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Sums of Involving the Harmonic Numbers and the Binomial Coefficients

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Abstract

Let the numbers $P(r,n,k)$ be defined by $P(r,n,k) := P_r\left(H_n^{(1)} - H_k^{(1)}, \cdots, H_n^{(r)} - H_k^{(r)}\right)$, where $P_r(x_1,\cdots,x_r) = (-1)^r Y_r(-0!x_1, -1!x_2, \cdots, -(r-1)!x_r)$ and Y_r are the exponential complete Bell polynomials. In this paper, by means of the methods of Riordan arrays, we establish general identities involving the numbers $P(r,n,k)$, binomial coefficients and inverse of binomial coefficients. From these identities, we deduce some identities involving binomial coefficients, Harmonic numbers and the Euler sum identities. Furthermore, we obtain the asymptotic values of some summations associated with the numbers $P(r,n,k)$ by Darboux's method.

Keywords

Harmonic Numbers, Euler Sum, Riordan Arrays, Asymptotic Values

1. Introduction and Preliminaries

Let Y_r be the exponential complete Bell polynomials and

In [1], Zave established the following series expansion:

$$P_r(x_1,\cdots,x_r) = (-1)^r Y_r(-0!x_1, -1!x_2, \cdots, -(r-1)!x_r)$$

$$\sum_{n=k}^{\infty} \binom{n}{k} P_r\left(H_n^{(1)} - H_k^{(1)}, \cdots, H_n^{(r)} - H_k^{(r)}\right) t^{n-k} = \frac{(-\ln(1-t))^r}{(1-t)^{k+1}} \tag{1}$$

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where $H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^r}$ for $n,r=1,2,\cdots$, $H_0^{(r)}=0$ and $H_n^{(1)}=H_n$.

Spiess [2] introduced the numbers $P(r,n,k)=P_r\left(H_n^{(1)}-H_k^{(1)},\cdots,H_n^{(r)}-H_k^{(r)}\right)$ and $P(1,n,0)=H_n$, $P(r,n,k)=0$ for $n<r+k$; then Equation (1.1) is equivalent to

$$\sum_{n=k+r}^{\infty}\binom{n}{k}P(r,n,k)t^{n-k}=\frac{\left(-\ln(1-t)\right)^r}{(1-t)^{k+1}}$$

$$\sum_{n=r}^{\infty}\binom{n+k}{k}P(r,n+k,k)t^{n}=\frac{\left(-\ln(1-t)\right)^r}{(1-t)^{k+1}}$$

where $P(0,n,k)=1$, $P(1,n,k)=H_n-H_k$, $P(2,n,k)=(H_n-H_k)^2-\left(H_n^{(2)}-H_k^{(2)}\right)$,

$$P(3,n,k)=(H_n-H_k)^3-3(H_n-H_k)\left(H_n^{(2)}-H_k^{(2)}\right)+2\left(H_n^{(2)}-H_k^{(2)}\right)$$

The paper is organized as follows. In Section 2, we obtain some for $P(r,n,k)$ and binomial coefficients by means of the Riordan arrays. In Section 3, we establish some identities involving the numbers $P(r,n,k)$ and inverse of binomial coefficients. Finally, in Section 4, we give the asymptotic expansions of some summations involving the numbers $P(r,n,k)$ by Darboux's method. Due to [3] [4], a Riordan array is a pair $(d(t),h(t))$ of formal power series with $h_0=h(0)$. It defines an infinite lower triangular array $(d_{n,k})_{n,k\in N}$ according to the rule

$$d_{n,k}=\left[t^n\right]d(t)(h(t))^k$$

Hence we write $\{d_{n,k}\}=(d(t),h(t))$. If $(d(t),h(t))$ is an Riordan array and $f(t)$ is the generating function of the sequence $\{f_k\}_{k\in N}$, *i.e.*, $f(t)=\sum_{k=0}^{\infty}f_kt^k$. Then we have

$$\sum_{k=0}^{\infty}d_{n,k}f_k=\left[t^n\right]d(t)f(h(t))=\left[t^n\right]d(t)\left[f(y)\middle|y=h(t)\right] \tag{2}$$

