_w \cdot v, S_h \rho) \\ &= (\eta, v_0) + \ell_w(v) - \theta_\rho(v), \end{aligned} \tag{18}$$

for all $v \in V_h^0$. Where ℓ_w and θ_ρ are two linear functionals on V_h^0 defined by

$$\ell_w(v) = \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, \nabla w \cdot n - R_h(\nabla w) \cdot n \rangle_{\partial T}$$

$$\theta_\rho(v) = \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (\rho - S_h \rho) n \rangle_{\partial T}$$

Proof. Together Lemma 3, Equation (5) and integration by parts. we obtain

$$\begin{aligned} &(\nabla_w (Q_h w), \nabla_w v)_T \\ &= (R_h(\nabla_w), \nabla_w v)_T \\ &= -(v_0, \nabla \cdot R_h(\nabla_w))_T + \langle v_b, \nabla \cdot R_h(\nabla_w) \cdot n \rangle_{\partial T} \\ &= (\nabla v_0, R_h(\nabla_w))_T - \langle v_0 - v_b, \nabla \cdot R_h(\nabla_w) \cdot n \rangle_{\partial T} \\ &= (\nabla w, \nabla v)_T - \langle v_0 - v_b, \nabla \cdot R_h(\nabla_w) \cdot n \rangle_{\partial T} \end{aligned} \tag{19}$$

Next, Combing Lemma 3 and Equation (8), the fact that $\sum_{T \in \mathcal{T}_h} \langle v_b, pn \rangle_{\partial T} = 0$, then using integration by parts, we obtain

$$\begin{aligned} &(\nabla_w \cdot v, S_h \rho) \\ &= -\sum_{T \in \mathcal{T}_h} \langle v_0, \nabla (S_h \rho) \rangle_T + \sum_{T \in \mathcal{T}_h} \langle v_b, (S_h \rho) n \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \langle \nabla \cdot v_0, S_h \rho \rangle_T - \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (S_h \rho) n \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \langle \nabla \cdot v_0, \rho \rangle_T - \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (S_h \rho) n \rangle_{\partial T} \\ &= -\sum_{T \in \mathcal{T}_h} \langle v_0, \nabla \rho \rangle_T + \sum_{T \in \mathcal{T}_h} \langle v_0, \rho n \rangle_{\partial T} - \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (S_h \rho) n \rangle_{\partial T} \\ &= -\sum_{T \in \mathcal{T}_h} \langle v_0, \nabla \rho \rangle_T + \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, \rho n \rangle_{\partial T} - \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (S_h \rho) n \rangle_{\partial T} \\ &= -(v_0, \nabla \rho) + \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (\rho - S_h \rho) n \rangle_{\partial T} \end{aligned}$$

We can imply that

$$(v_0, \nabla p) = -(\nabla_w \cdot v, S_h \rho) + \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, (\rho - S_h \rho) n \rangle_{\partial T} \tag{20}$$

Next, we test (17) by using v_0 in $v = \{v_0, v_b\} \in V_h^0$ to obtain, we can obtain

$$(w_t, v_0) - (\Delta w, v_0) + (\nabla \rho, v_0) = (\eta, v_0) \tag{21}$$

It follows from the usual integration by parts that

$$-(\Delta w, v_0) = \sum_{T \in \mathcal{T}_h} (\nabla w, \nabla v_0)_T - \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, \nabla w \cdot n \rangle_{\partial T}$$

Where we have used the fact that $\sum_{T \in \mathcal{T}_h} \langle v_b, \nabla w \cdot n \rangle_{\partial T} = 0$. using Equations (19) and (20), we have

$$-(\Delta w, v_0) = (\nabla_w (Q_h w), \nabla_w v) - \sum_{T \in \mathcal{T}_h} \langle v_0 - v_b, \nabla w \cdot n - R_h (\nabla w) \cdot n \rangle_{\partial T} \tag{22}$$

Substituting (20), (22) and $(Q_h w_t, v_0) = (w_t, v_0)$ into (21) yields

$$(Q_h w_t, v_0) + (\nabla_w (Q_h w), \nabla_w v) - (\nabla_w \cdot v, S_h \rho) = (\eta, v_0) + \ell_w (v) - \theta_\rho (v)$$

which completes the proof of the lemma.

