

Exact Traveling Wave Solutions for the System of Shallow Water Wave Equations and Modified Liouville Equation Using Extended Jacobian Elliptic Function Expansion Method

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Abstract

In this work, an extended Jacobian elliptic function expansion method is proposed for constructing the exact solutions of nonlinear evolution equations. The validity and reliability of the method are tested by its applications to the system of shallow water wave equations and modified Liouville equation which play an important role in mathematical physics.

Keywords

Extended Jacobian Elliptic Function Expansion Method, The System of Shallow Water Wave Equations, Modified Liouville Equation, Traveling Wave Solutions, Solitary Wave Solutions

1. Introduction

The nonlinear partial differential equations of mathematical physics are major subjects in physical science [1]. Exact solutions for these equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, optics, plasma physics and so on. Recently many new approaches for finding these solu-

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tions have been proposed, for example, tanh-sech method [2]-[4], extended tanh-method [5]-[7], sine-cosine method [8]-[10], homogeneous balance method [11] [12], F-expansion method [13]-[15], exp-function method [16], the modified simple equation method [17], the $\exp(-\phi(\xi))$ -expansion method [18], $\left(\frac{G'}{G}\right)$ -expansion method [19]-[22], Jacobi elliptic function method [23]-[26] and so on.

The objective of this article is to apply the extended Jacobian elliptic function expansion method for finding the exact traveling wave solution the system of shallow water wave equations and modified Liouville equation which play an important role in mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of the extended Jacobian elliptic function expansion method. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. Description of Method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \tag{1}$$

where F is polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [23]-[26].

Step 1. Using the transformation

$$u = u(\xi), \quad \xi = x - ct, \tag{2}$$

where c is wave speed, to reduce Equation (1) to the following ODE:

$$P(u, u', u'', u''', \dots) = 0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}'$.

Step 2. Making good use of ten Jacobian elliptic functions, we assume that (3) has the solutions in these forms:

$$u(\xi) = a_0 + \sum_{j=1}^N f_i^{j-1}(\xi) [a_j f_i(\xi) + b_j g_i(\xi)], \quad i = 1, 2, 3, \dots, \tag{4}$$

with

$$\begin{aligned} f_1(\xi) &= sn\xi, & g_1(\xi) &= cn\xi, \\ f_2(\xi) &= sn\xi, & g_2(\xi) &= dn\xi, \\ f_3(\xi) &= ns\xi, & g_3(\xi) &= cs\xi, \\ f_4(\xi) &= ns\xi, & g_4(\xi) &= ds\xi, \\ f_5(\xi) &= sc\xi, & g_5(\xi) &= nc\xi, \\ f_6(\xi) &= sd\xi, & g_6(\xi) &= nd\xi, \end{aligned} \tag{5}$$

where $sn\xi$, $cn\xi$, $dn\xi$, are the Jacobian elliptic sine function, the jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaisher's symbols and are generated by these three kinds of functions, namely

$$\begin{aligned} ns\xi &= \frac{1}{sn\xi}, \quad nc\xi = \frac{1}{cn\xi}, \quad nd\xi = \frac{1}{dn\xi}, \quad sc\xi = \frac{cn\xi}{sn\xi}, \\ cs\xi &= \frac{sn\xi}{cn\xi}, \quad ds\xi = \frac{dn\xi}{sn\xi}, \quad sd\xi = \frac{sn\xi}{dn\xi}, \end{aligned} \tag{6}$$

that have the relations

$$\begin{aligned} sn^2\xi + cn^2\xi &= 1, \quad dn^2\xi + m^2 sn^2\xi = 1, \quad ns^2\xi = 1 + cs^2\xi, \\ ns^2\xi &= m^2 + ds^2\xi, \quad sc^2\xi + 1 = nc^2\xi, \quad m^2 sd^2 + 1 = nd^2\xi, \end{aligned} \tag{7}$$

with the modulus m ($0 < m < 1$). In addition we know that

$$\frac{d}{d\xi} \operatorname{sn}\xi = \operatorname{cn}\xi \operatorname{dn}\xi, \quad \frac{d}{d\xi} \operatorname{cn}\xi = -\operatorname{sn}\xi \operatorname{dn}\xi, \quad \frac{d}{d\xi} \operatorname{dn}\xi = -m^2 \operatorname{sn}\xi \operatorname{cn}\xi. \quad (8)$$

