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A Common Fixed Point Theorem for Two Pairs of Mappings in Dislocated Metric Space

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Abstract

Dislocated metric space differs from metric space for a property that self distance of a point needs not to be equal to zero. This property plays an important role to deal with the problems of various disciplines to obtain fixed point results. In this article, we establish a common fixed point theorem for two pairs of weakly compatible mappings which generalize and extend the result of Brain Fisher [1] in the setting of dislocated metric space with replacement of contractive constant by contractive modulus for which continuity of mappings is not necessary and compatible mappings by weakly compatible mappings.

Keywords

d-Metric Space, Common Fixed Point, Weakly Compatible, Contractive Modulus, Cauchy Sequence

1. Introduction

In 1922, S. Banach [2] established a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by many authors and various generalizations of this theorem have been established. In 1982, S. Sessa [3] introduced the concept of weakly commuting maps and G. Jungck [4] in 1986, initiated the concept of compatibility. In 1998, Jungck and Rhoades [5] initiated the notion of weakly compatible maps and pointed that compatible maps were weakly compatible but not conversely.

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity. In 1986, S. G. Matthews [6] introduced the concept of dislocated metric space under the name of metric domains in domain theory. In 2000, P. Hitzler and A. K. Seda [7] generalized the famous

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Banach Contraction Principle in dislocated metric space. The study of dislocated metric plays very important role in topology, logic programming and in electronics engineering.

The purpose of this article is to establish a common fixed point theorem for two pairs of weakly compatible mappings in dislocated metric spaces which generalize and improve similar results of fixed point in the literature.

2. Preliminaries

We start with the following definitions, lemmas and theorems.

Definition 1. [7] *Let X be a non empty set and let* $d : X \times X \to [0,\infty)$ *be a function satisfying the following conditions*:

1) $d(x,y) = d(y,x)$.

2) $d(x,y) = d(y,x) = 0$ implies $x = y$.

3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called dislocated metric (or d-metric) on *X* and the pair (X,d) is called the dislocated metric space (or d-metric space).

Definition 2. [7] *A sequence* $\{x_n\}$ *in a d-metric space* (X,d) *is called a Cauchy sequence if for given* $\epsilon > 0$, *there corresponds* $n_0 \in N$ *such that for all* $m, n \ge n_0$ *, we have* $d(x_m, x_n) < \epsilon$.

Definition 3. [7] *A sequence in d-metric space converges with respect to d (or in d) if there exists* $x \in X$ *such that* $d(x_n, x) \to 0$ *as* $n \to \infty$.

In this case, *x* is called limit of $\{x_n\}$ (in d)and we write $x_n \to x$.

Definition 4. [7] *A d-metric space* (X,d) *is called complete if every Cauchy sequence in it is convergent with respect to d.*

Definition 5. [7] *Let* (X,d) *be a d-metric space. A map* $T : X \to X$ *is called contraction if there exists a number* λ *with* $0 \le \lambda < 1$ *such that* $d(Tx,Ty) \le \lambda d(x,y)$.

We state the following lemmas without proof.

Lemma 1. *Let* (X,d) *be a d-metric space. If* $T : X \to X$ *is a contraction function, then* $\{T^n(x_0)\}$ *is a Cauchy sequence for each* $x_0 \in X$.

Lemma 2. [7] *Limits in a d-metric space are unique.*

Theorem 1. [7] *Let* (X,d) *be a complete d-metric space and let* $T : X \to X$ *be a contraction mapping, then T has a unique fixed point.*

Definition 6. *Let A and S be two self mappings on a set X. Mappings A and S are said to be commuting if* $ASx = SAx \quad \forall x \in X$.

Definition 7. *Let A and S be two self mappings on a set X. If* $Ax = Sx$ *for some* $x \in X$ *, then x is called coincidence point of A and S.*

Definition 8. [5] *Let A and S be mappings from a metric space* (X,d) *into itself. Then, A and S are said to be weakly compatible if they commute at their coincident point; that is,* $Ax = Sx$ *for some* $x \in X$ *implies* $ASx = SAx$.

Definition 9. *A function* $\phi : [0,\infty) \to [0,\infty)$ *is said to be contractive modulus if* $\phi(t) < t$ *for* $t > 0$.