In what follows, we give the derivation of the error equation of (9).

Lemma 8. Let e_h and ε_h be the error of the weak Galerkin finite element solution arising from (9), as defined by (16). Then, we have

$$\begin{cases} (e_{h,t}, v) + a(e_h, v) - b(v, \varepsilon_h) = \varphi_{u,p}(v) \\ b(e_h, q) = 0, \end{cases} \tag{23}$$

for all $v \in V_h^0$ and $q \in W_h$, where $\varphi_{u,p}(v) = \ell_u(v) - \theta_p(v) + s(Q_h u, v)$ is a linear functional defined on V_h^0.

Proof. Since (u, p) satisfies the Equation (17) with $\eta = f$, then from Lemma 6 we have

$$(Q_h u_t, v_0) + (\nabla_w (Q_h u), \nabla_w v) - (\nabla_w \cdot v, S_h p) = (f, v_0) + \ell_u (v) - \theta_\rho (v)$$

Adding $s(Q_h u, v)$ to both side of the above equation give

$$(Q_h u_t, v_0) + a(Q_h u, v) - b(v, S_h p) = (f, v_0) + \ell_u (v) - \theta_\rho (v) + s(Q_h u, v) \tag{24}$$

The difference of (24) and (9) yields the following equation,

$$(e_{h,t}, v_0) + a(e_h, v) - b(v, \varepsilon_h) = \ell_u (v) - \theta_\rho (v) + s(Q_h u, v)$$

for all $v \in V_h^0$, where $e_h = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\}$. This completes the derivation of (23).

As to (24), we test Equation (1) by $q \in W_h$ and use (9) to obtain

$$0 = (\nabla \cdot u, q) = (\nabla_w \cdot Q_h u, q) = b(Q_h u, q) \tag{25}$$

The difference of (25) and (9) yields the following equation

$$b(e_h, q) = 0$$

for all $q \in W_h$.

Which completes the proof of the lemma.

In the following, the proof process of Lemma 9 refers to reference [10].

Lemma 9. *If* $(w,\rho,r)\in\left[H^{r+1}(\Omega)\right]^d\times H^r(\Omega)\times V_h$ *and* $1\le r\le k$, *with the precondition of regular-shape* $\mathcal{T}_h$, *we have the following estimation.*

$$\left|s(Q_h w,v)\right|\le Ch^r\|w\|_{r+1}|||v|||$$

$$\left|\ell_w(v)\right|\le Ch^r\|w\|_{r+1}|||v|||$$

$$\left|\theta_\rho(v)\right|\le Ch^r\|w\|_{r+1}|||v|||$$

6. Error Estimates

The following theorem is the main result of this paper.

Theorem 1. Let $(u,p)\in\left[H_0^1(\Omega)\cap H^{k+1}(\Omega)\right]^d\times\left[L_0^2(\Omega)\cap H^k(\Omega)\right]$ and $(u_h,p_h)\in V_h\times W_h$ be the solution of (1) and (9), respectively. the following error estimates is true.

$$\|Q_hu-u_h\|\le Ch^{k+1}\left(\|u\|_{k+1}+\|p\|_k+\int_0^t\left(\|u\|_{k+1}+\|p\|_k+\|u_t\|_{k+1}+\|p_t\|_k\right)\mathrm{d}\tau\right)$$

$$|||Q_hu-u_h|||\le Ch^k\left(\|u\|_{k+1}+\|p\|_k+\|u_t\|_{k+1}+\|p_t\|_k\right)$$

$$\begin{aligned}\|S_hp-p_h\|\le Ch^{k+1}&\Big(\|u\|_{k+1}+\|p\|_k+\|u_t\|_{k+1}+\|p_t\|_k\\&+\int_0^t\left(\|u\|_{k+1}+\|p\|_k+\|u_t\|_{k+1}+\|p_t\|_k+\|u_{tt}\|_{k+1}+\|p_{tt}\|_k\right)\mathrm{d}\tau\Big)\end{aligned}$$

Proof. Let

$$e_h=Q_hu-u_h=Q_hu-Q_h^1u+Q_h^1u-u_h=\theta+\eta$$

$$e_h(\cdot,0)=\theta(\cdot,0)=\eta(\cdot,0)=0$$

By the error of Equation (23), we have

$$\begin{aligned}&(\theta_t,v)+a(\theta,v)-b\left(v,S_hp-S_h^1p\right)\\&=\varphi_{u,p}(v)-(\eta_t,v)-a(\eta,v)+b\left(v,S_h^1p-p_h\right)\end{aligned}\tag{26}$$