The derivatives of other Jacobian elliptic functions are obtained by using Equation (8). To balance the highest order linear term with nonlinear term we define the degree of u as $D[u] = n$ which gives rise to the degrees of other expressions as

$$D\left[\frac{d^q u}{d\xi^q}\right] = n + q, \quad D\left[u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right] = np + s(n + q). \quad (9)$$

According to the rules, we can balance the highest order linear term and nonlinear term in Equation (3) so that n in Equation (4) can be determined.

Noticed that $\operatorname{sn}\xi \rightarrow \tanh \xi$, $\operatorname{cn}\xi \rightarrow \operatorname{sech} \xi$, $\operatorname{dn}\xi \rightarrow \operatorname{sech} \xi$ when the modulus $m \rightarrow 1$ and $\operatorname{sn}\xi \rightarrow \sin \xi$, $\operatorname{cn}\xi \rightarrow \cos \xi$, $\operatorname{dn}\xi \rightarrow 1$ when the modulus $m \rightarrow 0$, we can obtain the corresponding solitary wave solutions and triangle function solutions, respectively, while when therefore Equation (5) degenerate as the following forms

$$u(\xi) = a_0 + \sum_{j=1}^N \tanh^{j-1}(\xi) [a_j \tanh(\xi) + b_j \operatorname{sech}(\xi)], \quad (10)$$

$$u(\xi) = a_0 + \sum_{j=1}^N \coth^{j-1}(\xi) [a_j \coth(\xi) + b_j \coth(\xi)], \quad (11)$$

$$u(\xi) = a_0 + \sum_{j=1}^N \tan^{j-1}(\xi) [a_j \tan(\xi) + b_j \sec(\xi)], \quad (12)$$

$$u(\xi) = a_0 + \sum_{j=1}^N \cot^{j-1}(\xi) [a_j \cot(\xi) + b_j \csc(\xi)]. \quad (13)$$

Therefore the extended Jacobian elliptic function expansion method is more general than sine-cosine method, the tan-function method and Jacobian elliptic function expansion method.

3. Application

3.1. Example 1: The System of Shallow Water Wave Equations

We first consider the system of the shallow water wave equation [27] in order to demonstrate the $\exp(-\phi(\xi))$ -expansion method

$$\begin{cases} u_t + (uv)_x + v_{xxx} = 0, \\ v_t + u_x + vv_x = 0. \end{cases} \quad (14)$$

We use the wave transformation $u(x, t) = u(\xi)$, $\xi = x - ct$ to reduce Equations (14) to the following nonlinear system of ordinary differential equations:

$$\begin{cases} -cu' + vu' + uv' + v''' = 0, \\ u' - cv' + vv' = 0, \end{cases} \quad (15)$$

where by integrating once the second equation with zero constant of integration, we find

$$u = cv - \frac{v^2}{2} \quad (16)$$

substituting Equation (16) into the first equation of Equation (15) we obtain

$$v''' + \left(3cv - \frac{3v^2}{2} - c^2\right)v' = 0. \quad (17)$$

Integrating Equation (17) with zero constant of integration, we find

$$v'' + \frac{3}{2}cv^2 - \frac{1}{2}v^3 - c^2v = 0. \tag{18}$$

Balancing v'' and v^3 in Equation (18) yields, $N + 2 = 3N \Rightarrow N = 1$. This suggests the choice of $v(\xi)$ in Equation (18) as

$$u = a_0 + a_1sn + b_1cn, \tag{19}$$

where a_0, a_1 and b_1 are constant such that $a_1 \neq 0$ or $b_1 \neq 0$. From (19), it is easy to see that

$$u' = a_1cndn - b_1snda, \tag{20}$$

$$u'' = -m^2sna_1 + 2a_1sn^3m^2 + 2m^2sn^2cnb_1 - a_1sn - b_1cn. \tag{21}$$

Substituting Equations (19) and (21) into Equation (18) and equating all coefficients of $sn^3, sn^2cn, sn^2, sn cn, sn, cn, sn^0$ respectively to zero, we obtain:

$$2a_1m^2 - \frac{1}{2}a_1^3 + \frac{3}{2}a_1b_1^2 = 0, \tag{22}$$

$$2m^2b_1 - \frac{3}{2}a_1^2b_1 + \frac{1}{2}b_1^3 = 0, \tag{23}$$

$$\frac{3}{2}ca_1^2 - \frac{3}{2}cb_1^2 - \frac{3}{2}a_0a_1^2 + \frac{3}{2}a_0b_1^2 = 0, \tag{24}$$

$$3ca_1b_1 - 3a_0a_1b_1 = 0, \tag{25}$$

$$-a_1m^2 - a_1 + 3ca_0a_1 - \frac{3}{2}a_0^2a_1 - \frac{3}{2}a_1b_1^2 - c^2a_1 = 0, \tag{26}$$

$$-b_1 + 3ca_0b_1 - \frac{3}{2}a_0^2b_1 - \frac{1}{2}b_1^3 - c^2b_1 = 0, \tag{27}$$

$$\frac{3}{2}c(a_0^2 + b_1^2) - \frac{1}{2}a_0^3 - \frac{3}{2}a_0b_1^2 - c^2a_0 = 0. \tag{28}$$

Solving the above system with the aid of Maple or Mathematica, we have the following solution:

Case 1.

$$c = a_0 = \pm\sqrt{2m^2 + 2}, a_1 = \pm 2m, b_1 = 0.$$

So that the solution of Equation (18) can be written as

$$u = \pm\sqrt{2m^2 + 2} \pm 2msn, \tag{29}$$

when $m = 1$, the solution can be in the form

$$u = \pm 2 \pm 2 \tanh(\xi). \tag{30}$$

Case 2.

$$c = a_0 = \pm\sqrt{2 - m^2}, a_1 = \pm m, b_1 = \pm im.$$

So that the solution of Equation (18) can be written as

$$u = \pm\sqrt{2 - m^2} \pm msn \pm imcn, \tag{31}$$

when $m = 1$, the solution can be in the form

$$u = \pm 1 \pm \tanh(\xi) \pm i \operatorname{sech}(\xi). \tag{32}$$

Case 3.

$$c = a_0 = \pm\sqrt{2 - 4m^2}, a_1 = 0, b_1 = \pm 2im.$$

So that the solution of Equation (18) can be written as

$$u = \pm\sqrt{2-4m^2} \pm 2imcn, \quad (33)$$

when $m = 1$, the solution can be in the form

$$u = \pm\sqrt{-2} \pm 2\text{isech}(\xi). \quad (34)$$

3.2. Example 2: Modified Liouville Equation

Now, let us consider the modified Liouville equation [28].

$$a^2u_{xx} - u_{tt} + be^{\beta u} = 0, \quad (35)$$

respectively, where a , β and b are non zero and arbitrary coefficients. Using the wave transformation $u(x, t) = u(\xi)$, $\xi = kx + \omega t$, $v = e^{\beta u}$, to reduce Equation (35) to be in the form:

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)v''v - \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)v'^2 + bv^3 = 0. \quad (36)$$

Balancing $v''v$ and v^3 in Equation (36) yields, $N + 2 + N = 3N \Rightarrow N = 2$. Consequently, we have the formal solution:

$$u(\xi) = a_0 + a_1sn + b_1cn + a_2sn^2 + b_2sncn, \quad (37)$$

where a_0 , a_1 , a_2 are constants to be determined, such that $a_2 \neq 0$ or $b_2 \neq 0$. It is easy to see that

$$u' = a_1cndn - b_1sndn + 2dna_2sncn - 2dnb_2sn^2 + دنب_2, \quad (38)$$

$$u'' = -m^2sna_1 + 2a_1sn^3m^2 + 2m^2sn^2cnb_1 - 4a_2m^2sn^2 + 6a_2sn^4m^2 + 6m^2sn^3cnb_2 - m^2sncnb_2 - a_1sn - b_1cn + 2a_2 - 4a_2sn^2 - 4b_2sncn. \quad (39)$$