Definition 10. *A real valued function* ϕ *defined on* $X \subseteq \mathbb{R}$ *is said to be upper semicontinuous if*

$$\lim_{n\to\infty} \phi(t_n) \le \phi(t)$$

for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.

It is clear that every continuous function is upper semicontinuous but converse may not be true.

In 1983, B. Fisher [1] established the following theorem in metric space.

Theorem 2. *Suppose that S, P, T and Q are four self maps of a complete metric space* (*X*, *d*) *satisfying the following conditions*

1) $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$.

2) Pairs (*S*, *P*) and (*T*, *Q*) are commuting.

3) One of *S*, *P*, *T* and *Q* is continuous.

4) $d(Sx,Ty) \le \lambda m(x,y)$ where $m(x,y) = \max\{d(Px,Qy), d(Px,Sx), d(Qy,Ty)\}$ for all $x, y \in X$ and $0 \le \lambda < 1$

Then *S*, *P*, *T* and *Q* have a unique common fixed point $z \in X$. Also, *z* is the unique common fixed point of pairs (*S*, *P*) and (*T*, *Q*).

3. Main Results

Theorem 3. *Let* (*X*, *d*) *be a complete d-metric space. Suppose that A, B, S and T are four self mappings of X satisfying the following conditions*

i) $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$

ii) $d(Sx,Ty) \le \phi(m(x,y))$ where ϕ is an upper semicontinuous contractive modulus and

$$m(x,y) = \max\left\{d(Ax,By), d(Ax,Sx), d(By,Ty), \frac{1}{2}d(Ax,Ty), \frac{1}{2}d(By,Sx)\right\}$$

iii) The pairs (S,A) and (T,B) are weakly compatible, then *A*, *B*, *S* and *T* have an unique common fixed point.

Proof. Let x_0 be an arbitrary point of *X* and define a sequence $\{y_n\}$ in *X* such that

$$y_n = Sx_n = Bx_{n+1} \quad \text{and} \quad y_{n+1} = Tx_{n+1} = Ax_{n+2}$$

Now by condition ii), we have

$$d(y_n, y_{n+1}) = d(Sx_n, Tx_{n+1}) \le \phi(m(x_n, x_{n+1}))$$

where

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\left\{d(Ax_n, Bx_{n+1}), d(Ax_n, Sx_n), d(Bx_{n+1}, Tx_{n+1}), \frac{1}{2}d(Ax_n, Tx_{n+1}), \frac{1}{2}d(Bx_{n+1}, Sx_n)\right\} \\ &= \max\left\{d(Tx_{n-1}, Sx_n), d(Tx_{n-1}, Sx_n), d(Sx_n, Tx_{n+1}), \frac{1}{2}d(Tx_{n-1}, Tx_{n+1}), \frac{1}{2}d(Sx_n, Sx_n)\right\} \\ &= \max\left\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}d(y_{n-1}, y_{n+1}), \frac{1}{2}d(y_n, y_n)\right\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \end{aligned}$$

$m(x_n, x_{n+1}) = d(y_n, y_{n+1})$ is not possible since ϕ is a contractive modulus, so

$$d(y_n, y_{n+1}) \le \phi(d(y_{n-1}, y_n)) \tag{1}$$

Since ϕ is upper semicontinuous, contractive modulus the Equation (1) implies that the sequence $\{d(y_{n+1}, y_n)\}$ is monotonic decreasing and continuous.

Hence there exists a real number $t \ge 0$ such that

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = t$$

Taking limit in (1) we obtain $t \le \phi(t)$ which is possible if $t = 0$, sice ϕ is contractive modulus. therfore

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

We claim that $\{y_n\}$ is a cauchy sequence.

if possible, let $\{y_n\}$ is not a cauchy sequence. Then there exists a real number $\varepsilon > 0$ and subsequences q_i and p_i such that $p_i < q_i < p_{i+1}$ and

$$d(y_{p_i}, y_{q_i}) \ge \varepsilon \quad \text{and} \quad d(y_{p_i}, y_{q_{i-1}}) < \varepsilon \tag{2}$$

so that

$$\varepsilon \le d\left(y_{p_i}, y_{q_i}\right) \le d\left(y_{p_i}, y_{q_{i-1}}\right) + d\left(y_{q_{i-1}}, y_{q_i}\right) < \varepsilon + d\left(y_{q_{i-1}}, y_{q_i}\right)$$