Substituting (13) into (26), we obtain

$$(\theta_t,v)+a(\theta,v)-b\left(v,S_hp-S_h^1p\right)=-(\eta_t,v)\tag{27}$$

Let $v=\theta=Q_hu-Q_h^1u$, combing the Equation (25) and (14), we have

$$b\left(\theta,S_hp-S_h^1p\right)=0$$

That is

$$(\theta_t,\theta)+a(\theta,\theta)=-(\eta_t,\theta)$$

By Lemma 2 and Cauchy inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2+|||\theta|||^2\le\|\eta_t\|\|\theta\|\le\|\eta_t\|^2+\|\theta\|^2\tag{28}$$

By Gronwall Lemma, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2+|||\theta|||^2\le\left(Ch^{k+1}\left(\|u\|_{k+1}+\|p\|_k+\|u_t\|_{k+1}+\|p_t\|_k\right)\right)^2\tag{29}$$

By Cauchy inequality, we have

$$\|\theta\| \le Ch^{k+1}\left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k\right) \tag{30}$$

Then take the integration about t of both side of Equation (28)

$$\|\theta(\cdot,t)\|^2 + 2\int_0^t \|\theta\|^2 \mathrm{d}\tau \le \|\theta(\cdot,0)\|^2 + C\left(\int_0^t \|\eta_t\| \mathrm{d}\tau\right)^2 + \frac{1}{4}\sup_{\tau\le t}\|\theta(\tau)\|^2$$

Since $\|\theta(\cdot,0)\| = 0$, then

$$\begin{aligned}\|\theta\| &\le \sup_{\tau\le t}\|\theta(\tau)\| \le C\int_0^t \|\eta_t\| \mathrm{d}\tau \\ &\le Ch^{k+1}\int_0^t \left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k\right)\mathrm{d}\tau\end{aligned} \tag{31}$$

Combing the Equations (15), (29), (30) and triangle inequality, we have

$$\|Q_h u - u_h\| \le Ch^{k+1}\left(\|u\|_{k+1} + \|p\|_k + \int_0^t \left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k\right)\mathrm{d}\tau\right) \tag{32}$$

$$\|Q_h u - u_h\| \le Ch^{k}\left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k\right) \tag{33}$$

Next, we proof the error estimate of pressure approximation $\|S_h p - p_h\|$, by using error Equation (23), we have

$$b(v, S_h p - p_h) = (e_{h,t}, v) + a(e_h, v) - \varphi_{u,p}(v)$$

By using Lemma 2, Lemma 5 and Lemma 9, we obtain

$$\begin{aligned}b(v, S_h p - p_h) &\le \|e_{h,t}\|\|v\| + \|e_h\|\|v\| + Ch^{k+1}\left(\|u\|_{k+1} + \|p\|_k\right)\|v\| \\ &\le C\|e_{h,t}\|\|v\| + \|e_h\|\|v\| + Ch^{k+1}\left(\|u\|_{k+1} + \|p\|_k\right)\|v\|\end{aligned}$$

By Lemma 4, we have

$$\|S_h p - p_h\| \le C\|e_{h,t}\| + \|e_h\| + Ch^{k+1}\left(\|u\|_{k+1} + \|p\|_k\right) \tag{34}$$

Next we seek error estimate $\|e_{h,t}\|$, then take the derivation about t of both sides of Equation (27)

$$(\theta_{tt}, v) + a(\theta_t, v) - b\left(v, S_h p_t - S_h^1 p_t\right) = -(\eta_{tt}, v)$$

Let $v = \theta_t$, take the derivation about t of both side of Equations (14) and (25), we obtain

$$b\left(\theta_t, S_h p_t - S_h^1 p_t\right) = 0$$

That is

$$(\theta_{tt}, \theta_t) + a(\theta_t, \theta_t) = -(\eta_{tt}, \theta_t)$$

By Lemma 2 and Cauchy inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\theta_t\|^2 + \|\theta_t\|^2 \le \|\eta_{tt}\|\|\theta_t\|$$