Based on the generating function (1), we obtain the next Riordan arrays, to which we pay particular attention in the present paper:

$$\left\{\binom{n}{k}P(r,n,k)\right\}=\left(\frac{\left(-\ln(1-t)\right)^r}{(1-t)},\frac{t}{1-t}\right) \tag{3}$$

Lemma 1 (see [5]) *Let* α *be a real number and* $L(z)=\ln\left(\frac{1}{1-z}\right)$. *When* $n\to\infty$,

$$\left[z^n\right](1-z)^{\alpha}L^k(z)\sim\frac{1}{\Gamma(-\alpha)}n^{-\alpha-1}\ln^k n,\quad\left(\alpha\notin\{0,1,2,\cdots\}\right)$$

$$\left[z^n\right](1-z)^{m}L^k(z)\sim(-1)^m km!n^{-m-1}\ln^{k-1}n,\quad\left(m\in\mathbb{Z}_{\ge0},k\in\mathbb{Z}_{\ge1}\right)$$

2. Identities Involving the Numbers $P(r,n,k)$ and Binomial Coefficients

Theorem 1. *Let* r, $k\ge0$, $n\ge1$, *then*

$$\sum_{j=r}^{n-1}\binom{j+k}{k}\frac{P(r,k+j,k)}{n-j}=\binom{n+k}{k}P(r+1,n+k,k) \tag{4}$$

Proof. By (1), we have

$$\sum_{n=r}^{\infty}\binom{k+n}{k}P(r+1,n+k,k)t^n=\frac{(-\ln(1-t))^{r+1}}{(1-t)^{k+1}}=\frac{(-\ln(1-t))^{r}}{(1-t)^{k+1}}(-\ln(1-t))=\sum_{n=r}^{\infty}\binom{k+n}{k}P(r,n+k,k)t^n\sum_{n=1}^{\infty}\frac{t^n}{n}. \tag{5}$$

Comparing the coefficients of t^n on both sides of (5), we completes the proof of Theorem 1.

Recall that $P(0,n,k)=1$ Thus, setting $r=0,1,2$ in Theorem 1 gives the next three identities, respectively.

Corollary 1. *Let* $n\ge 1$, $k\ge 0$, *the following relations hold*

$$\sum_{j=0}^{n-1}\binom{j+k}{k}\frac{1}{n-j}=\binom{n+k}{k}(H_{n+k}-H_k),$$

$$\sum_{j=0}^{n-1}\binom{j+k}{k}\frac{H_{j+k}}{n-j}=\binom{n+k}{k}\left(H_{n+k}^2-H_{n+k}H_k-H_{n+k}^{(2)}+H_k^{(2)}\right),$$

$$\sum_{j=0}^{n-1}\binom{j+k}{k}\frac{H_{k+j}^2-H_{k+j}^{(2)}}{n-j}=\binom{n+k}{k}\left((H_{n+k}-H_k)\left(H_{n+k}^2-H_{n+k}^{(2)}\right)-2H_{n+k}\left(H_{n+k}^{(2)}-H_k^{(2)}\right)+2\left(H_{n+k}^{(2)}-H_k^{(2)}\right)\right).$$

Theorem 2. *Let* $r\ge 0$, $n\ge 1$, *then*

$$\sum_{j_1=r}^{n-1}\sum_{j_2=r-1}^{j_1-1}\cdots\sum_{j_r=0}^{j_{r-1}-1}\binom{j_r+k}{k}\frac{1}{(n-j_1)(j_1-j_2)\cdots(j_{r-1}-j_r)}=\binom{n+k}{k}P(r+1,n+k,k) \tag{6}$$

Proof. To obtain the result, make use of the Theorem 1.

Theorem 3. *Let* $n,r\ge 0$, $m\ge k$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}P(r,n,k)=\binom{n+m}{m}P(r,n+m,m) \tag{7}$$

Proof. Applying the summation property (2) to the Riordan arrays (3), we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}P(r,n,k)=[t^n]\frac{(-\ln(1-t))^r}{(1-t)}\left((1+y)^m\middle|\,y=\frac{t}{1-t}\right)=[t^n]\frac{(-\ln(1-t))^r}{(1-t)^{m+1}}=\binom{n+m}{m}P(r,n+m,m),$$

which is just the desired result.

Setting $m=n$ in Theorem 3 gives the next Corollary.