Hence

$$\lim_{n\to\infty} d\left(y_{p_i}, y_{q_i}\right) = \varepsilon$$

Now

$$d\left(y_{p_{i-1}}, y_{q_{i-1}}\right) \le d\left(y_{p_{i-1}}, y_{p_i}\right) + d\left(y_{p_i}, y_{q_i}\right) + d\left(y_{q_i}, y_{q_{i-1}}\right)$$

Taking limit as $i \to \infty$ we have

$$\lim_{i\to\infty} d\left(y_{p_{i-1}}, y_{q_{i-1}}\right) = \varepsilon.$$

So by contractive condition ii) and (2)

$$\varepsilon \le d\left(y_{p_i}, y_{q_i}\right) = d\left(Sx_{p_i}, Tx_{q_i}\right) \le \phi\left(m\left(x_{p_i}, x_{q_i}\right)\right) \tag{3}$$

where

$$\begin{aligned} m\left(x_{p_i}, x_{q_i}\right) &= \max\left\{d\left(Ax_{p_i}, Bx_{q_i}\right), d\left(Ax_{p_i}, Sx_{p_i}\right), d\left(Bx_{q_i}, Tx_{q_i}\right), \frac{1}{2}d\left(Ax_{p_i}, Tx_{q_i}\right), \frac{1}{2}d\left(Bx_{q_i}, Sx_{p_i}\right)\right\} \\ &= \max\left\{d\left(Tx_{p_{i-1}}, Sx_{q_{i-1}}\right), d\left(Tx_{p_{i-1}}, Sx_{p_i}\right), d\left(Sx_{q_{i-1}}, Tx_{q_i}\right), \frac{1}{2}d\left(Tx_{p_{i-1}}, Tx_{q_i}\right), \frac{1}{2}d\left(Sx_{q_{i-1}}, Sx_{p_i}\right)\right\} \\ &= \max\left\{d\left(y_{p_{i-1}}, y_{q_{i-1}}\right), d\left(y_{p_{i-1}}, y_{p_i}\right), d\left(y_{q_{i-1}}, y_{q_i}\right), \frac{1}{2}d\left(y_{p_{i-1}}, y_{q_i}\right), \frac{1}{2}d\left(y_{q_{i-1}}, y_{p_i}\right)\right\} \end{aligned}$$

Now taking limit as $i \to \infty$ we get

$$\lim_{i\to\infty} m\left(x_{p_i}, x_{q_i}\right) = \max\left\{\varepsilon, 0, 0, \frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right\} = \varepsilon$$

Therefore from (3) we have $\varepsilon \le \phi(\varepsilon)$ which is a contradiction, since ϕ is contractive modulus.
Hence $\{y_n\}$ is a cauchy sequence.
Since X is complete, there exists a point u in X such that $\lim_{n\to\infty} y_n = u$. So,

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_{n+1} = u \quad \& \quad \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Ax_{n+2} = u.$$

Hence, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_{n+1} = \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Ax_{n+2} = u.$

Since $T(X) \subseteq A(X)$ there exists a point $v \in X$ such that $u = Av$. Now by condition ii)

$$d(Sv, u) \le d(Sv, Tx_{n+1}) + d(Tx_{n+1}, u) \le \phi\left(m(v, x_{n+1})\right) + d(Tx_{n+1}, u),$$

where

$$\begin{aligned} m(v, x_{n+1}) &= \max\left\{d(Av, Bx_{n+1}), d(Av, Sv), d(Bx_{n+1}, Tx_{n+1}), \frac{1}{2}d(Av, Tx_{n+1}), \frac{1}{2}d(Bx_{n+1}, Sv)\right\} \\ &= \max\left\{d(u, Sx_n), d(u, Sv), d(Sx_n, Tx_{n+1}), \frac{1}{2}d(u, Tx_{n+1}), \frac{1}{2}d(Sx_n, Sv)\right\} \end{aligned}$$

Taking limit as $n \to \infty$ we have

$$m(v, x_{n+1}) = \max\left\{d(u, Sv), \frac{1}{2}d(u, Sv)\right\} = d(u, Sv)$$

Thus $n \to \infty$ implies $d(u, Sv) \le \phi\left(d(u, Sv)\right)$ which is a contradiction, since ϕ is a contractive modulus. Thus $Sv = u$. Hence $Av = Sv = u$ which represents that v is the coincidence point of A and S.