That is

$$\|\theta_t(\cdot,t)\|^2 \le \|\theta_t(\cdot,0)\|^2 + C\left(\int_0^t \|\eta_{tt}\| \mathrm{d}\tau\right)^2 + \frac{1}{4}\sup_{\tau\le t}\|\theta_t(\tau)\|^2 \tag{35}$$

Since $\|\theta_t(\cdot,0)\| = 0$, that is

$$\|\theta_t\| \le \sup_{\tau\le t}\|\theta_t(\cdot,t)\| \le Ch^{k+1}\int_0^t \left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k + \|u_{tt}\|_{k+1} + \|p_{tt}\|_k\right)\mathrm{d}\tau \quad (36)$$

Combing the Equations (15) and triangle inequality, we have

$$\begin{aligned}\|e_{h,t}\| \le Ch^{k+1}\Big(&\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k \\ &+ \int_0^t \left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k + \|u_{tt}\|_{k+1} + \|p_{tt}\|_k\right)\mathrm{d}\tau\Big)\end{aligned} \quad (37)$$

Substituting (33) and (36) into (34), we have

$$\begin{aligned}\|S_h p - p_h\| \le Ch^{k+1}\Big(&\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k \\ &+ \int_0^t \left(\|u\|_{k+1} + \|p\|_k + \|u_t\|_{k+1} + \|p_t\|_k + \|u_{tt}\|_{k+1} + \|p_{tt}\|_k\right)\mathrm{d}\tau\Big)\end{aligned}$$

This completes the proof. Thus, the error estimates of *Theorem 1* hold. Optimal-order error estimates are established for the corresponding numerical approximation in an H^1 norm for the velocity, and L^2 norm for both the velocity and the pressure by use of the Stokes projection.

References

[1] Babuska, I. (1973) The Finite Element Method with Lagrangian Multiplier. *Numerische Mathematik*, **20**, 179-192. https://doi.org/10.1007/BF01436561

[2] Brenner, S. and Scott, R. (2008) Mathematical Theory of Finite Element Methods. *Texts in Applied Mathematics*, **15**. https://doi.org/10.1007/978-0-387-75934-0

[3] Brezzi, F. (1974) On the Existence, Uniqueness and Approximation of Saddle-Point Problems Arising from Lagrangian Multipliers. *Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique*, **8**, 129-151.

[4] Brezzi, F. and Fortin, M. (1991) Mixed and Hybrid Finite Element Methods. *Springer Series in Computational Mathematics*, **15**. https://doi.org/10.1007/978-1-4612-3172-1

[5] Crouzeix, M. and Raviart, P.A. (1973) Conforming and Nonconforming Finite Element Methods for Solving the Stationary Stokes Equations. *Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique*, **7**, 33-75.

[6] Girault, V. and Raviart, P.A. (1986) Finite Element Methods for the Navier-Stokes Equations: Theory and Algorithms. Springer Publishing Company, Berlin. https://doi.org/10.1007/978-3-642-61623-5

[7] Gunzburger, M.D. (1989) Finite Element Methods for Viscous Incompressible Flows, A Guide to Theory, Practice and Algorithms.

[8] Wang, J. and Ye, X. (2015) A Weak Galerkin Finite Element Method for the Stokes Equations. *Journal of Computational and Applied Mathematics*, **275**, 79-90. https://doi.org/10.1016/j.cam.2014.08.006

[9] Wang, J. and Ye, X. (2013) Weak Galerkin Methods for Second Order Elliptic Interface Problems. *Journal of Computational Physics*, **10**, 106-125.

[10] Wang, J. and Ye, X. (2014) A Weak Galerkin Mixed Finite Element Method for Second-Order Elliptic Problems. *Mathematics of Computation*, **8**, 2101-2126. https://doi.org/10.1090/S0025-5718-2014-02852-4

[11] Mu, L., Wang, J. and Ye, X. (2012) Weak Galerkin Finite Element Methods on Polytopal Meshes. *International Journal of Numerical Analysis and Modeling*, **12**.

[12] Qiao, L., Li, H. and Wang, J.P. (2013) Weak Galerkin Finite Element Methods for Parabolic Equations. *Numerical Methods for Partial Differential Equatons*, **29**, 2004-2024.