Substituting (37) and (39) into Equation (36) and equating all the coefficients of sn^6 , sn^5cn , sn^5 , sn^4cn , sn^4 , sn^3cn , sn^3 , sn^2cn , sn^2 , $sncn$, sn , cn , sn^0 to zero, we deduce respectively

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-2m^2b_2^2 + 2a_2^2m^2) + b(a_2^3 - 3a_2b_2^2) = 0, \quad (40)$$

$$4\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)a_2m^2b_2 + b(3a_2^2b_2 - b_2^3) = 0, \quad (41)$$

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-4m^2b_1b_2 + 4a_1m^2a_2) + b(3a_1a_2^2 - 6b_1a_2b_2 - 3a_1b_2^2) = 0, \quad (42)$$

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(4a_1m^2b_2 + 4m^2b_1a_2) + b(-3b_1b_2^2 + 3b_1a_2^2 + 6a_1a_2b_2) = 0, \quad (43)$$

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-m^2b_1^2 + 2m^2b_2^2 + a_1^2m^2 + 6a_2m^2a_0) + b(3a_1^2a_2 + 3a_0a_2^2 - 6a_1b_1b_2 + 3a_2b_2^2 - 3b_1^2a_2 - 3a_0b_2^2) = 0, \quad (44)$$

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(2a_1m^2b_1 + 6m^2b_2a_0 - a_2m^2b_2) + b(-3b_1^2b_2 + 3a_1^2b_2 + b_2^3 + 6a_0a_2b_2 + 6a_1b_1a_2) = 0, \quad (45)$$

$$\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(b_1b_2 - a_1a_2 - a_1m^2a_2 + 2a_1m^2a_0 + 7m^2b_1b_2) + b(a_1^3 - 3a_1b_1^2 - 6a_0b_1b_2 + 6a_0a_1a_2 + 3a_1b_2^2 + 6b_1a_2b_2) = 0, \quad (46)$$

$$\left(\frac{k^2 a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-b_1 a_2 - a_1 b_2 + 2m^2 b_1 a_0 - 4m^2 b_1 a_2) + b(3a_1^2 b_1 - b_1^3 + 3b_1 b_2^2 + 6a_0 a_1 b_2 + 6a_0 b_1 a_2) = 0, \quad (47)$$

$$\left(\frac{k^2 a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-4a_2 a_0 + 2m^2 b_1^2 - 4a_2 m^2 a_0 - 2a_2^2) + b(3a_0 a_1^2 - 3a_0 b_1^2 + 3a_2 a_0^2 + 3b_1^2 a_2 + 3a_0 b_2^2 + 6a_1 b_1 b_2) = 0, \quad (48)$$

$$\left(\frac{k^2 a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-2a_2 b_2 - 4b_2 a_0 - m^2 b_2 a_0 - a_1 m^2 b_1) + b(6a_0 a_1 b_1 + 3b_2 a_0^2 + 3b_1^2 b_2) = 0, \quad (49)$$

$$\left(\frac{k^2 a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-a_0 b_1 + 2b_1 a_2 - 2a_1 b_2) + b(3a_0^2 b_1 + b_1^3) = 0, \quad (50)$$

$$\left(\frac{k^2 a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-a_0 a_1 - 3b_1 b_2 - 2a_1 a_2 - a_1 m^2 a_0 - m^2 b_1 b_2) + b(3a_0^2 a_1 + 3a_1 b_1^2 + 6a_0 b_1 b_2) = 0, \quad (51)$$

$$\left(\frac{k^2 a^2}{\beta} - \frac{\omega^2}{\beta}\right)(-b_2^2 - b_1^2 + 2a_2 a_0 - a_1^2) + b(a_0^3 + 3a_0 b_1^2) = 0, \quad (52)$$

Solving the above system with the aid of Maple or Mathematica, we have the following solution:

$$a = a, b = \frac{-2(k^2 a^2 - \omega^2)}{\beta a_2}, k = k, m = \pm 1, \beta = \beta, a_0 = -a_2, a_1 = 0, a_2 = a_2, b_1 = b_2 = 0.$$