Corollary 2 *Let* $n,r\ge 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}^2P(r,n,k)=\binom{2n}{n}P(r,2n,n)$$

Corollary 3 *Let* $n\ge 0$, $m\ge k$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}=\binom{n+m}{m},$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}H_k=\binom{n+m}{m}H_n-H_{n+m}+H_m,$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\left(H_k^2+H_k^{(2)}\right)=\binom{n+m}{m}\left((H_n-H_{n+m}+H_m)^2+H_n^{(2)}-H_{n+m}^{(2)}+H_m^{(2)}\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\left(H_k^3+3H_kH_k^{(2)}+2H_k^{(2)}\right)=\binom{n+m}{m}\Big(\left(H_n-H_{n+m}+H_m\right)^3+3\left(H_n-H_{n+m}\right)\left(H_m^{(2)}-H_{n+m}^{(2)}++H_m^{(2)}\right)$$
$$+2\left(H_n^{(2)}-H_{n+m}^{(2)}+H_m^{(2)}\right)+3H_m\left(H_n^{(2)}-H_{m+n}^{(2)}\right)\Big).$$

Proof. Setting $r=0,1,2,3$ in Theorem 3 gives Corollary 3.

Corollary 4. *Let* $n\geq 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n},$$
$$\sum_{k=0}^{n}\binom{n}{k}^2H_k=\binom{2n}{n}\left(2H_n-H_{2n}\right),$$
$$\sum_{k=0}^{n}\binom{n}{k}^2\left(H_k^2+H_k^{(2)}\right)=\binom{2n}{n}\left(\left(2H_n-H_{2n}\right)^2+2H_n^{(2)}-H_{2n}^{(2)}\right),$$
$$\sum_{k=0}^{n}\binom{n}{k}^2\left(H_k^3+3H_kH_k^{(2)}+2H_k^{(2)}\right)=\binom{2n}{n}\Big(\left(2H_n-H_{2n}\right)^3+2\left(2H_n^{(2)}-H_{2n}^{(2)}\right)$$
$$+3\left(H_n-H_{2n}\right)\left(2H_n^{(2)}-H_{2n}^{(2)}\right)+3H_n\left(H_n^{(2)}-H_{2n}^{(2)}\right)\Big).$$

Proof. Setting $r=0,1,2,3$ in Corollary 2 yields Corollary 4.

Theorem 4. *Let* $n\geq r$, $m\geq 2$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}kP(r,n,k)=m\binom{n+m-1}{n+1}P(r,n+m-1,m-2) \tag{8}$$

Proof. which is just the desired result.

Setting $m=n$ in Theorem 4 gives the next Corollary.

Corollary 5. *Let* $n,r\geq 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}^2kP(r,n,k)=n\binom{2n-1}{n+1}P(r,2n-1,n-2)$$

Corollary 6. *The substitutions* $r=0,1,2,3$ *in Theorem* 4 *gives the next four identities, respectively.*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}k=m\binom{n+m-1}{n+1},$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}kH_k=m\binom{n+m-1}{n+1}\left(H_n-H_{n+m-1}+H_{m-2}\right),$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}k\left(H_k^2+H_k^{(2)}\right)=m\binom{n+m-1}{n+1}\left(\left(H_n-H_{n+m-1}+H_{m-2}\right)^2+H_n^{(2)}-H_{n+m-1}^{(2)}+H_{m-2}^{(2)}\right),$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}k\left(H_k^3+3H_kH_k^{(2)}+2H_k^{(2)}\right)$$
$$=m\binom{n+m-1}{n+1}\Big(\left(H_n-H_{n+m-1}+H_{m-2}\right)^3+2\left(H_n^{(2)}-H_{n+m-1}^{(2)}+H_{m-2}^{(2)}\right)$$
$$+3H_{n+m-1}\left(H_{n+m-1}^{(2)}-H_n^{(2)}\right)+3\left(H_n^{(2)}-H_{n+m-1}^{(2)}+H_{m-2}^{(2)}\right)\left(H_n+H_{m-2}\right)\Big).$$

Setting $r=0,1,2,3$ in Corollary 5 gives the next four identities, respectively.