Since the pair (S,A) are weakly compatible, so $SAv = ASv \Rightarrow Su = Au$

Again, since $S(X) \subseteq B(X)$ there exists a point $w \in X$ such that $u = Bw$. Then by condition ii) we have,

$$d(u,Tw) = d(Sv,Tw) \le \phi(m(v,w)),$$

where

$$\begin{aligned} m(v,w) &= \max\left\{d(Av,Bw), d(Av,Sv), d(Bw,Tw), \frac{1}{2}d(Av,Tw), \frac{1}{2}d(Bw,Sv)\right\} \\ &= \max\left\{d(u,u), d(u,u), d(u,Tw), \frac{1}{2}d(u,Tw), \frac{1}{2}d(u,u)\right\} \\ &= \max\left\{d(u,u), d(u,Tw)\right\} \end{aligned}$$

If $m(v,w) = d(u,u)$ then $m(v,w) \le 2d(u,Tw)$ which implies

$$d(u,Tw) \le \phi(2d(u,Tw)) < 2d(u,Tw)$$

a contradiction, since ϕ is a contractive modulus.

Again if $m(v,w) = d(u,Tw)$ then

$$d(u,Tw) \le \phi(d(u,Tw)) < d(u,Tw)$$

a contradiction. Hence, $d(u,Tw) = 0$. Which implies $u = Tw$. Therefore $Tw = Bw = u$. Thus w is the coincidence point of B and T.

Since the pair (B,T) are weakly compatible, so $BTw = TBw \Rightarrow Bu = Tu$. Now we show that u is the fixed point of S.

By condition ii), we have

$$d(Su,u) = d(Su,Tw) \le \phi(m(u,w))$$

where,

$$\begin{aligned} m(u,w) &= \max\left\{d(Au,Bw), d(Au,Su), d(Bw,Tw), \frac{1}{2}d(Au,Tw), \frac{1}{2}d(Bw,Su)\right\} \\ &= \max\left\{d(Su,u), d(Su,Su), d(u,u), \frac{1}{2}d(Su,u), \frac{1}{2}d(u,Su)\right\} \\ &= \max\left\{d(Su,u), d(Su,Su), d(u,u)\right\} \end{aligned}$$

If $m(u,w) = d(Su,u)$ then,

$$d(Su,u) \le \phi(m(u,w)) = \phi(d(Su,u)) < d(Su,u)$$

a contradiction since ϕ is contractive modulus.

If $m(u,w) = d(Su,Su)$ or $m(u,w) = d(u,u)$, one can observe that there are contradictions for both cases. Hence we conclude that $d(Su,u) = 0$ which implies that $Su = u$.

Therefore, $Su = Au = u$.

Now we show that u is the fixed point of T. Again by condition ii),

$$d(u,Tu) = d(Su,Tu) \le \phi(m(u,u))$$

where,

$$\begin{aligned} m(u,u) &= \max\left\{d(Au,Bu), d(Au,Su), d(Bu,Tu), \frac{1}{2}d(Au,Tu), \frac{1}{2}d(Bu,Su)\right\} \\ &= \max\left\{d(u,Tu), d(u,u), d(Tu,Tu), \frac{1}{2}d(u,Tu), \frac{1}{2}d(Tu,u)\right\} \\ &= \max\left\{d(u,Tu), d(u,u), d(Tu,Tu)\right\} \end{aligned}$$

If $m(u,u)=d(u,Tu)$ then,

$$d(u,Tu)\le\phi(m(u,u))=\phi(d(u,Tu))<d(u,Tu)$$

a contradiction.