So that the solve of Equation (36) can be written in the form

$$v = \frac{2(k^2 a^2 - \omega^2)}{\beta b} - \frac{2(k^2 a^2 - \omega^2)}{\beta b} sn^2, \quad (53)$$

$$u = \frac{1}{\beta} \ln \left(\frac{2(k^2 a^2 - \omega^2)}{\beta b} - \frac{2(k^2 a^2 - \omega^2)}{\beta b} sn^2 \right). \quad (54)$$

When $m = 1$, the solution can be in the form

$$v = \frac{2(k^2 a^2 - \omega^2)}{\beta b} - \frac{2(k^2 a^2 - \omega^2)}{\beta b} \tanh^2(\xi), \quad (55)$$

$$u = \frac{1}{\beta} \ln \left(\frac{2(k^2 a^2 - \omega^2)}{\beta b} - \frac{2(k^2 a^2 - \omega^2)}{\beta b} \tanh^2(\xi) \right). \quad (56)$$

4. Conclusions

We establish exact solutions for the system of shallow water wave equations and modified Liouville equation which are two of the most fascinating problems of modern mathematical physics.

The extended Jacobian elliptic function expansion method has been successfully used to find the exact traveling wave solutions of some nonlinear evolution equations. As an application, the traveling wave solutions for the system of shallow water wave equations and modified Liouville equation, have been constructed using the extended Jacobian elliptic function expansion method. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: our results of the system of shallow water wave equations and modified Liouville equation are new and different from those obtained in [27] and [28] and **Figure 1** and **Figure 2** show the solitary wave solution of Equations

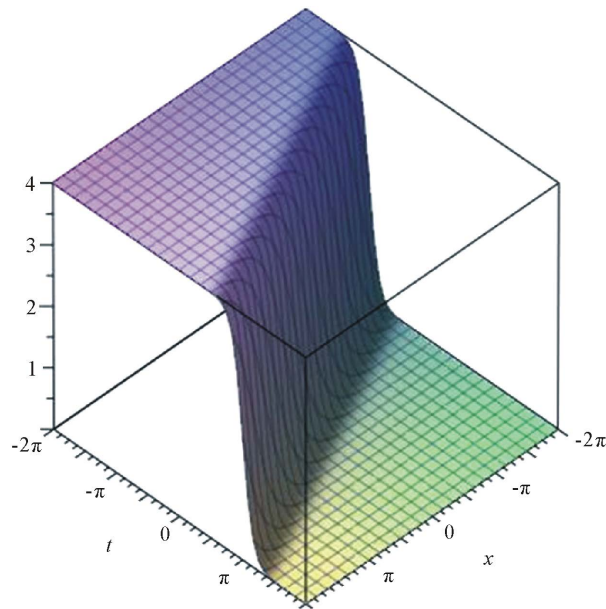


Figure 1. Solitary wave solution of Equation (30).

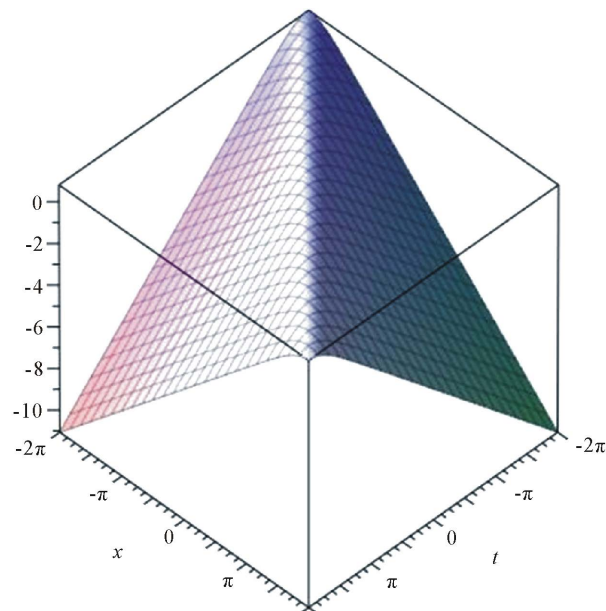


Figure 2. Solitary wave solution of Equation (56).

(30) and (56). It can be concluded that this method is reliable and proposes a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations.

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