Corollary 7. *Let* $n\geq 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}^2 k = n\binom{2n-1}{n+1},$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 kH_k = n\binom{2n-1}{n+1}\left(H_n - H_{2n-1} + H_{n-2}\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 k\left(H_k^2 + H_k^{(2)}\right) = n\binom{2n-1}{n+1}\left(\left(H_n - H_{2n-1} + H_{n-2}\right)^2 + H_n^{(2)} - H_{2n-1}^{(2)} + H_{n-2}^{(2)}\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 \left(H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(2)}\right) = n\binom{2n-1}{n+1}\Big(\left(H_n - H_{2n-1} + H_{n-2}\right)^3 + 2\left(H_n^{(2)} - H_{2n-1}^{(2)} + H_{n-2}^{(2)}\right)$$
$$+3H_{2n-1}\left(H_{2n-1}^{(2)} - H_n^{(2)}\right) + 3\left(H_n^{(2)} - H_{2n-1}^{(2)} + H_{n-2}^{(2)}\right)\left(H_n + H_{n-2}\right)\Big).$$

Theorem 5. *Let* $n, r \ge 0$, $m \ge k$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{P(r,n,k)}{k+1} = \frac{1}{m+1}\left(\binom{n+m+1}{m}P(r,n+m+1,m) - \frac{r!\left|s(n+1,r)\right|}{(n+1)!}\right) \tag{9}$$

where $s(n,h)$ are the Stirling numbers of the first kind.

Proof. By (1) and (2), we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{P(r,n,k)}{k+1} = \left[t^n\right]\frac{\left(-\ln(1-t)\right)^r}{(1-t)}\left[\frac{(1+y)^{m+1}-1}{y(m+1)}\Bigg| y = \frac{t}{1-t}\right] = \left[t^{n+1}\right]\frac{1}{m+1}\left(\frac{\left(-\ln(1-t)\right)^r}{(1-t)^{m+1}} - \left(-\ln(1-t)\right)^r\right)$$
$$= \frac{1}{m+1}\left(\binom{n+m+1}{m}P(r,n+m+1,m) - \frac{r!\left|s(n+1,r)\right|}{(n+1)!}\right),$$

which is just the desired result.

Setting $m = n$ in Theorem 5 gives the next Corollary.

Corollary 8. *Let* $n, r \ge 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{P(r,n,k)}{k+1} = \frac{1}{n+1}\left(\binom{2n+1}{n}P(r,2n+1,n) - \frac{r!\left|s(n+1,r)\right|}{(n+1)!}\right)$$

Setting $r = 0,1,2,3$ in Theorem 6 gives the next Corollary.

Corollary 9. *Let* $n \ge 0$, $m \ge k$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{1}{k+1} = \frac{1}{m+1}\binom{n+m+1}{m},$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{H_k}{k+1} = \frac{1}{m+1}\binom{n+m+1}{m}\left(H_n - H_{n+m+1} + H_m\right) + \frac{1}{(m+1)(n+1)},$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{H_k^2 + H_k^{(2)}}{k+1} = \frac{1}{m+1}\binom{n+m+1}{m}\left(\left(H_n - H_{n+m+1} + H_m\right)^2 + H_n^{(2)} - H_{n+m+1}^{(2)} + H_m^{(2)}\right) + \frac{2H_n}{(m+1)(n+1)},$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(2)}}{k+1} = \frac{1}{m+1}\binom{n+m+1}{m}\Big(\left(H_n - H_{n+m+1} + H_m\right)^3$$
$$+\left(2 + 3\left(H_n + H_m\right)\right)\left(H_n^{(2)} - H_{n+m+1}^{(2)} + H_m^{(2)}\right) + 3H_{n+m+1}\left(H_{n+m+1}^{(2)} - H_m^{(2)}\right)\Big)$$
$$+\frac{3\left(H_n^{(2)} + H_m^{(2)}\right)}{(m+1)(n+1)}.$$

We give four applications of Corollary 9:

Corollary 10. *Let* $n \ge 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{1}{k+1}=\frac{1}{n+1}\binom{2n+1}{n},$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{H_k}{k+1}=\frac{1}{n+1}\binom{2n+1}{n}\left(2H_n-H_{2n+1}\right)+\frac{1}{(n+1)^2},$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{H_k^2+H_k^{(2)}}{k+1}=\frac{1}{n+1}\binom{2n+1}{n}\left(\left(2H_n-H_{2n+1}\right)^2+2H_n^{(2)}-H_{2n+1}^{(2)}\right)+\frac{2H_n}{(n+1)^2},$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{H_k^3+3H_kH_k^{(2)}+2H_k^{(2)}}{k+1}=\frac{1}{n+1}\binom{2n+1}{n}\Bigg(\left(2H_n-H_{2n+1}\right)^3+\frac{6H_n^{(2)}}{(n+1)^2}$$
$$+\left(2+6H_n\right)\left(2H_n^{(2)}-H_{2n+1}^{(2)}\right)+3H_{2n+1}\left(H_{2n+1}^{(2)}-H_n^{(2)}\right)\Bigg).$$

3. Identities Involving $P(r,n,k)$ and Inverse of Binomial Coefficients

For identities involving Harmonic numbers and inverse of binomial coefficients $\sum_{n=1}^{\infty}\frac{H_{2n}}{n\binom{n+k}{k}}$ in given in [6].

In Section, we obtain some for $P(r,n,k)$ and binomial coefficients by means of the Riordan arrays. From these identities, we deduce some identities involving binomial coefficients, Harmonic numbers and identities related to $\zeta(2)$, $\zeta(3)$

In [7], the inverse of a binomial coefficient is related to an integral, as follows

$$\binom{n}{k}^{-1}=(n+1)\int_0^1 t^k(1-t)^{n-k}\,\mathrm{d}t \tag{10}$$

From the generating function of $P(r,n,k)$ and (10), we have

Theorem 6. *For* $r\ge 0$ *be any integer, then*

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\binom{n+k}{k}^{-1}\frac{P(r,n,k-1)}{n+k+1}=-\frac{r!}{k}\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^j}{j^r} \tag{11}$$

Proof. From (1) and (10), we obtain

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\binom{n+k}{k}^{-1}\frac{P(r,n,k-1)}{n+k+1}=\sum_{n=k-1}^{\infty}\binom{n}{k-1}P(r,n,k-1)\int_0^1 t^k(1-t)^n\,\mathrm{d}t$$
$$=\int_0^1 t^k\frac{(1-t)^{k-1}\left(-\ln\left(1-(1-t)\right)\right)^r}{\left(1-(1-t)\right)^k}\mathrm{d}t$$
$$=\int_0^1(1-t)^{k-1}(-\ln t)^r\,\mathrm{d}t$$
$$=-\frac{r!}{k}\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^j}{j^r}.$$

This gives (11).

Corollary 11 *Setting* $r=0,1,2,3$ *in Theorem* 6, *The following relation holds*:

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\binom{n+k}{k}^{-1}\frac{1}{n+k+1}=\frac{1}{k} \tag{12}$$

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\binom{n+k}{k}^{-1}\frac{H_n}{n+k+1}=\frac{1}{k}\left(H_k+H_{k-1}\right) \tag{13}$$

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\binom{n+k}{k}^{-1}\frac{H_n^2-H_n^{(2)}}{n+k+1}=\frac{1}{k}\left(\left(H_k+H_{k-1}\right)^2+H_k^{(2)}-H_{k-1}^{(2)}\right) \tag{14}$$

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\binom{n+k}{k}^{-1}\frac{H_n^3+2H_n^{(2)}-3H_nH_n^{(2)}}{n+k+1}$$
$$=\frac{1}{k}\left(\left(H_k+H_{k-1}\right)^3+2\left(H_{k-1}^{(2)}+H_k^{(3)}\right)+3\left(H_k+H_{k-1}\right)\left(H_k^{(2)}-H_{k-1}^{(2)}\right)\right) \tag{15}$$

Setting $k=1,2$ in Corollary 11, gives the next identities.
Corollary 12 *The following relation holds*

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}=1 \tag{16}$$

$$\sum_{n=0}^{\infty}\frac{H_n}{(n+1)(n+2)}=1 \tag{17}$$

$$\sum_{n=0}^{\infty}\frac{H_n^2-H_n^{(2)}}{(n+1)(n+2)}=2 \tag{18}$$

$$\sum_{n=0}^{\infty}\frac{H_n^3+2H_n^{(2)}-3H_nH_n^{(2)}}{(n+1)(n+2)}=6 \tag{19}$$