If $m(u,u)=d(u,u)$ or $m(u,u)=d(Tu,Tu)$ one can observe that there are contradictions for both cases. Hence we conclude that $d(u,Tu)=0$ which implies that $Tu=u$.

Therefore $Tu=Bu=u$

Hence, $Au=Bu=Su=Tu=u$ *i.e.* u is the common fixed point of the mappings A,B,S and T.

Uniqueness:

If possible let u and z $(u\ne z)$ are two common fixed points of the mappings A,B,S and T. By condition ii) we have,

$$d(u,z)=d(Su,Tz)\le\phi(m(u,z))$$

where,

$$\begin{aligned} m(u,z)&=\max\left\{d(Au,Bz),d(Au,Su),d(Bz,Tz),\frac{1}{2}d(Au,Tz),\frac{1}{2}d(Bz,Su)\right\}\\ &=\max\left\{d(u,z),d(u,u),d(z,z),\frac{1}{2}d(u,z),\frac{1}{2}d(z,u)\right\}\\ &=\max\{d(u,z),d(u,u),d(z,z)\}\end{aligned}$$

If $m(u,z)=d(u,z)$ then,

$$d(u,z)\le\phi(m(u,z))=\phi(d(u,z))<d(u,z)$$

a contradiction, since ϕ is a contractive modulus.

Again if $m(u,z)=d(u,u)$ or $m(u,z)=d(z,z)$ one can observe that there are contradictions for both cases. Hence we conclude that $d(u,z)=0$ which implies that $u=z$.

Therefore, u is the unique common fixed point of the four mappings A,B,S and T. This completes the proof of the theorem.

Now we have the following corollaries:

Corollary 1. *Let* (X, d) *be a complete dislocated metric space. Suppose that* A, S *and* T *are three self mappings of* X *satisfying the following conditions*:

1) $T(X)\subseteq A(X)$ and $S(X)\subseteq A(X)$.

2) $d(Sx,Ty)\le\phi(m(x,y))$ where ϕ is an upper semicontinuous contractive modulus and

$$m(x,y)=\max\left\{d(Ax,Ay),d(Ax,Sx),d(Ay,Ty),\frac{1}{2}d(Ax,Ty),\frac{1}{2}d(Ay,Sx)\right\}.$$

3) The pairs (S,A) and (T,A) are weakly compatible, then A, S and T have an unique common fixed point.

Proof. If we take $A=B$ in theorem (3) and follow the similar proof we get the required result.

Corollary 2. *Let* (X, d) *be a complete dislocated metric space. Suppose that* A *and* S *are two self mappings of* X *satisfying the following conditions.*

1) $S(X)\subseteq A(X)$.

2) $d(Sx,Sy)\le\phi(m(x,y))$ where ϕ is an upper semicontinuous contractive modulus and

$$m(x,y)=\max\left\{d(Ax,Ay),d(Ax,Sx),d(Ay,Sy),\frac{1}{2}d(Ax,Sy),\frac{1}{2}d(Ay,Sx)\right\}.$$

3) The pair (S,A) is weakly compatible, then A and S have an unique common fixed point.

Proof. If we take $A=B$ and $S=T$ in theorem (3) and follow the similar proof we get the required result.

Corollary 3. *Let (X, d) be a complete dislocated metric space. Suppose that S and T are two self mappings of X satisfying the following conditions*

1) $d(Sx,Ty) \le \phi(m(x,y))$ where ϕ is an upper semicontinuous contractive modulus and

$$m(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}d(x,Ty), \frac{1}{2}d(y,Sx)\right\}.$$

2) The pairs (S,I) and (T,I) are weakly compatible, then *S* and *T* have an unique common fixed point.

Proof. If we take $A = B = I$ in theorem (3) and follow the similar proof we get the required result.

Corollary 4 *Let (X, d) be a complete dislocated metric space. Let* $S : X \to X$ *be a map satisfying the following conditions*

$d(Sx,Sy) \le \phi(m(x,y))$ where ϕ is an upper semicontinuous contractive modulus and

$$m(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Sy), \frac{1}{2}d(x,Sy), \frac{1}{2}d(y,Sx)\right\}$$

then the map S has a unique fixed point.

Proof. If we take $T = S$ in corollary (3) and follow the similar proof we get the required result.

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