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)(n+3)}=\frac{1}{4} \tag{20}$$

$$\sum_{n=1}^{\infty}\frac{nH_n}{(n+1)(n+2)(n+3)}=\frac{5}{8} \tag{21}$$

$$\sum_{n=1}^{\infty}\frac{n\left(H_n^2-H_n^{(2)}\right)}{(n+1)(n+2)(n+3)}=\frac{13}{8} \tag{22}$$

$$\sum_{n=1}^{\infty}\frac{n\left(H_n^3+2H_n^{(2)}-3H_nH_n^{(2)}\right)}{(n+1)(n+2)(n+3)}=\frac{87}{16} \tag{23}$$

Corollary 13. *The following relation holds*

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{4} \tag{24}$$

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+2)(n+3)}=\frac{1}{8} \tag{25}$$

$$\sum_{n=1}^{\infty}\frac{H_n^2-H_n^{(2)}}{(n+1)(n+2)(n+3)}=\frac{1}{8} \tag{26}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3+2H_n^{(2)}-3H_nH_n^{(2)}}{(n+1)(n+2)(n+3)}=\frac{3}{16} \tag{27}$$

Proof. (16) minus(20) give (24); (17) minus (21), (18) minus (22) and (19) minus (23), yields (25), (26) and (27), respectively.

Leonhard Euler (1707-1783) had already stated the equation

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$$

Recall the Euler sum identities [8] [9].

$$\sum_{n=k}^{\infty}\frac{H_n}{n^2}=2\zeta(3),\quad \sum_{n=k}^{\infty}\frac{H_n}{n^3}=\frac{5}{4}\zeta(4),\quad \sum_{n=k}^{\infty}\frac{H_n}{n(n+1)}=\zeta(2),\quad \sum_{n=k}^{\infty}\frac{H_n}{(n+1)^2}=\zeta(3)$$

The next, we gives identities related to $\zeta(2)$, $\zeta(3)$

For completeness we supply proofs:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+2)}&=\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\left(H_{n-1}+\frac{1}{n}\right)^2-\frac{H_n^2}{n+2}\right)=\sum_{n=1}^{\infty}\left(\frac{H_{n-1}^2}{n+1}-\frac{H_n^2}{n+2}+\frac{2H_{n-1}}{n(n+1)}+\frac{1}{n^2(n+1)}\right)\\&=2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+2)}+\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1+\zeta(2).\end{aligned}\tag{28}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(n+3)}&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}+\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{H_n}{(n+3)}-\frac{H_n}{(n+1)}\right)\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}+\frac{1}{4}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+3)}-\sum_{n=1}^{\infty}\frac{1}{(n+1)}\left(H_{n-2}+\frac{1}{(n-1)}+\frac{1}{n}\right)\right)\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}-\frac{1}{4}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)n}-\sum_{n=2}^{\infty}\frac{1}{(n+1)(n-1)}\right)=\frac{1}{2}\zeta(2)-\frac{7}{16}.\end{aligned}\tag{29}$$

Similarly, we obtain summation formulas related $\zeta(2)$, $\zeta(3)$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+3)}=\frac{1}{2}(1+\zeta(2))\tag{30}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)(n+2)}=1+2\zeta(2)+4\zeta(3)\tag{31}$$

By (18) and (28), (19) and (31), we have

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)(n+2)}=\zeta(2)-1\tag{32}$$

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{(n+1)(n+2)}=\frac{4}{3}(\zeta(2)+\zeta(3))-1\tag{33}$$

Similarly, for completeness we supply a proof:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)(n+2)(n+3)}&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)(n+2)}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^3}{(n+2)(n+3)}\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)(n+2)}-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+2}\left(H_{n-1}+\frac{1}{n}\right)^3-\frac{H_n^3}{n+3}\right)\\&=\frac{1}{2}(1+2\zeta(2)+4\zeta(3))-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+3)}-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(n+3)}\\&=\frac{1}{4}\left(\zeta(2)+5\zeta(3)+\frac{13}{8}\right).\end{aligned}\tag{34}$$

By (28) minus (30), we get

$$\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+2)(n+3)}=\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(n+2)}-\sum_{n=1}^{\infty}\frac{H_n^2}{(n+2)(n+3)}=\frac{1}{2}\left(1+\zeta(2)\right) \tag{35}$$

Applying (25) and (34), (26) and (32), we have

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)(n+2)(n+3)}=\frac{1}{2}\zeta(2)+\frac{3}{8},\quad \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{(n+1)(n+2)(n+3)}=\frac{5}{12}\left(\zeta(2)+\zeta(3)\right)+\frac{31}{96}$$

4. Asymptotics

Theorem 7 *For* $r\geq 1$ *be any integer, as* $n\to\infty$, *we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}P(r,n,k)\sim\begin{cases}\dfrac{n^{m-2}\ln^r n}{\Gamma(m-1)}, & m\notin\{1,0,-1-2\cdots\};\\[2ex] \dfrac{(-1)^{-m+1}r(-m+1)!\ln^{r-1}n}{n^{2-m}}, & m\in\{1,0,-1-2\cdots\}.\end{cases} \tag{36}$$

Proof. By Lemma 1, we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}P(r,n,k)=\left[t^n\right]\frac{\left(-\ln(1-t)\right)^r}{(1-t)^{m+1}}\sim\begin{cases}\dfrac{n^{m-2}\ln^r n}{\Gamma(m-1)}, & m\notin\{1,0,-1-2\cdots\};\\[2ex] \dfrac{(-1)^{-m+1}r(-m+1)!\ln^{r-1}n}{n^{2-m}}, & m\in\{1,0,-1-2\cdots\}.\end{cases}$$

and this complete the proof.

Similarly, we can obtain the next Theorem.

Theorem 8. *Let* $r\geq 1$ *be any integer, as* $n\to\infty$, *we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}kP(r,n,k)\sim\begin{cases}\dfrac{mn^{m-2}\ln^r n}{\Gamma(m-1)}, & m\notin\{1,0,-1-2\cdots\};\\[2ex] \dfrac{m(-1)^{-m+1}r(-m+1)!\ln^{r-1}n}{n^{2-m}}, & m\in\{1,0,-1-2\cdots\}.\end{cases} \tag{37}$$

Theorem 9. *For* $r\geq 1$ *be any integer, as* $n\to\infty$, *we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{P(r,n,k)}{k+1}\sim\begin{cases}\dfrac{1}{m+1}\left(\dfrac{n^{m-2}\ln^r n}{\Gamma(m-1)}-\dfrac{r\ln^{r-1}n}{n+1}\right), & m\notin\{1,0,-1-2\cdots\};\\[2ex] \dfrac{1}{m+1}\left(\dfrac{(-1)^{-m+1}r(-m+1)!\ln^{r-1}n}{n^{2-m}}-\dfrac{r\ln^{r-1}n}{n+1}\right), & m\in\{1,0,-1-2\cdots\}.\end{cases} \tag{38}$$

Proof. By Lemma 1, we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{P(r,n,k)}{k+1}=\left[t^{n+1}\right]\frac{1}{m+1}\left(\frac{\left(-\ln(1-t)\right)^r}{(1-t)^{m+1}}-\left(-\ln(1-t)\right)^r\right)$$

$$\sim\begin{cases}\dfrac{1}{m+1}\left(\dfrac{n^{m-2}\ln^r n}{\Gamma(m-1)}-\dfrac{r\ln^{r-1}n}{n+1}\right), & m\notin\{1,0,-1-2\cdots\};\\[2ex] \dfrac{1}{m+1}\left(\dfrac{(-1)^{-m+1}r(-m+1)!\ln^{r-1}n}{n^{2-m}}-\dfrac{r\ln^{r-1}n}{n+1}\right), & m\in\{1,0,-1-2\cdots\}.\end{cases}$$

this give (38).

Theorem 10. *For* $r \geq 0$ *be any integer, as* $k \to \infty$, *we have*

$$\sum_{n=k-1}^{\infty} \binom{n}{k-1} \binom{n+k}{k}^{-1} \frac{P(r,n,k-1)}{n+k+1} \sim \frac{1}{k} \sum_{j=0}^{r} \binom{r}{j} (-1)^j \Gamma^{(j)}(1) \ln^{r-j} k + O\left(\frac{r! \ln^r k}{k^2}\right) \quad (39)$$

Proof. By Corollary 3 of [10], immediately complete the proof of Theorem 10.

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